Iranian Journal of Mathematical Chemistry

IJMC

DOI: 10.22052/IJMC.2023.253926.1789
Vol. 15, No. 2, 2024, pp. 65-78
Research Paper

The Laplacian Spectrum of the Generalized n-Prism Networks

Mehdi Eliasi[1]*

[1]Department of Mathematics, Khansar Faculty, University of Isfahan, Iran

Keywords:
Laplacian spectra,
Prism network,
Spanning tree,
Kirchhoff index

AMS Subject Classification (2020):

05C50; 05C30; 05C31

Article History:

Received: 1 December 2023
Accepted: 19 December 2023

Abstract

The Laplacian eigenvalues and polynomials of the networks play an essential role in understanding the relations between the topology and the dynamic of networks. Generally, computation of the Laplacian spectrum of a network is a hard problem and there are just a few classes of graphs with the property that their spectra have been completely computed. Laplacian spectrum for n-prism networks was investigated in [Liu et al., Neurocomputing 198 (2016) 69-73]. In this paper, we give a method for calculating the eigenvalues and characteristic polynomial of the Laplacian matrix of a generalized n-prism network. We show how such large networks can be constructed from small graphs by using graph products. Moreover, our results are used to obtain the Kirchhoff index and the number of the spanning trees in the generalized n-prism networks. We also give some examples of applications, that explain the usefulness and efficiency of the proposed method.

1 Introduction

The Internet, the social networks, polymer networks, and the World Wide Web are examples of networks that indicate the importance of networks in science and our daily life [1, 2]. Usually, a useful representation of a network is a graph, where the components of the network are represented by vertices and their mutual interactions by the edges of the graph. This model allows us to apply graph theory tools and methods for studying complex networks [3–5]. According to a great deal of research, the structural and dynamical properties of a network are determined by the eigenvalue spectrum of its associated matrices[6].

In this study, we consider the Laplacian matrix of the networks [7]. More precisely, the mathematical structure underlying any network is a graph $G = (V(G), E(G))$ consisting of a set $V(G) = \{v_1, \ldots, v_n\}$ of vertices (sometimes called nodes) and a set E(G) of edges (sometimes

*Corresponding author
E-mail address: m.eliasi@khc.ui.ac.ir (M. Eliasi)
Academic Editor: Gholam Hossein Fath-Tabar

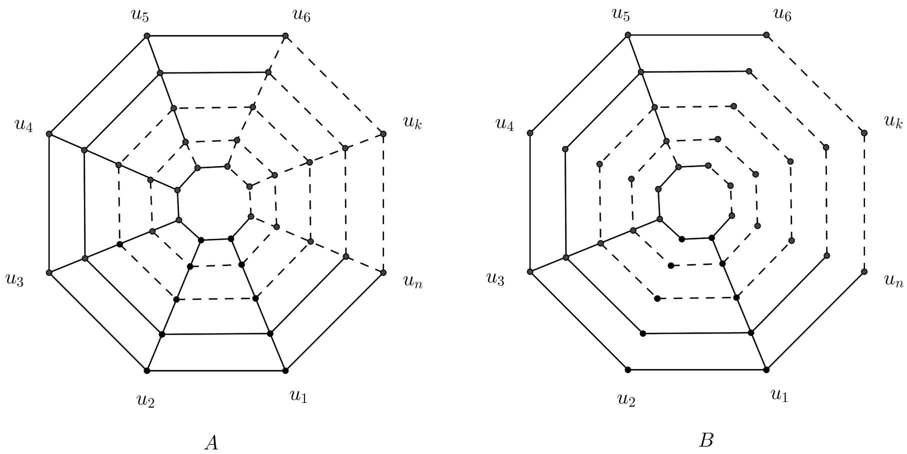

Figure 1: Networks A: An n-prism (or $P(g, n)$) and B: the generalized n-prism $P(g, n, \{u_1, u_3, u_5\})$.

called links) connecting the vertices. The adjacency matrix of G is defined to be the matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and 0 otherwise. We denote the degree of the vertex v_i of G by $\deg_G(v_i)$. The Laplacian matrix of G, denoted by $L(G) = D(G) - A(G)$, is the difference of $D(G) = diag(\deg_G(v_1), \ldots, \deg_G(v_n))$, the diagonal matrix of vertex degrees, and the adjacency matrix of G. The spectrum and characteristic of this matrix are called the Laplacian spectra and Laplacian polynomial of G, respectively. The study of the Laplacian spectra of the graph networks plays an essential role in estimating important structural properties, which provides information on the topological characteristics of the corresponding networks [8, 9]. In addition, the Laplacian matrix has many applications in graph isomorphism problems, computational techniques for differential equations, physical chemistry, biochemistry, computer science, and the design of statical experiments [10–16].

Due to a lack of effective methods, determining the Laplacian spectra of large networks can generally be challenging for mathematicians. Among the existing methods, the decomposition of large networks into smaller networks is very common. For example, to build a n-prism network, Liu and Cao have introduced an iterative way [17]. Let $P(1, n)$ be an n-polygon whose vertices are $\{u_1, \cdots, u_n\}$. Assume that for a positive integer $g \geq 2$, the network $P(g-1, n)$ is defined. For $g \geq 3$; $P(g, n)$ is obtained from $P(g-1, n)$ as follows: Every innermost node in $P(g-1, n)$ gives birth to a new node and these n new nodes construct a new n-polygon. Now, by connecting each new node with its corresponding 'mother' node, as illustrated in Figure 1 A, we obtain $P(g, n)$. As a generalization, suppose that $\emptyset \neq U \subseteq \{u_1, \cdots, u_n\}$. Then $P(g, n, U)$, is called the generalized n-prism network, obtained from $p(g, n)$ by deleting the edges on the interior bisectors of angles that do not belong to U (see, Figure 1 B). By this definition, a generalized n-prism is a subnetwork of an n-prism network. Note that a $P(g, n, U)$ network has gn vertices and $ng + (g-1)|U|$ edges.

The Laplacian spectra of the 3-prism networks and n-prism networks have been studied in [17] and [18], respectively. The Kirchhoff index of a network is defined as the sum of its resistance distances between all pairs of vertices and has many applications in physics, chemistry, complex networks, graph theory, etc. [19]. Also, the number of spanning trees characterizes the reliability of a network and is closely related to its optimal synchronization and the study of random walks

[20, 21].

In this paper, we explain a method for computing the spectra Laplacian matrix of the generalized hierarchical product of two networks. Moreover, we indicates how this method can be used to obtain the number of the spanning trees and the Kirchhoff index of $P(g, n, U)$-networks.

2 Preliminaries

The Kronecker product $A \otimes B$ of two matrices $A = (a_{i,j})$ and B of orders $m \times p$ and $n \times q$, respectively, is the partitioned matrix $(a_{ij}B)$ of order $mn \times pq$. If C and D are matrices of such size that one can form the matrix products AC and BD, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. By I_n, we denote the identity matrix of size n. We denote the characteristic polynomial of the square matrix M by $\phi_M(x)$. In particular, if G is a network consisting of a set $V(G) = \{v_1, \ldots, v_n\}$ of nodes and $B = L(G)$, the Laplacian matrix of G, then we write $\phi_{L(G)}(x)$ by $\Phi_G(x)$. Similarly, $eig(M)$ (eigenvalue spectrum of M) and $eig(G)$ (Laplacian spectrum of G) denote the set of eigenvalues of M and the set of eigenvalues of $L(G)$, respectively. Suppose that $\mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$ are the Laplacian spectra of G. It is known that $L(G)$ is a positive semidefinit matrix and $\mu_1(G) = 0$. Particularly, $\mu_2(G) > 0$ if and only if G is a connected network. We recall that if $\tau(G)$ is the number of spanning trees of G, then by Matrix Tree Theorem

$$\prod_{i=2}^{n} \mu_i(G) = |V(G)|\tau(G). \tag{1}$$

The resistance distance between vertices v_i and v_j in G, denoted by r_{ij}, is defined to be the effective resistance between nodes v_i and v_j as computed by Ohm's law in electrical network theory. The Kirchhoff index of G, denoted by $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices in G. Gutman and Mohar [22] proved the following identity:

Lemma 2.1. *Let G be a connected network with $n \geq 2$ vertices. Then*

$$Kf(G) = \sum_{i=2}^{n} \frac{n}{\mu_i(G)}.$$

Theorem 2.2. *Let G be a network with $n \geq 3$ vertices. If $\Phi(G) = f_1(x)f_2(x)\ldots f_t(x)$, where $f_i(x)$ is a polynomial, for $i = 1, \ldots t$, then*

$$\begin{aligned}
\tau(G) &= \frac{(-1)^{n-1} \sum_{i=1}^{t} f_i'(0) \prod_{j=1, j\neq i}^{t} f_j(0)}{n}, \\
\frac{Kf(G)}{n} &= -\frac{\sum_{i=1}^{t} f_i''(0) + \sum_{i=1}^{t} \sum_{j=1, j\neq i}^{t} f_i'(0) f_j'(0) \prod_{k=1, k\neq i,j}^{t} f_k(0)}{2\sum_{i=1}^{t} f_i'(0) \prod_{j=1, j\neq i}^{t} f_j(0)}, \\
Kf(G) &= (-1)^n \frac{\sum_{i=1}^{t} f_i''(0) + \sum_{i=1}^{t} \sum_{j=1, j\neq i}^{t} f_i'(0) f_j'(0) \prod_{k=1, k\neq i,j}^{t} f_k(0)}{2\tau(G)}.
\end{aligned}$$

Proof. Assume that $\Phi(G) = f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x$. Then [17]

$$\begin{aligned}
\frac{Kf(G)}{n} &= -\frac{a_{n-2}}{a_{n-1}}, \\
\tau(G) &= \frac{(-1)^{n-1}a_{n-1}}{n}.
\end{aligned}$$

From $a_{n-1} = f^{'}(0)$, $2a_{n-2} = f^{''}(0)$ and $f(x) = f_1(x)f_2(x)\ldots f_t(x)$, we deduce that

$$a_{n-1} = \sum_{i=1}^{t} f_i^{'}(0) \prod_{j=1, j\neq i}^{t} f_j(0), \tag{2}$$

$$2a_{n-2} = \sum_{i=1}^{t} f_i^{''}(0) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} f_i^{'}(0)f_j^{'}(0) \prod_{k=1, k\neq i,j}^{n} f_k(0). \tag{3}$$

By using (2) and (3), we obtain the desirable result. ■

In some cases, a pair of matrices A and B can be simultaneously diagonalized by the same matrix. Here we mention an important theorem for the case that A and B are symmetric.

Theorem 2.3. *Let A and B two symmetric real matrices. If $AB = BA$, then there exists an orthogonal matrix Q , $Q'Q = I$, such that $Q'AQ$ and $Q'BQ$ are diagonal.*

Proof. See [23]. ■

Through this paper, if $d \geq 1$, then P_d is the path on d vertices $\{u_1, u_2, \cdots, u_d\}$ such that $u_i u_{i+1} \in E(P_d)$, for $i = 1, \ldots, d-1$. Note that $eig(P_d) = \{4\sin^2(\frac{\pi i}{2d}) | i = 0, \cdots, d-1\}$ [7]. Also, by joining u_1 and u_d in P_d, we obtain C_d, a d-polygon.

The following trigonometric identity has many applications in this paper.

Proposition 2.4. *Let d be a positive integer. Then,*

$$\prod_{j=1}^{d-1} 4\sin^2(\frac{\pi j}{2d}) = d.$$

Proof. Obviously $\tau(P_d) = 1$. By (1), we obtain

$$\prod_{i=2}^{d} \mu_i(P_d) = |V(P_d)|\tau(P_d) = d.$$

This yields $\prod_{j=1}^{d-1} 4\sin^2(\frac{\pi j}{2d}) = d$. ■

3 The spectra Laplacian of generalized n- prism networks

Consider $P(g, n)$ $(g \geq 2)$ and let at $g = 1$, $H := P(1, n)$ be the n-polygon with $V(H) = \{u_1, \cdots, u_n\}$. For $U \subseteq V(H)$, we give a method to obtain $eig(P(g, n, U))$ and $\Phi_{P(g,n,U)}$. It is easy to check that

$$L(P(g, n, U)) = I_g \otimes L(H) + L(P_g) \otimes D(U),$$

where $D(U) = diag(\chi_U(h_1), \cdots, \chi_U(h_n))$ and χ_U denotes the characteristic function of the set U.

Theorem 3.1. *Let $K = P(g, n, U)$ and $eig(P_g) = \{\mu_1(G), \cdots, \mu_g(G)\}$. Then,*

$$eig(K) = \bigcup_{i=1}^{g} eig\big(L(H) + \mu_i D(U)\big) \quad \text{and} \quad \Phi_K(x) = \prod_{i=1}^{g} \Phi_{L(H)+\mu_i D(U)}(x).$$

Proof. Since $L(P_g)$ and I_n are commuting symmetric matrices, Theorem 2.3 shows that there exists an orthogonal matrix Q such that $Q^{'}L(G)Q$ and $Q^{'}I_mQ$ are simultaneously diagonalizable. Without loss of generality, we can assume that
$Q^{'}L(P_g)Q = diag(\mu_1(P_g), \ldots, \mu_g(P_g))$. Let $T := Q \otimes I_n$. Then, $T^{'} = (Q \otimes I_n)^{'} = Q^{'} \otimes I_n$ and $T^{'}T = Q^{'}Q \otimes I_nI_n = I_g \otimes I_n = I_{gn}$. Therefor,

$$\begin{aligned}
T^{'}L(K)T &= (Q^{'} \otimes I_n)\big(I_g \otimes L(H) + L(P_g) \otimes D(U)\big)(P \otimes I_n) \\
&= \big(Q^{'}I_g \otimes I_nL(H) + Q^{'}L(P_g) \otimes I_nD(U)\big)(Q \otimes I_g) \\
&= Q^{'}I_gQ \otimes I_nL(H)I_n + Q^{'}L(P_g)Q \otimes I_nD(U)I_n \\
&= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \otimes L(H) + \begin{bmatrix} \mu_1(P_g) & & \\ & \ddots & \\ & & \mu_g(P_g) \end{bmatrix} \otimes D(U) \\
&= \begin{bmatrix} L(H) & & \\ & \ddots & \\ & & L(H) \end{bmatrix} + \begin{bmatrix} \mu_1(G)D(U) & & \\ & \ddots & \\ & & \mu_g(P_g)D(U) \end{bmatrix} \\
&= \begin{bmatrix} L(H) + \mu_1(P_g)D(U) & & \\ & \ddots & \\ & & L(H) + \mu_g(P_g)D(U) \end{bmatrix}.
\end{aligned}$$

Thus, if

$$M := \begin{bmatrix} L(H) + \mu_1(P_g)D(U) & & \\ & \ddots & \\ & & L(H) + \mu_g(P_g)D(U) \end{bmatrix},$$

then M and $L(K)$ are similar. This yields $\Phi_{L(K)}(x) = \phi_M(x)$. Because M is a diagonal block matrix, we obtain $\phi_M(x) = \prod_{i=1}^{g} \phi_{L(H)+\mu_i(P_g)D(U)}(x)$. In addition, the eigenvalues of $L(K)$ are the roots of $\Phi_K(x)$. Hence, $eig(K) = \bigcup_{i=1}^{g} eig\big(L(H) + \mu_i(G)D(U)\big)$. ■

Theorem 3.2. *Let $K = P(g, n, U)$. Then,*

$$eig(C_n) \subseteq eig(K), \qquad \textit{and} \qquad \Phi_{C_n}(x)|\Phi_K(x).$$

Proof. Since $0 \in eig(P_g)$, Theorem 3.1 implies that, $eig(L(H) + 0D(U)) \subseteq eig(K)$. Therefore $eig(H) \subseteq eig(K)$, and hence $\Phi_H(x)|\Phi_K(x)$. But $H = P(1, n) \cong C_n$, which gives the result. ■

4 Some examples and applications

Theorem 3.1 provides a method for calculating the eigenvalues and characteristic polynomial of the Laplacian matrix of weakly prism networks. In the remainder of this article, we explain this method.

Example 4.1. Assume that $H = C_3$, and $V(H) = \{u_1, u_2, u_3\}$. We consider the three following cases.

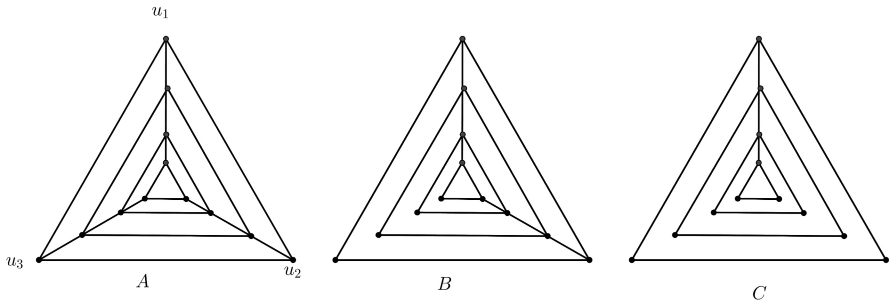

Figure 2: Networks A: A 3-prism network, B: $P(g,3,\{u_1,u_2\})$ network and C: $P(g,3,\{u_1\})$ network.

I. For $P(g,3,\{u_1,u_2\})$, illustrated in Figure 2 B, Theorem 3.1 yields

$$\begin{aligned}
& eig(P(g,3,\{u_1,u_2\})) \\
= & \bigcup_{i=0}^{g-1} eig\Big(L(H) + 4\sin^2(\frac{\pi i}{2g}) D(\{u_1,u_2\}) \Big) \\
= & \bigcup_{i=0}^{g-1} eig\left(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & -1 \\ -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 \\ -1 & -1 & 2 \end{bmatrix} \right) \\
= & \bigcup_{i=0}^{g-1} \left\{ 3+4\sin^2(\frac{\pi i}{2g}), \frac{3}{2}+2\sin^2(\frac{\pi i}{2g}) \pm \frac{1}{2}\sqrt{9-8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})} \right\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Phi_{P(g,3,\{u_1,u_2\})} &= \prod_{i=0}^{g-1} \Big[x^3 - (6+8\sin^2(\frac{\pi i}{2g}))x^2 + (9+32\sin^2(\frac{\pi i}{2g}) \\
&+ 16(\sin(\frac{\pi i}{2g}))^4)x - 24\sin^2(\frac{\pi i}{2g}) - 32\sin^4(\frac{\pi i}{2g}) \Big].
\end{aligned}$$

Therefore, by using Proposition 2.4, we obtain:

$$\begin{aligned}
\tau(P(g,3,\{u_1,u_2\})) &= \frac{\prod_{i=0}^{g-1}(3+4\sin)^2(\frac{\pi i}{2g}) \times 3 \times \prod_{i=1}^{g-1}(8\sin^2(\frac{\pi i}{2g}))}{3g} \\
&= \frac{\prod_{i=0}^{g-1}(3+4\sin^2(\frac{\pi i}{2g})) \times 3 \times 2^{g-1}g}{3g} \\
&= 2^{g-1}\prod_{i=0}^{g-1}(3+4\sin^2(\frac{\pi i}{2g})),
\end{aligned}$$

and

$$\begin{aligned}
&Kf(P(g,3,\{u_1,u_2\}))\\
=\ & \sum_{i=0}^{g-1}\frac{3g}{3+4\sin^2(\frac{\pi i}{2g})}+\frac{3g}{3}+\sum_{i=1}^{g-1}\frac{3g}{\frac{3}{2}+2\sin^2(\frac{\pi i}{2g})+\frac{1}{2}\sqrt{9-8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})}}\\
+\ & \sum_{i=1}^{g-1}\frac{3g}{\frac{3}{2}+2\sin^2(\frac{\pi i}{2g})-\frac{1}{2}\sqrt{9-8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})}}\\
=\ & \sum_{i=0}^{g-1}\frac{3g}{3+4\sin^2(\frac{\pi i}{2g})}+g+\sum_{i=1}^{g-1}\frac{3g(3+4\sin^2(\frac{\pi i}{2g}))}{8\sin(\frac{\pi i}{2g})^2}.
\end{aligned}$$

II. For $P(g,\{u_1\})$, illustrated in Figure 2 C, Theorem 3.1 implies

$$\begin{aligned}
eig(P(g,3,\{u_1\})) &= \bigcup_{i=0}^{g-1} eig\Big(L(H)+4\sin^2(\frac{\pi i}{2g})D(\{u_1\})\Big)\\
&= \bigcup_{i=0}^{g-1} eig\Big(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\Big)\\
&= \bigcup_{i=0}^{g-1}\Big\{3, \frac{3}{2}+2\sin^2(\frac{\pi i}{2g})\pm\frac{1}{2}\sqrt{9+8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})}\Big\}.
\end{aligned}$$

Moreover,

$$\Phi_{P(g,3,\{u_1\})} = \prod_{i=0}^{g-1}\Big[x^3-(6+4\sin^2(\frac{\pi i}{2g}))x^2+(9+16\sin^2(\frac{\pi i}{2g}))x-12\sin^2(\frac{\pi i}{2g})\Big].$$

Consequently, by using Proposition 2.4, we get:

$$\begin{aligned}
\tau(P(g,3,\{u_1,u_2\})) &= \frac{\prod_{i=0}^{g-1}(3)\times 3\times\prod_{i=1}^{g-1}(4\sin^2(\frac{\pi i}{2g}))}{3g}\\
&= \frac{3^g\times 3\times g}{3g}=3^g,
\end{aligned}$$

and

$$\begin{aligned}
&Kf(P(g,3,\{u_1\}))\\
=\ & \sum_{i=0}^{g-1}\frac{3g}{3}+\frac{3g}{3}+\sum_{i=1}^{g-1}\frac{3g}{\frac{3}{2}+2\sin^2(\frac{\pi i}{2g})+\frac{1}{2}\sqrt{9+8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})}}\\
+\ & \sum_{i=1}^{g-1}\frac{3g}{\frac{3}{2}+2\sin^2(\frac{\pi i}{2g})-\frac{1}{2}\sqrt{9+8\sin^2(\frac{\pi i}{2g})+16\sin^4(\frac{\pi i}{2g})}}\\
=\ & g^2+g+\sum_{i=1}^{g-1}\frac{3g(3+4\sin^2(\frac{\pi i}{2g}))}{4\sin(\frac{\pi i}{2g})^2)}.
\end{aligned}$$

III. For $P(g,\{u_1,u_2,u_3\})$, or 3-prism network, illustrated in Figure 2 A, Theorem 3.1 gives

$$\begin{aligned} eig(P(g,3,\{u_1,u_2,u_3\})) &= \bigcup_{i=0}^{g-1} eig\Big(L(H)+4\sin^2(\frac{\pi i}{2g})D(\{u_1,u_2\})\Big) \\ &= \bigcup_{i=0}^{g-1} eig\left(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & -1 \\ -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 \\ -1 & -1 & 2+4\sin^2(\frac{\pi i}{2g}) \end{bmatrix}\right) \\ &= \bigcup_{i=0}^{g-1} \Big\{4\sin^2(\frac{\pi i}{2g}), 3+4\sin^2(\frac{\pi i}{2g}), 3+4\sin^2(\frac{\pi i}{2g})\Big\}. \end{aligned}$$

Moreover,

$$\begin{aligned} \Phi_{P(g,3,\{u_1,u_2,u_3\})} &= \prod_{i=0}^{g-1}\Big[x^3-(6+12\sin^2(\frac{\pi i}{2g}))x^2+(9+48\sin^2(\frac{\pi i}{2g})+48\sin^4(\frac{\pi i}{2g}))x \\ &- 36\sin^2(\frac{\pi i}{2g})-96\sin^4(\frac{\pi i}{2g})-64\sin^6(\frac{\pi i}{2g})\Big]. \end{aligned}$$

Therefore, by using Proposition 2.4, we have:

$$\begin{aligned} \tau(P(g,3,\{u_1,u_2,u_3\})) &= \frac{\prod_{i=1}^{g-1}(4\sin^2(\frac{\pi i}{2g}))\times\prod_{i=0}^{g-1}[3+4\sin^2(\frac{\pi i}{2g})]^2}{3g} \\ &= \frac{\prod_{i=0}^{g-1}[3+4\sin^2(\frac{\pi i}{2g})]^2}{3} \end{aligned}$$

and

$$Kf(P(g,3,\{u_1,u_2,u_3\})) = \sum_{i=1}^{g-1}\frac{3g}{4\sin^2(\frac{\pi i}{2g})}+\sum_{i=0}^{g-1}\frac{6g}{3+4\sin^2(\frac{\pi i}{2g})}.$$

If we consider $G := P(g,3,\{u_1,u_2\}$, then Figure 3 shows the time consumption of Maple software, for computing $\tau(G)$, before using our method. After using this method (see Case I in Example 4.1), this time becomes almost zero. We show our computations in Table 1.

Example 4.2. Now, assume that $H = C_4$, and $V(H) = \{u_1,u_2,u_3,u_4\}$. Then we consider the following cases:

I. Let $U = \{u_1,u_2\}$. Then, for $P(g,U)$, illustrated in Figure 4 C by Theorem 3.1, we deduce that

$$\begin{aligned} & eig(P(g,4,\{u_1,u_2\})) \\ &= \bigcup_{i=0}^{g-1} eig\Big(L(H)+4\sin^2(\frac{\pi i}{2g})D(\{u_1,u_2\})\Big) \\ &= \bigcup_{i=0}^{g-1} eig\left(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & 0 & -1 \\ -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\right) \\ &= \bigcup_{i=0}^{g-1}\Big\{1+2\sin^2(\frac{\pi i}{2g})\pm\sqrt{1+4\sin^4(\frac{\pi i}{2g})}, 3+2\sin^2(\frac{\pi i}{2g})\pm\sqrt{1+4\sin^4(\frac{\pi i}{2g})}\Big\}. \end{aligned}$$

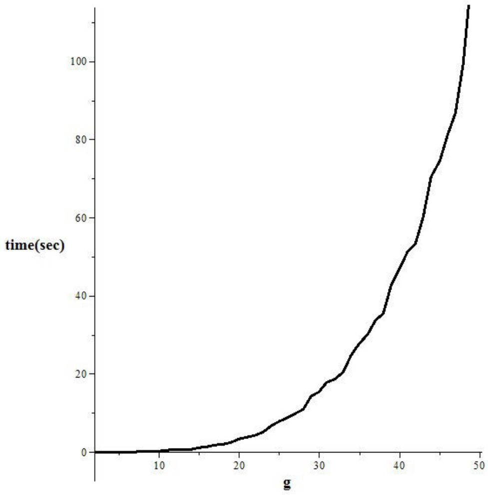

Figure 3: Time consumption of Maple software, for computing τ, before using our method: Horizontal g in $P(g,3,\{u_1,u_2\})$.

Hence, by using Proposition 2.4, we obtain:

$$\begin{aligned}\tau(P(g,4,\{u_1,u_2\})) &= \frac{2\times\prod_{i=1}^{g-1}(4\sin^2(\frac{\pi i}{2g}))\times\prod_{i=0}^{g-1}[8+12\sin^2(\frac{\pi i}{2g})]}{4g}\\ &= \frac{\prod_{i=0}^{g-1}[8+12\sin^2(\frac{\pi i}{2g})]}{2} = 2^{2g-1}\prod_{i=0}^{g-1}[2+3\sin^2(\frac{\pi i}{2g})],\end{aligned}$$

Table 1: The number of spanning trees and Kirchhoff index for $P(g,3,\{u_1,u_2\})$-Prism.

Weakly prism	The number of spanning trees	Kirchhoff index
$P(3,3,\{u_1,u_2\})$	288	36.75
$P(4,3,\{u_1,u_2\})$	2760	78.6173912952
$P(5,3,\{u_1,u_2\})$	26448	142.7949182
$P(6,3,\{u_1,u_2\})$	253440	233.7818182
$P(7,3,\{u_1,u_2\})$	2428608	356.0780298
$P(10,3,\{u_1,u_2\})$	2136998402	955.8225098
$P(20,3,\{u_1,u_2\})$	13951688857603276802	6842.575755
$P(60,3,\{u_1,u_2\})$	$0.2534644899\times10^{59}$	169698.896
$P(100,3,\{u_1,u_2\})$	$0.4604764957\times10^{98}$	771450.1082
$P(1000,3,\{u_1,u_2\})$	$0.9936653997\times10^{981}$	752153687.9

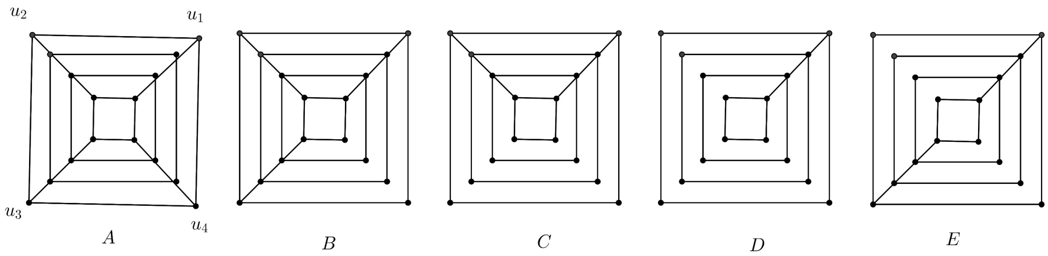

Figure 4: Networks A: A 4-prism network, B: $P(g,4,\{u_1,u_2,u_4\})$, C: $P(g,3,\{u_1,u_2\})$, D: $P(g,4,\{u_1\})$ and E: $P(g,4,\{u_2,u_4\})$.

$$
\begin{aligned}
& Kf(P(g,4,\{u_1,u_2\})) \\
= \ & \frac{4g}{2} + \sum_{i=1}^{g-1} \frac{4g}{1+2\sin^2(\frac{\pi i}{2g}) + \sqrt{1+4\sin^4(\frac{\pi i}{2g})}} + \sum_{i=1}^{g-1} \frac{4g}{1+2\sin^2(\frac{\pi i}{2g}) - \sqrt{1+4\sin^4(\frac{\pi i}{2g})}} \\
+ \ & \sum_{i=0}^{g-1} \frac{4g}{3+2\sin^2(\frac{\pi i}{2g}) + \sqrt{1+4\sin^4(\frac{\pi i}{2g})}} + \sum_{i=0}^{g-1} \frac{4g}{3+2\sin^2(\frac{\pi i}{2g}) - \sqrt{1+4\sin^4(\frac{\pi i}{2g})}} \\
= \ & 2g + \sum_{i=1}^{g-1} \frac{2g(1+2\sin^2(\frac{\pi i}{2g}))}{\sin^2(\frac{\pi i}{2g})} + \sum_{i=0}^{g-1} \frac{2g(3+2\sin^2(\frac{\pi i}{2g}))}{2+3\sin^2(\frac{\pi i}{2g})}.
\end{aligned}
$$

II. Let $U=\{u_1,u_3\}$. Then, for $P(g,U)$, illustrated in Figure 4 E, Theorem 3.1 yields

$$
\begin{aligned}
eig(P(g,4,\{u_1,u_3\})) &= \bigcup_{i=0}^{g-1} eig\Big(L(H) + 4\sin^2(\frac{\pi i}{2g})D(\{u_1,u_3\})\Big) \\
&= \bigcup_{i=0}^{g-1} eig\left(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\right) \\
&= \bigcup_{i=0}^{g-1} \left\{2, 2+4\sin^2(\frac{\pi i}{2g}), 2+2\sin^2(\frac{\pi i}{2g}) \pm 2\sqrt{1+\sin^4(\frac{\pi i}{2g})}\right\}.
\end{aligned}
$$

Therefore, by using Proposition 2.4, we obtain:

$$
\begin{aligned}
\tau(P(g,4,\{u_1,u_3\})) &= \frac{2^g \times \prod_{i=0}^{g-1}[2+4\sin^2(\frac{\pi i}{2g})] \times 4 \times \prod_{i=1}^{g-1}[8\sin^2(\frac{\pi i}{2g})]}{4g} \\
&= 2^{3g-1}\prod_{i=0}^{g-1}[1+2\sin^2(\frac{\pi i}{2g})] \quad ,
\end{aligned}
$$

and

$$\begin{aligned}
Kf(P(g,4,\{u_1,u_3\})) &= \sum_{i=0}^{g-1}\frac{4g}{2}+\sum_{i=0}^{g-1}\frac{4g}{2+4\sin^2(\frac{\pi i}{2g})}+\frac{4g}{4} \\
&+ \sum_{i=1}^{g-1}\frac{4g}{2+2\sin^2(\frac{\pi i}{2g})+2\sqrt{1+\sin^4(\frac{\pi i}{2g})}} \\
&+ \sum_{i=1}^{g-1}\frac{4g}{2+2\sin^2(\frac{\pi i}{2g})-2\sqrt{1+\sin^4(\frac{\pi i}{2g})}} \\
&= 2g^2+\sum_{i=0}^{g-1}\frac{2g}{1+2\sin^2(\frac{\pi i}{2g})}+g+\sum_{i=1}^{g-1}\frac{2g(1+\sin^2(\frac{\pi i}{2g}))}{\sin^2(\frac{\pi i}{2g})}.
\end{aligned}$$

III. Let $U=\{u_1,u_2,u_3,u_4\}$. Then, for $P(g,4,U)$, 4-prism network, illustrated in Figure 4 A, Theorem 3.1 implies

$$\begin{aligned}
& eig(P(g,4,\{u_1,u_2,u_3,u_4\})) \\
=& \bigcup_{i=0}^{g-1} eig\Big(L(H)+4\sin^2(\frac{\pi i}{2g})D(\{u_1,u_2,u_3,u_4\})\Big) \\
=& \bigcup_{i=0}^{g-1} eig\Big(\begin{bmatrix} 2+4\sin^2(\frac{\pi i}{2g}) & -1 & 0 & -1 \\ -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 & 0 \\ 0 & -1 & 2+4\sin^2(\frac{\pi i}{2g}) & -1 \\ -1 & 0 & -1 & 2+4\sin^2(\frac{\pi i}{2g}) \end{bmatrix}\Big) \\
=& \bigcup_{i=0}^{g-1}\Big\{4\sin^2(\frac{\pi i}{2g}), 4+4\sin^2(\frac{\pi i}{2g}), 2+4\sin^2(\frac{\pi i}{2g}), 2+4\sin^2(\frac{\pi i}{2g})\Big\}.
\end{aligned}$$

Consequently, by using Proposition 2.4, we get:

$$\begin{aligned}
\tau(P(g,4,\{u_1,u_2,u_3,u_4\})) &= \frac{\prod_{i=1}^{g-1}[4\sin^2(\frac{\pi i}{2g})]\prod_{i=0}^{g-1}[4+4\sin^2(\frac{\pi i}{2g})]\prod_{i=0}^{g-1}[2+4\sin^2(\frac{\pi i}{2g})]^2}{4g} \\
&= 2^{4g-2}\prod_{i=0}^{g-1}[1+\sin^2(\frac{\pi i}{2g})]\prod_{i=0}^{g-1}[1+2\sin^2(\frac{\pi i}{2g})]^2.
\end{aligned}$$

$$Kf(P(g,4,\{u_1,u_3\})) = \sum_{i=1}^{g-1}\frac{g}{\sin^2(\frac{\pi i}{2g})}+\sum_{i=0}^{g-1}\frac{g}{1+\sin^2(\frac{\pi i}{2g})}+\sum_{i=0}^{g-1}\frac{4g}{1+2\sin^2(\frac{\pi i}{2g})}.$$

IV. For $U=\{u_1\}$ or $U=\{u_1,u_2,u_3\}$, see Figure 4 D and B, we can compute the Laplacian polynomial of $P(g,4,U)$ as follows:

$$\begin{aligned}
\Phi_{P(g,4,\{u_1\})} &= \prod_{i=0}^{g-1}\Big[x^4-(8+4\sin^2(\frac{\pi i}{2g}))x^3+(20+24\sin^2(\frac{\pi i}{2g}))x^2 \\
&- (16+40\sin^2(\frac{\pi i}{2g}))x+16\sin^2(\frac{\pi i}{2g})\Big].
\end{aligned}$$

$$\begin{aligned}\Phi_{P(g,4,\{u_1,u_2,u_3\})} &= \prod_{i=0}^{g-1}\Big[x^4-(8+12\sin^2(\frac{\pi i}{2g}))x^3+(20+72\sin^2(\frac{\pi i}{2g})\\ &+ 48\sin^4(\frac{\pi i}{2g}))x^2-(16+120\sin^2(\frac{\pi i}{2g})+192\sin^4(\frac{\pi i}{2g})\\ &+ 64\sin^6(\frac{\pi i}{2g}))x+48\sin^2(\frac{\pi i}{2g})+160\sin^4(\frac{\pi i}{2g})+128\sin^6(\frac{\pi i}{2g})\Big].\end{aligned}$$

Therefore, by Theorem 2.2, we have:

$$\begin{aligned}\tau(P(g,4,\{u_1\})) &= (-1)^{4g-1}\frac{\sum_{i=0}^{g-1}(-16-40\sin^2(\frac{\pi i}{2g}))\prod_{j=0,j\neq i}^{g-1}16\sin^2(\frac{\pi j}{2g})}{4g}\\ &= -\frac{(-16)\prod_{j=1}^{g-1}16\sin^2(\frac{\pi i}{2g})}{4g}=4^g.\end{aligned}$$

Also, based on Proposition 2.4, we obtain:

$$\begin{aligned}Kf(P(g,4,\{u_1\})) &= \frac{(-1)^{4g}}{2\tau(P(g,4,\{u_1\}))}\Bigg(\sum_{i=0}^{g-1}(40+48\sin^2(\frac{\pi i}{2g}))\\ &+ \sum_{i=0}^{g-1}\sum_{j=0,j\neq i}^{g-1}(16+40\sin^2(\frac{\pi i}{2g}))(16+40\sin^2(\frac{\pi j}{2g}))\prod_{k=0,k\neq i,j}^{g-1}16\sin^2(\frac{\pi k}{2g})\Bigg)\\ &= \frac{1}{2\times 4^g}\Bigg(64g-24+32\sum_{j=1}^{g-1}(16+40\sin^2(\frac{\pi j}{2g}))\prod_{k=1,k\neq j}^{g-1}16\sin^2(\frac{\pi k}{2g})\Bigg).\end{aligned}$$

$$\begin{aligned}\tau(P(g,4,\{u_1,u_2,u_3\})) &= \frac{(-1)^{4g-1}}{4g}\Bigg(\sum_{i=0}^{g-1}-(16+120\sin^2(\frac{\pi i}{2g})+192\sin^4(\frac{\pi i}{2g})\\ &+ 64\sin^6(\frac{\pi i}{2g}))\prod_{j=0,j\neq i}^{g-1}[48\sin^2(\frac{\pi j}{2g})+160\sin^4(\frac{\pi j}{2g})+128\sin^6(\frac{\pi j}{2g})]\Bigg)\\ &= \frac{(-1)^{4g-1}}{4g}(-16)\times\Bigg(\prod_{j=1}^{g-1}[48\sin^2(\frac{\pi j}{2g})+160\sin^4(\frac{\pi j}{2g})+128\sin^6(\frac{\pi j}{2g})]\Bigg)\\ &= \frac{4}{g}\Bigg(\prod_{j=1}^{g-1}(4\sin^2(\frac{\pi j}{2g}))\times 4^{g-1}\times\prod_{j=1}^{g-1}[3+10\sin^2(\frac{\pi j}{2g})+8\sin^4(\frac{\pi j}{2g})]\Bigg)\\ &= 4^g\prod_{j=1}^{g-1}[3+10\sin^2(\frac{\pi j}{2g})+8\sin^4(\frac{\pi j}{2g})].\end{aligned}$$

Conflicts of Interest. The author declares that he has no conflicts of interest regarding the publication of this article.

References

[1] R. Albert, A. L. Barabási, Statistical mechanics of complex networks, *Rev. Mod. Phys.* **74** (2002) p. 47, https://doi.org/10.1103/RevModPhys.74.47.

[2] M. Newman, A. L. Barabasi and D. J. Watts, *The Structure and Dynamics of Networks*, Princeton University Press, 2006.

[3] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez and A. Arenas, Mathematical formulation of multilayer networks, *Phys. Rev. X* **3** (2013) p. 041022, https://doi.org/10.1103/PhysRevX.3.041022.

[4] R. Guimera, L. Danon, A. Diaz-Guilera, F. Giralt and A. Arenas, Self-similar community structure in a network of human interactions, *Phys. Rev. E* **68** (2003) p. 065103, https://doi.org/10.1103/PhysRevE.68.065103.

[5] M. E. J. Newman, The structure and function of complex networks, *SIAM Rev.* **45** (2003) 167–256, https://doi.org/10.1137/S003614450342480.

[6] P. V. Mieghem, *Graph Spectra for Complex Networks*, Cambridge University Press, 2011.

[7] W. N. Anderson Jr and T. D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra* **18** (1985) 141–145, https://doi.org/10.1080/03081088508817681.

[8] B. Y. Hou, H. J. Zhang and L. Liu, Applications of Laplacian spectra for extended Koch networks, *Eur. Phys. J. B* **85** (2012) 311–317, https://doi.org/10.1140/epjb/e2012-30373-x.

[9] A. N. Samukhin, S. N. Dorogovtsev and J. F. F. Mendes, Laplacian spectra of complex networks and random walks on them: Are scale-free architectures really important?, *Phys. Rev. E* **77** (2008) p. 036115, https://doi.org/10.1103/PhysRevE.77.036115.

[10] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Applications*, Third edition. Johann Ambrosius Barth, Heidelberg, 1995.

[11] R. Merris, Laplacian matrices of graphs; a survay, *Linear Algebra Appl.* **197-198** (1994) 143–176, https://doi.org/10.1016/0024-3795(94)90486-3.

[12] H. R. Ellahi, R. Nasiri, G. H. Fath-Tabar and A. Gholami, The Signless Laplacian Estrada Index of Unicyclic Graphs, *Math. Interdisc. Res.* **2** (2017) 155–167, https://doi.org/10.22052/MIR.2017.57775.1038.

[13] B. Mohar and S. Poljak, *Eigenvalues in combinatorial optimization*: in Combinatorial and Graph-Theoretical Problems in Linear Algebra, R. A. Brualdi, S. Friedland, V. Klee, Eds., IMA Volumes in Mathematics and Its Applications, **50**, Springer-Verlag, 1993, 107–151.

[14] Ş. Burcu Bozkurt Altındağ, Note on the sum of powers of normalized signless Laplacian eigenvalues of graphs, *Math. Interdisc. Res.* **4** (2019) 171 –182, https://doi.org/10.22052/MIR.2019.208991.1180.

[15] R. B. Bapat, The Laplacian matrix of a graphs, *Math. Student* **65** (1996) 214–223.

[16] S. Barik, R. B. Bapat and S. Pati, On the Laplacian spectra of product graphs, *Appl. Anal. Discrete Math.* **9** (2015) 39–58, https://doi.org/10.2298/AADM150218006B.

[17] J. Liu, J. Cao, A. Alofi, A. Al-Mazrooei and A. Elaiw, Applications of Laplacian spectra for n-prism networks, *Neurocomputing* **198** (2016) 69–73, https://doi.org/10.1016/j.neucom.2015.06.109.

[18] Q. Ding, W. Sun and F. Chen, Applications of Laplacian spectra on a 3-prism graph, *Modern Phys. Lett. B* **28** (2014) p. 1450009, https://doi.org/10.1142/S0217984914500092.

[19] Z. Zhang, Some physical and chemical indices of clique-inserted lattices, *J. Stat. Mech. Theory Exp.* **2013** (2013) p. P10004, https://doi.org/10.1088/1742-5468/2013/10/P10004.

[20] C. J. Colbourn, *The Combinatorics of Network Reliability*, Oxford University Press, New York, 1987.

[21] P. Marchal, Loop-erased random walks, spanning trees and Hamiltonian cycles, *Electron. Comm. Probab.* **5** (2000) 39–50, https://doi.org/10.1214/ECP.v5-1016.

[22] I. Gutman and B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, *J. Chem. Inf. Comput. Sci.* **36** (1996) 982–985, https://doi.org/10.1021/ci960007t.

[23] Y. P. Hong and R. A. Horn and C. R. Johnson, On the reduction of pairs of hermitian or symmetric matrices to diagonal form by congruence, *Linear Algebra Appl.* **73** (1986) 213–226, https://doi.org/10.1016/0024-3795(86)90241-7.