Iranian Journal of Mathematical Chemistry

IJMC

DOI: 10.22052/IJMC.2023.252990.1723
Vol. 15, No. 2, 2024, pp. 79-90
Research Paper

Computation of Some Graph Energies of the Zero-Divisor Graph Associated with the Commutative Ring $\mathbb{Z}_{p^2}[x]/\langle x^2\rangle$

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Keywords:
Zero-divisor graph,
Commutative ring,
Adjacency matrix,
Laplacian matrix,
Laplacian energy

AMS Subject Classification (2020):

05C25; 05C50; 05C90

Article History:
Received: 23 May 2023
Accepted: 22 August 2023

Abstract

Let $\mathcal{R}$ be the commutative ring $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ with identity and $Z^*(\mathcal{R})$ be the set of all non-zero zero-divisors of $\mathcal{R}$. Then, $\Gamma(\mathcal{R})$ is said to be a zero-divisor graph if and only if $a \cdot b = 0$ where $a, b \in V(\Gamma(\mathcal{R})) = Z^*(\mathcal{R})$ and $(a, b) \in E(\Gamma(\mathcal{R}))$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix of $\Gamma(\mathcal{R})$. Then we discuss the energy $\mathcal{E}(\Gamma(\mathcal{R})) = \sum_{i=1}^{n}|\lambda_i|$ and the Laplacian energy $\mathcal{LE}(\Gamma(\mathcal{R})) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$ where n and m are the order and size of $\Gamma(\mathcal{R})$.

1 Introduction

A graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is a non-empty set whose elements are called vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called edges of the graph G. In the graph G, the number of edges that are incident to a vertex v is called its degree, and it is denoted by $d(v)$.The zero-divisors are one of the most intriguing aspects of a ring. Rare's zero-divisors are defined as two non-zero elements of $\mathcal{R}$, say x and y, whose product is zero. An integral domain is a ring that does not have any non-zero zero divisors. Meanwhile, probability theory has been widely applied to determine certain properties of finite groups. The results are then used to calculate a type of probability in rings, namely the likelihood that two elements of a ring have a product of zero [1–3]. Let $\mathcal{R}$ be a commutative ring with $Z^*(\mathcal{R})$ is the set of all non-zero divisors considered vertices, and there is an edge defined between x and y by having a zero product. Istvan Beck [4] presented the idea of the zero-divisor graphs $\Gamma_0(\mathcal{R})$ of commutative rings and included 0 in the definition of the term. His primary focus was on the various hues that these rings could take on. Later on, authors of [5, 6] modified the definition of the zero-divisor graphs $\Gamma(\mathcal{R})$ by excluding 0 of the ring from the zero-divisor set. Additionally, they defined the edges between

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Academic Editor: Saeid Alikhani

two non-zero zero-divisors if and only if their product is zero. In the field of mathematics, the concept of matrix energy has been discussed for quite some time [7–9]. Furthermore, it has been demonstrated to have a number of practical applications in other branches of science, such as chemistry and physics. Given the prevalence of matrices in the field of research on complex networks, it is only natural to investigate whether or not matrix energies can be applied to the context of networks.

Matrix computations have many uses outside of mathematics. It is helpful when attempting to solve linear equations. Matrices are extremely valuable objects that can be discovered in many different places. Matrix mathematics has applications in many branches of science and mathematics. Engineering mathematics affects nearly every facet of our daily lives.

In this article, we will define matrices, discuss their applications, and explore some common matrix-based problem-solving techniques. In the field of computer graphics, they are employed to bring a three-dimensional object into the realm of a two-dimensional display. In probability theory and statistics, stochastic matrices are used to explain collections of probabilities; for instance, they are used in the page rank algorithm that determines how websites are ranked in a Google search. Separate research was conducted on algebraic structures due to the close connections they share with representation theory and number theory, as well as the significant role they play in combinatorics. As a direct result of the extensive mathematical research that has been conducted in this field, finite rings and fields have received a significant amount of attention for the applications that they have to cryptography and coding theory [10–13]. Ring theory has been studied in many research projects in mathematics, primarily in the area of algebra, and it has been applied in various fields such as [14, 15] computer science, cryptography, and image segmentation.

2 Results and discussion

2.1 Energy

In 1978, Ivan Gutman defined [16] graph energy for the first time. However, the motivation behind his definition came much earlier, in the 1930s, when Erich Huckel proposed the now-famous Huckel Molecular Orbital Theory. Chemists can make accurate estimates of the energies connected to the electron orbitals in the conjugated hydrocarbon subclass of molecules using Huckel's method [17]. The method works on the assumption that the Hamiltonian operator is a simple linear combination of specific orbitals, and it solves for the desired energies using the time-independent Schrodinger equation. In 1956, Gunthard and Primes made the discovery that the matrix used in the Huckel method is a first-degree polynomial of the adjacency matrix associated with a particular graph that is related to the molecule that is being investigated.
The spectrum of the ordinary graph spectrum, also known as the adjacency matrix spectrum, is what is meant when people talk about the graph energy [16, 17]. The adjacency matrix $\mathcal{A}(\Gamma(\mathcal{R}))$ represented by graph vertices, with value 1 or 0 in (a_i, b_j) represents whether a_i and b_j are adjacent or not. As said before, the concept of graph energy $\mathcal{E}(\Gamma(\mathcal{R}))$ is defined as the sum of the adjacency matrix's absolute eigenvalues. It is related to the total $\pi-$electron energy [16] in Huckel theory by a molecular graph, then $\mathcal{E}(\Gamma(\mathcal{R})) = \sum_{i=1}^{n} |\lambda_i|$.
During the preceding decades, the parameter $\mathcal{E}(\Gamma(\mathcal{R}))$ and its bounds were the subject of extensive research and study. For illustration purposes, the reader is directed to [16, 17]. McClelland [3] obtained the first graph-energy result in 1971 that addressed the upper bound

for $\mathcal{E}(\Gamma(\mathcal{R}))$ with the order n and the size m:

$$\mathcal{E}(\Gamma(\mathcal{R})) \leq \sqrt{2mn}.$$

Koolen and Moulton [18] made some improvements to the bound in the above equation. They came to the conclusion that if $\Gamma(\mathcal{R})$ is a graph with an order of n and a size of m, and if $2m > n$, then:

$$\mathcal{E}(\Gamma(\mathcal{R})) \leq \frac{2m}{n} + \sqrt{(n-1)\left(2m - \left(\frac{2m}{n}\right)^2\right)}.$$

McClelland also derived a lower bound for $\mathcal{E}(\Gamma(\mathcal{R}))$ in terms of n, m and $|A(\Gamma(\mathcal{R}))|$ (determinant of adjacency matrix $A(\Gamma(\mathcal{R}))$) in [3]:

$$\mathcal{E}(\Gamma(\mathcal{R})) \geq \sqrt{2m + n(n-1)|A(\Gamma(\mathcal{R}))|^{\frac{2}{n}}}.$$

A simplified lower bound involving only the number of edges m in [19] was obtained almost thirty years later by Caporossi G et al. :

$$\mathcal{E}(\Gamma(\mathcal{R})) \geq 2\sqrt{m}.$$

Theorem 2.1. *Let $\Gamma(\mathcal{R})$ be the zero-divisor graph of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ with prime $p \geq 3$, then*

$$\mathcal{A}(\Gamma(\mathcal{R})) = \begin{pmatrix} N_1 & N & N & N & \dots & N \\ N & N_2 & O & O & \dots & O \\ N & O & N_2 & O & \dots & O \\ N & O & O & N_3 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N & O & O & O & \dots & N_3 \end{pmatrix},$$

where $N_1 = \big(N - I\big)_{(p-1)}$, $N_2 = \big(N - I\big)_{p(p-1)}$, $N_3 = \begin{pmatrix} O_{p(p-1)} & N_{p(p-1)} \\ N_{p(p-1)} & O_{p(p-1)} \end{pmatrix}_{2p(p-1)}$, O is zero matrix and N is a matrix of ones.

Proof. Let $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^4\rangle$ for prime $p \geq 3$, then the set $\{ax + b; \quad a, b \in \mathbb{Z}_{p^2}\}$ has p^4 elements.
Now, $\phi(p^4) = p^3(p-1)$. Let $Z^*(\mathcal{R}) - \{0\}$ be set of all non-zero zero divisors of $\mathcal{R} = p^4 - p^3(p-1) - 1 = p^3 - 1$ and let the edge set be defined by $E(\Gamma(\mathcal{R})) = \{a \cdot b = 0; \forall a, b \in Z^*(\mathcal{R})\}$.
Then the zero-divisor graph $\Gamma(\mathcal{R})$ is of order $(p^3 - 1)$ and of size $\frac{1}{2}(p^5 + p^4 - 3p^3 - 2p^2 + p + 2)$, and we have:

$$V(\Gamma(\mathcal{R})) = \{p, 2p, ..., (p-1)p, x, 2x, 3x, ..., (p^2-1)x, x+p, x+2p, ..., x+(p-1)p,$$
$$2x+p, 2x+2p, ..., 2x+(p-1)p, ..., (p^2-1)x+p, (p^2-1)x+2p, ..., (p^2-1)x+(p-1)p\}.$$

Thus we obtain $|V(\Gamma(\mathcal{R}))| = p^3 - 1$. Now the vertex set has been divided by

$$V_1 = \{lpx | l = 1, 2, \dots (p-1) \quad \& \quad p \nmid l\},$$
$$V_2 = \{lp, kx, lpx + lp | k = 1, 2, \dots (p^2-1) \quad \& \quad p \nmid k\},$$
$$V_3 = \{kx + lp| \quad \& \quad p \nmid l, k\}.$$

Also V_2 and V_3 are subdivided by

$$\begin{aligned}
&V_{21} = \{kx\},\\
&V_{22} = \{lp, lpx + lp\},\\
&V_{31} = \{kx + p, kx + p(p-1)\},\\
&V_{32} = \{kx + 2p, kx + p(p-2)\},\\
&\vdots\\
&V_{3(p-1)/2} = \{kx + (p-1)/2p, kx + p(p+1)/2\}.
\end{aligned}$$

The following conditions define criteria for determining adjacency among various sets of vertices for the zero-divisor graph $\Gamma(\mathcal{R})$:

1. For V_1: If the product of any two distinct elements lpx and $l'px$ is divisible by x^2 for all $1 \leq l < l' \leq p-1$, then every vertex is adjacent to each other in V_1. This means that any pair of distinct vertices in V_1 is adjacent to each other.

2. For V_{21}: Like V_1, if the product of any two distinct elements kx and $k'x$ is divisible by x^2 for all $1 \leq k < k' \leq p^2 - 1$, then every vertex is adjacent to each other in V_{21}.

3. For V_{22}: Here, if the product of any two distinct elements $lpx + lp$ and $l'px + l'p$ is either divisible by x^2 or p^2 for all $1 \leq l < l' \leq p-1$, then every vertex is adjacent to each other in V_{22}. This implies that any pair of distinct vertices in V_{22} is adjacent to each other, and the product of their elements is divisible by either x^2 or p^2.

4. For V_{31} to $V_{3(p-1)/2}$: Finally, if the product of any two distinct elements $kx + p$ and $k'x + p(p-1)$ is either divisible by x^2 or p^2 for all $1 \leq k < k' \leq p^2 - 1$, then $kx + p$ is adjacent to $k'x + p(p-1)$ in V_{31}. Additionally, no two $kx + p$, $k'x + p$, or $kx + p(p-1)$, $k'x + p(p-1)$ have a product divisible by x^2 or p^2 in V_{31}, ensuring non-adjacency. Similar arguments extend to V_{32}, V_{33}, and so on, up to $V_{3(p-1)/2}$.

As a result, we have an adjacency matrix by first considering the elements of V_1, V_2 and then V_3. Therefore, the adjacency matrix of $\Gamma(\mathcal{R})$ is

$$\mathcal{A}(\Gamma(\mathcal{R})) = \begin{array}{c} \\ V_1 \\ V_{21} \\ V_{22} \\ V_{31} \\ \vdots \\ V_{3(p-1/2)} \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_{21} & V_{22} & V_{31} & \dots & V_{3(p-1/2)} \end{array} \\ \begin{pmatrix} N_1 & N & N & N & \dots & N \\ N & N_2 & O & O & \dots & O \\ N & O & N_2 & O & \dots & O \\ N & O & O & N_3 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N & O & O & O & \dots & N_3 \end{pmatrix} \end{array} \tag{1}$$

$$= \begin{pmatrix} N_1 & N_{(p-1)\times p(p-1)} & N_{(p-1)\times p(p-1)} & N_{(p-1)\times 2p(p-1)} & \dots & N_{(p-1)\times 2p(p-1)} \\ N_{p(p-1)\times(p-1)} & N_2 & O_{p(p-1)\times p(p-1)} & O_{p(p-1)\times 2p(p-1)} & \dots & O_{p(p-1)\times 2p(p-1)} \\ N_{p(p-1)\times(p-1)} & O_{p(p-1)\times p(p-1)} & N_2 & O_{p(p-1)\times 2p(p-1)} & \dots & O_{p(p-1)\times 2p(p-1)} \\ N_{2p(p-1)\times(p-1)} & O_{2p(p-1)\times p(p-1)} & O_{2p(p-1)\times p(p-1)} & N_3 & \dots & O_{2p(p-1)\times 2p(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{2p(p-1)\times(p-1)} & O_{2p(p-1)\times p(p-1)} & O_{2p(p-1)\times p(p-1)} & O_{2p(p-1)\times 2p(p-1)} & \dots & N_3 \end{pmatrix}.$$

where $N_1 = \big(N - I\big)_{(p-1)}$, $N_2 = \big(N - I\big)_{p(p-1)}$, $N_3 = \begin{pmatrix} O_{p(p-1)} & N_{p(p-1)} \\ N_{p(p-1)} & O_{p(p-1)} \end{pmatrix}_{2p(p-1)}$, O is zero matrix and N is a matrix of ones. ■

Theorem 2.2. *Let $\Gamma(\mathcal{R})$ be the zero-divisor graph of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ with prime $p \geq 3$, then $\mathcal{E}(\Gamma(\mathcal{R})) \geq p^3 + 2p^2 - p - 8$.*

Proof. Let $\Gamma(\mathcal{R})$ be zero-divisor graph with prime $p \geq 3$ of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$. Then, by using (1) and Theorem 2.1, the characteristic equation of $\mathcal{A}(\Gamma(\mathcal{R}))$ can be written as:

$$\begin{vmatrix} N_1 - \lambda\mathcal{I} & N & N & N & \dots & N \\ N & N_2 - \lambda\mathcal{I} & O & O & \dots & O \\ N & O & N_2 - \lambda\mathcal{I} & O & \dots & O \\ N & O & O & N_3 - \lambda\mathcal{I} & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N & O & O & O & \dots & N_3 - \lambda\mathcal{I} \end{vmatrix} = 0.$$

Thus $\lambda^{p^3-2p^2+1}[\lambda+1]^{2p^2-p-4}[\lambda-(p^2-p)]^{\frac{p-3}{2}}[\lambda+(p^2-p)]^{\frac{p-1}{2}}[\lambda-(p^2-p-1)][\lambda^3-(2p^2-p-3)\lambda^2+(p^3-5p^2+3p+2)\lambda+p(p-1)(p^4-2p^3+p^2+2p-3)] = 0$.

Since $\lambda^3-(2p^2-p-3)\lambda^2+(p^3-5p^2+3p+2)\lambda+p(p-1)(p^4-2p^3+p^2+2p-3) = 0$, we have $\alpha+\beta+\gamma = (2p^2-p-3)$ and also we know $|\alpha|+|\beta|+|\gamma| > \alpha+\beta+\gamma$. So,

$$\begin{aligned} \mathcal{E}(\Gamma(\mathcal{R})) &= \sum_{i=1}^{n} |\lambda_i| \\ &= (2p^2-p-4)|1| + \frac{p-3}{2}\left|p^2-p\right| + \frac{p-1}{2}\left|p^2-p\right| + \left|p^2-p-1\right| + |\alpha| + |\beta| + |\gamma|. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{E}(\Gamma(\mathcal{R})) &\geq (p^3-5) + \alpha+\beta+\gamma \\ &\geq (p^3-5) + (2p^2-p-3) = p^3+2p^2-p-8. \end{aligned}$$

■

2.2 Laplacian energy

The spectral theory [20, 21] of the Laplacian matrix is another well-developed aspect of algebraic graph theory. Gutman and Zhou defined a graph's Laplacian energy in 2006 as the sum of the absolute deviations (i.e., distance from the mean) of its Laplacian matrix's [22] eigenvalues. Laplacian energy has been used in image processing and information theory, as well as theoretical organic chemistry. Let $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix [21, 22] $L(G)$ is the difference between the diagonal matrix's vertex degrees and the adjacency matrix (i.e., $L(G) = D(G) - A(G)$). Also let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of the Laplacian matrix of the zero-divisor graph $\Gamma(\mathcal{R})$. The Laplacian energy [23, 24] of a zero-divisor graph is defined as $\mathcal{LE}(\Gamma(\mathcal{R})) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$.

Let $\Gamma(\mathcal{R})$ be a simple undirected graph of oder n and size m. The eigenvalues of the adjacency matrix $\mathcal{A}(\Gamma(\mathcal{R}))$ of graph are known as eigenvalues of graph $\Gamma(\mathcal{R})$. The set of eigenvalues of the graph with their multiplicities is known as the spectrum of the graph. Hence

$$Spec(\mathcal{A}(\Gamma(\mathcal{R}))) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

Also, the set of Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph with their multiplicities is known as the Laplacian spectrum of the graph $\Gamma(\mathcal{R})$. Hence

$$Spec(\mathcal{L}(\Gamma(\mathcal{R}))) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

The ordinary graph's eigenvalues can meet the following conditions:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} (\lambda_i)^2 = 2m.$$

The relations that are analogous for the Laplacian eigenvalues are as follows:

$$\sum_{i=1}^{n} \gamma_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i)^2 = 2M,$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$ with d_i is the degree of the i^{th} vertex of $\Gamma(\mathcal{R})$. It is easy to see that $M \geq m$ for all graphs $\Gamma(\mathcal{R})$, and that $M = m$ holds if and only if $\Gamma(\mathcal{R})$ is a regular graph. We have some bounds of Laplacian energy [20, 24] for $\Gamma(\mathcal{R})$ as follows:

1. $\mathcal{LE}(\Gamma(\mathcal{R})) \leq \sqrt{2Mn}$.
2. $\mathcal{LE}(\Gamma(\mathcal{R})) \leq \frac{2m}{n} + \sqrt{(n-1)[2M - (\frac{2m}{n})^2]}$.
3. $2\sqrt{M} \leq \mathcal{LE}(\Gamma(\mathcal{R})) \leq 2M$, where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

Theorem 2.3. *Let $\Gamma(\mathcal{R})$ be the zero-divisor graph of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ with prime $p \geq 3$, then*

$$\mathcal{L}(\Gamma(\mathcal{R})) = \begin{pmatrix} \mathcal{D}_1 - N_1 & -N & -N & -N & \ldots & -N \\ -N & \mathcal{D}_2 - N_2 & O & O & \ldots & O \\ -N & O & \mathcal{D}_2 - N_2 & O & \ldots & O \\ -N & O & O & \mathcal{D}_3 - N_3 & \ldots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -N & O & O & O & \ldots & \mathcal{D}_3 - N_3 \end{pmatrix}.$$

Proof. Let $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$, then the set of all zero-divisors $Z^*(\mathcal{R}) - \{0\}$ is considered as vertices, and there is an edge defined between x and y by having a zero product. Then $\Gamma(\mathcal{R})$ is of order $(p^3 - 1)$ and size $\frac{1}{2}(p^5 + p^4 - 3p^3 - 2p^2 + p + 2)$.
It's clear to show that if $a \in V_1$ then $d(a) = p^3 - 2$. Also, if $a \in V_2$ then $d(a) = p^2 - 2$. Moreover, if $a \in V_3$ then $d(a) = p^2 - 1$.

The adjacency matrix of $\Gamma(\mathcal{R})$ is given in Equation (1). Also, the diagonal matrix of $\Gamma(\mathcal{R})$ is

$$\mathcal{D}(\Gamma(\mathcal{R})) = \begin{array}{c} \\ V_1 \\ V_{21} \\ V_{22} \\ V_{31} \\ \vdots \\ C_(p-1/2) \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_{21} & V_{22} & V_{31} & \dots & V_{3(p-1)/2} \end{array} \\ \begin{pmatrix} \mathcal{D}_1 & O & O & O & \dots & O \\ O & \mathcal{D}_2 & O & O & \dots & O \\ O & O & \mathcal{D}_2 & O & \dots & O \\ O & O & O & \mathcal{D}_3 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \dots & \mathcal{D}_3 \end{pmatrix} \end{array}.$$

where

$$\mathcal{D}_1 = \big((p^3-2)I\big)_{(p-1)},\ \mathcal{D}_2 = \big((p^2-2)I\big)_{p(p-1)},\ \mathcal{D}_3 = \begin{pmatrix} (p^3-1)I & N_{p(p-1)} \\ N_{p(p-1)} & (p^2-1)I \end{pmatrix}_{2p(p-1)}.$$

This completes the proof. ■

Theorem 2.4. *Let $\Gamma(\mathcal{R})$ be the zero-divisor graph of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ with prime $p \geq 3$, then $\mathcal{LE}(\Gamma(\mathcal{R})) = \frac{(p-1)(2p^5+p^4-3p^3+p^2+5p+3)}{p^2+p+1}$.*

Proof. Let $\Gamma(\mathcal{R})$ be the zero-divisor graph with prime $p \geq 3$ of $\mathcal{R} = \mathbb{Z}_{p^2}[x]/\langle x^2\rangle$. Then by Theorem 2.3, the characteristic equation of $\mathcal{L}(\Gamma(\mathcal{R}))$ is

$$\begin{vmatrix} \mathcal{D}_1 - N_1 - \mu I & -N & -N & -N & \dots & -N \\ -N & \mathcal{D}_2 - N_2 - \mu I & O & O & \dots & O \\ -N & O & \mathcal{D}_2 - N_2 - \mu I & O & \dots & O \\ -N & O & O & \mathcal{D}_3 - N_3 - \mu I & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -N & O & O & O & \dots & \mathcal{D}_3 - N_3 - \mu I \end{vmatrix} = 0.$$

Thus $\mu(\mu-p+1)^{\frac{(p-1)}{2}}(\mu-p^2+1)^{(2p^3-2p-1)}(\mu-2p^2+p+1)^{\frac{(p+1)}{2}}(\mu-p^3+1)^{(p-1)} = 0$. So,

$$Spec(\mathcal{L}(\Gamma(\mathcal{R}))) = \begin{pmatrix} 0 & p-1 & p^2-1 & 2p^2-p-1 & p^3-1 \\ 1 & \frac{p+1}{2} & p^3-2p-1 & \frac{p-1}{2} & p-1 \end{pmatrix}.$$

Since $m = \frac{1}{2}(p^5+p^4-3p^3-2p^2+p+2)$, and $n = p^3-1$, we have

$$\begin{aligned} \mathcal{LE}(\Gamma(\mathcal{R})) &= \sum_{i=1}^{n} \left|\mu_i - \tfrac{2m}{n}\right| \\ &= \left|0 - \tfrac{2p^4+2p^3-p^2-3p-2}{p^2+p+1}\right| + \frac{p+1}{2}\left|(p-1) - \tfrac{2p^4+2p^3-p^2-3p-2}{p^2+p+1}\right| \\ &\quad + (p^3-2p-1)\left|(p^2-1) - \tfrac{2p^4+2p^3-p^2-3p-2}{p^2+p+1}\right| + \frac{p-1}{2}\left|(2p^2-p-1) - \tfrac{2p^4+2p^3-p^2-3p-2}{p^2+p+1}\right| \\ &\quad + (p-1)\left|(p^3-1) - \tfrac{2p^4+2p^3-p^2-3p-2}{p^2+p+1}\right| \\ &= \frac{(p-1)(2p^5+p^4-3p^3+p^2+5p+3)}{p^2+p+1}. \end{aligned}$$

■

2.3 Discussion of applications

Matrix energy is a concept that has been around in mathematics for a while, and it has been shown to be useful in many different areas of science, including chemistry and physics. Given the prevalence of matrices in the study of complex networks, it is only natural to investigate the applicability of matrix energies in the context of networks. Complex networks, and social networks in particular, exhibit a number of intriguing topological characteristics.
For example, If $p = 3$ then we got a graph $\Gamma(\mathbb{Z}_9[x]/\langle x^2\rangle)$ (see Figure 1). The vertex set $\Gamma(\mathcal{R})$

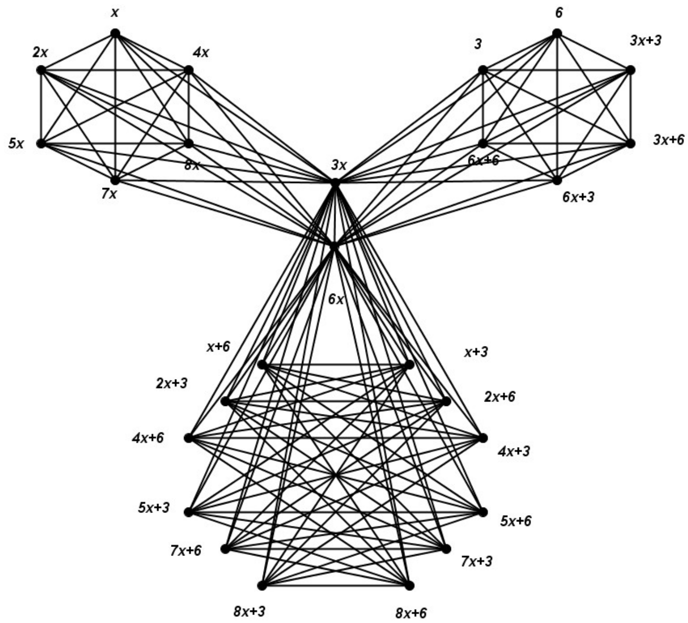

Figure 1: $\Gamma(\mathbb{Z}_9[x]/\langle x^2\rangle)$.

has been divided by

$$
\begin{aligned}
V_1 &= \{3x, 6x\},\\
V_{21} &= \{x, 2x, 4x, 5x, 7x, 8x\},\\
V_{22} &= \{3, 6, 3x+3, 3x+6, 6x+3, 6x+6\},\\
V_{31} &= \{2x+3, 5x+3, 8x+3, x+6, 4x+6, 7x+6\},\\
V_{32} &= \{x+3, 4x+3, 7x+3, 2x+6, 5x+6, 8x+6\}.
\end{aligned}
$$

It is clear that the adjacency matrix for the zero-divisor graph $\Gamma(\mathcal{R})$ is

$$\mathcal{A}(\Gamma(\mathcal{R})) = \begin{array}{c} \\ V_1 \\ V_{21} \\ V_{22} \\ V_{31} \\ V_{32} \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_{21} & V_{22} & V_{31} & V_{32} \end{array} \\ \begin{pmatrix} N_1 & N_{2\times 6} & N_{2\times 6} & N_{2\times 12} & N_{2\times 12} \\ N_{6\times 2} & N_2 & O_{6\times 6} & O_{6\times 12} & O_{6\times 12} \\ N_{6\times 2} & O_{6\times 6} & N_2 & O_{6\times 12} & O_{6\times 12} \\ N_{12\times 2} & O_{12\times 6} & O_{12\times 6} & N_3 & O_{12\times 12} \\ N_{12\times 2} & O_{12\times 6} & O_{12\times 6} & O_{12\times 12} & N_3 \end{pmatrix} \end{array}.$$

Thus $\lambda^{10}[\lambda+1]^{11}[\lambda+6][\lambda-5][\lambda^3-12\lambda^2-7\lambda+234]=0$, and

$$Spec(\mathcal{A}(\Gamma(\mathcal{R}))) = \begin{pmatrix} 0 & -1 & -6 & 5 & -4.0435 & 5.4767 & 10.5667 \\ 10 & 11 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \text{ and } \mathcal{E}(\Gamma(\mathcal{R})) = 42.0869.$$

Also,

$$\mathcal{L}(\Gamma(\mathcal{R})) = \begin{array}{c} \\ V_1 \\ V_{21} \\ V_{22} \\ V_{31} \\ V_{32} \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_{21} & V_{22} & V_{31} & V_{32} \end{array} \\ \begin{pmatrix} \mathcal{D}_1 - N_1 & -N_{2\times 6} & -N_{2\times 6} & -N_{2\times 12} & -N_{2\times 12} \\ -N_{6\times 2} & \mathcal{D}_2 - N_2 & O_{6\times 6} & O_{6\times 12} & O_{6\times 12} \\ -N_{6\times 2} & O_{6\times 6} & \mathcal{D}_2 - N_2 & O_{6\times 12} & O_{6\times 12} \\ -N_{12\times 2} & O_{12\times 6} & O_{12\times 6} & \mathcal{D}_3 - N_3 & O_{12\times 12} \\ -N_{12\times 2} & O_{12\times 6} & O_{12\times 6} & O_{12\times 12} & \mathcal{D}_3 - N_3 \end{pmatrix} \end{array}.$$

Thus $\mu(\mu-p+1)^{\frac{p-1}{2}}(\mu-p^2+1)^{(2p^3-2p-1)}(\mu-2p^2+p+1)^{\frac{p+1}{2}}(\mu-p^3+1)^{p-1}=0$,

$$Spec(\mathcal{L}(\Gamma(\mathcal{R}))) = \begin{pmatrix} 0 & 2 & 8 & 14 & 26 \\ 1 & 2 & 20 & 1 & 2 \end{pmatrix}, \text{ and } \mathcal{LE}(\Gamma(\mathcal{R})) = 78.9231.$$

In the past, architects, animators, and engineers would draw their creations by hand; today, they use computer graphics. The linear transformation of objects is very conveniently represented by square matrices. In computer graphics, they allow for the projection of three-dimensional images onto two-dimensional screens. A digital image is first viewed as a matrix when working with graphics. The matrix's rows and columns represent rows and columns of pixels, while the numerical entries represent the colour values of the pixels.

Mathematical techniques like using matrices to manipulate a point are widely used in game graphics. Graphs can also be expressed in the form of a matrix. Each cell in a matrix represents a node in a graph, and the strength of the connection between two nodes is represented by the value of their intersection. In graphics, matrix operations like translate, rotate, and pack are frequently used. Data encryption through cryptography ensures that only authorised parties have access to sensitive information. Video signals were not encrypted until recently. After owners of satellites began losing money because videos could be viewed by anyone with a dish, they began encrypting the video signals so that only those with video cyphers could decode them.

If the key used to encrypt the data is itself not invertible, the resulting signal cannot be decrypted and restored to its original form. Matrix algebra is used for this purpose. First, a digital audio or video signal is converted into a numerical sequence that represents the time-varying air-pressure changes that make up an acoustic audio signal. Filtering methods based on matrix multiplication are employed. Wireless signals are modelled and optimised using matrices. Information contained in signal matrices is extracted and processed for use in detection. Signal estimation and detection issues rely heavily on matrices. They are implemented in adaptive filter design and the processing of signals from sensor arrays. Digital image processing and representation both benefit from the use of matrices.

We are all aware of the significance of wireless communication to the telecommunications sector. Sensor array signal processing has significant implications in many fields, including radar signals and underwater surveillance, due to its emphasis on signal enumeration and source location applications. Detecting and localising the radiating sources using the collected temporal and spatial information from the sensors is the primary challenge in sensor array signal processing. A dynamic graph is one that is, among other things, dynamic, heterogeneous, highly transitive, has relatively short average distances between vertices, and is degree associative. In addition, these characteristics vary widely among the many vertices that make up the network's components. Graph energy is the energetic potential of a network modelled as a symmetric matrix. Different types of matrices (such as adjacency matrices, distance matrices, and Laplacian matrices) produce different forms of energy when used to describe a network. The informational value of graph energy is drastically reduced, and its interpretation becomes murky, when applied to the matrix representation of an entire network. One could argue that high-energy networks have a lot of potential storage space. Because of the stationary distribution of allocation, such networks make it possible to allocate a sizable amount of resources without disruption. The scientific community needs to dig deeper into this hypothesis. It should be obvious that the usefulness of graph energy in describing the topology of chemical compounds or other relatively small graphs does not naturally transfer to the realm of large complex networks. The topology of moderately compact graphs is best described using graph energy.

3 Conclusion

In this paper, we studied the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2}[x]/\langle x^2\rangle)$ over the commutative ring $\mathbb{Z}_{p^2}[x]/\langle x^2\rangle$ and discussed adjacency matrix and Laplacian matrix for zero-divisor graph. Finally, we investigated the graph energy and Laplacian energy of $\Gamma(\mathbb{Z}_{p^2}[x]/\langle x^2\rangle)$ over the commutative ring $\mathbb{Z}_{p^2}[x]/\langle x^2\rangle$.

Author Contributions. Clement Johnson R contributes for conceptualization, methodology, software, and writing-original draft preparation. Ravi Sankar J contributes for supervision, validation, formal analysis, writing-review and editing. All authors discussed the results and contributed to the final manuscript.

Conflicts of interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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