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Research Paper

Degree-Based Function Index of Graphs with Given Connectivity

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Abstract

We investigate the index $I_f(G) = \sum_{vw \in E(G)} f(d_G(v), d_G(w))$ of a graph G, where f is a symmetric function of two variables satisfying certain conditions, $E(G)$ is the edge set of G, and $d_G(v)$ and $d_G(w)$ are the degrees of vertices v and w in G, respectively. Those conditions are satisfied by functions that can be used to define the general sum-connectivity index χ_a, general Randić index R_a, general reduced second Zagreb index GRM_a for some $a \in \mathbb{R}$, general Sombor index $SO_{a,b}$, general augmented Zagreb index $AZI_{a,b}$ and by one other generalization $M_{a,b}$ for some $a, b \in \mathbb{R}$. The general augmented Zagreb index is a new index defined in this paper.

We obtain a sharp upper bound on I_f for graphs with given order and connectivity, and a sharp lower bound on I_f for 2-connected graphs with given order. Our upper bound holds for $M_{a,b}$ and $SO_{a,b}$ where $a, b \geq 1$; χ_a and R_a where $a \geq 1$; and GRM_a where $a > -1$. Our lower bound holds for $M_{a,b}$ where $a \geq 0$ and $b \geq -a$; $SO_{a,b}$ where $a, b \geq 0$ or $a, b \leq 0$; $AZI_{a,b}$ where $a \geq -2$ and $b \geq 0$; χ_a and R_a where $a \geq 0$; and GRM_a where $a > -2$.

1 Introduction

Let $V(G)$ and $E(G)$ be the vertex set and the edge set of a connected graph G. The order of G is the number of vertices in $V(G)$. The degree of $v \in V(G)$, denoted by $d_G(v)$, is the number of vertices adjacent to v. The vertex connectivity or just the connectivity of a connected graph G is the smallest number of vertices whose removal from G disconnects G. For $k \geq 1$, a graph is k-connected if its connectivity is at least k.

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For a graph G, we study degree-based indices defined as

$$I_f(G) = \sum_{vw \in E(G)} f(d_G(v), d_G(w)),$$

where f is a real-valued symmetric function of two variables. If $f(d_G(v), d_G(w)) = [d_G(v) + d_G(w)]^a$ where $a \in \mathbb{R}$, we obtain the general sum-connectivity index

$$\chi_a(G) = \sum_{vw \in E(G)} [d_G(v) + d_G(w)]^a,$$

of G defined by Zhou and Trinajstić [1]. From $\chi_a(G)$ we obtain the reciprocal sum-connectivity index if $a = \frac{1}{2}$, first Zagreb index if $a = 1$ and first hyper-Zagreb index if $a = 2$.

If $f(d_G(v), d_G(w)) = [d_G(v)d_G(w)]^a$ where $a \in \mathbb{R}$, we obtain the general Randić index

$$R_a(G) = \sum_{vw \in E(G)} [d_G(v)d_G(w)]^a,$$

of a graph G which was first investigated by Bollobás and Erdős [2]. From $R_a(G)$ we get the reciprocal Randić index if $a = \frac{1}{2}$, the second Zagreb index if $a = 1$, and the second hyper-Zagreb index if $a = 2$.

We can generalize the general Randić index and general sum-connectivity index even more by using $f(d_G(v), d_G(w)) = [d_G(v)d_G(w)]^a[d_G(v) + d_G(w)]^b$ where $a, b \in \mathbb{R}$. We obtain the generalization

$$M_{a,b}(G) = \sum_{vw \in E(G)} [d_G(v)d_G(w)]^a[d_G(v) + d_G(w)]^b,$$

(see [3]). From $M_{a,b}(G)$ we get the third redefined Zagreb index also called second Gourava index (see [4]) if $a = 1$ and $b = 1$, second redefined Zagreb index also known as inverse sum indeg index if $a = 1$ and $b = -1$, second hyper-Gourava index (see [5]) if $a = 2$ and $b = 2$, general Randić index if $b = 0$ and general sum-connectivity index if $a = 0$.

We also consider the general Sombor index of a graph G,

$$SO_{a,b}(G) = \sum_{vw \in E(G)} ([d_G(v)]^a + [d_G(w)]^a)^b,$$

defined for $a, b \in \mathbb{R}$; see [6]. We obtain $SO_{a,b}(G)$ from $I_f(G)$ if $f(d_G(v), d_G(w)) = ([d_G(v)]^a + [d_G(w)]^a)^b$. From $SO_{a,b}(G)$ we get the classical Sombor index if $a = 2$ and $b = \frac{1}{2}$ (see [7]), forgotten index if $a = 2$ and $b = 1$, and general sum-connectivity index if $a = 1$.

If $f(d_G(v), d_G(w)) = (d_G(v) + a)(d_G(w) + a)$ where $a \in \mathbb{R}$, we obtain the general reduced second Zagreb index

$$GRM_a(G) = \sum_{vw \in E(G)} (d_G(v) + a)(d_G(w) + a),$$

of a graph G from $I_f(G)$. This index was defined in [8]. From $GRM_a(G)$ we get the second Zagreb index if $a = 0$ and reduced second Zagreb index if $a = -1$.

For $a, b \in \mathbb{R}$ where $a > -3$, we introduce the general augmented Zagreb index

$$AZI_{a,b}(G) = \sum_{vw \in E(G)} \left(\frac{d_G(v)d_G(w)}{d_G(v) + d_G(w) + a} \right)^b,$$

of a graph G. We obtain $AZI_{a,b}(G)$ from $I_f(G)$ if $f(d_G(v), d_G(w)) = \left(\frac{d_G(v)d_G(w)}{d_G(v)+d_G(w)+a}\right)^b$. We call it "general augmented Zagreb index", because for $a = -2$ and $b = 3$, we get the classical augmented Zagreb index.

Indices are usually studied for connected graphs G of order $n \geq 3$. The reason for defining the general augmented Zagreb index for $a > -3$ is that $d_G(v) + d_G(w)$ is 3 if G contains an edge vw incident with vertices having degrees 1 and 2. In that case, if $a = -3$, we would have $d_G(v) + d_G(w) + a = 0$ in the denominator of $\frac{d_G(v)d_G(w)}{d_G(v)+d_G(w)+a}$.

Indices of graphs are investigated due to their extensive applications, especially in chemistry. Indices using a degree-based edge-weight function were investigated by Hu et al. [9], who presented extremal results for graphs with given order and size. Degree-based indices called bond incident degree indices were investigated for example in [10–14]. Ali and Dimitrov [10] studied graphs with a small number of cycles, Ali et al. [11] considered graphs with given order and size, Liu et al. [12] studied complex structures in drugs, Ye et al. [13] investigated polygonal cacti and Zhou et al. [14] studied graphs with a given number of pendant vertices. General degree-based indices were studied also in [15–22] and some related indices in [23, 24]. Chen and Guo [25] obtained bipartite graphs with prescribed connectivity having the maximum Zagreb indices. Tomescu, Arshad, and Jamil [26] presented the graph of given order and connectivity having the maximum χ_a and R_a for $a \geq 1$, and the 2-connected graph having the minimum χ_a and R_a for $a > 0$.

For a function f satisfying certain conditions, we obtain a sharp upper bound on I_f for graphs with given order and connectivity, and a sharp lower bound on I_f for 2-connected graphs with given order. Our upper bound holds for $M_{a,b}$ and $SO_{a,b}$ where $a, b \geq 1$; χ_a and R_a where $a \geq 1$; and GRM_a where $a > -1$. Our lower bound holds for $M_{a,b}$ where $a \geq 0$ and $b \geq -a$; $SO_{a,b}$ where $a, b \geq 0$ or $a, b \leq 0$; $AZI_{a,b}$ where $a \geq -2$ and $b \geq 0$; χ_a and R_a where $a \geq 0$; and GRM_a where $a > -2$.

2 Preliminary results

We investigate degree-based indices with the help of Definition 2.1.

Definition 2.1. A symmetric function $f(x, y)$ of two variables x and y having property Q is any function satisfying the following conditions:

(i) $f(x, y) > 0$ for $x, y \geq 2$,

(ii) $f(x_1, y_1) \leq f(x_2, y_2)$ for $2 \leq x_1 \leq x_2$ and $2 \leq y_1 \leq y_2$.

There are many functions that have the property Q. In Lemma 2.2 we present those ones which can be used to obtain some well-known indices. In the proof of Lemma 2.2, we consider the functions $(xy)^a(x+y)^b$ and $(x^a+y^a)^b$ for $x, y \geq 1$, because we use those values in Lemma 2.5.

Lemma 2.2. *The following functions of two variables x and y have property Q:*

- $(xy)^a(x + y)^b$ *for* $a \geq 0$, $b \geq -a$,
- $(x^a + y^a)^b$ *for* $a, b \geq 0$ *or* $a, b \leq 0$,
- $(\frac{xy}{x+y+a})^b$ *for* $a \geq -2$, $b \geq 0$,
- $(x + a)(y + a)$ *for* $a > -2$.

Proof. We show that $f(x, y) = (xy)^a(x+y)^b$ has property Q for $a \geq 0$ and $b \geq -a$. Let $x, y \geq 1$.

(i) We get $(xy)^a(x+y)^b > 0$.

(ii) Let $b = c - a$ where $a, c \geq 0$. Then

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= a(xy)^{a-1}y(x+y)^{c-a} + (c-a)(xy)^a(x+y)^{c-a-1} \\ &= a(xy)^{a-1}(x+y)^{c-a-1}[y(x+y) - xy] + c(xy)^a(x+y)^{c-a-1} \\ &= ay^2(xy)^{a-1}(x+y)^{c-a-1} + c(xy)^a(x+y)^{c-a-1} \\ &\geq 0.\end{aligned}$$

Since $f(x,y)$ is symmetric, we get $\frac{\partial f(x,y)}{\partial y} \geq 0$. Thus, for $1 \leq x_1 \leq x_2$ and $1 \leq y_1 \leq y_2$, we have $f(x_1, y_1) \leq f(x_2, y_2)$.

Let $f(x,y) = (x^a + y^a)^b$ where both $a, b \geq 0$ or both $a, b \leq 0$. Let $x, y \geq 1$.

(i) We obtain $(x^a + y^a)^b > 0$.

(ii) We get

$$\frac{\partial f(x,y)}{\partial x} = b(x^a + y^a)^{b-1}ax^{a-1} \geq 0,$$

since $(x^a + y^a)^{b-1} > 0$ and $x^{a-1} > 0$. Similarly, $\frac{\partial f(x,y)}{\partial y} > 0$, so part (ii) holds.

We show that $f(x,y) = (\frac{xy}{x+y+a})^b$ has property Q for $a \geq -2$ and $b \geq 0$. Let $x, y \geq 2$.

(i) We get $xy \geq 4$ and $x + y + a \geq 2$, thus $(\frac{xy}{x+y+a})^b > 0$.

(ii) We obtain

$$\frac{\partial f(x,y)}{\partial x} = b\left(\frac{xy}{x+y+a}\right)^{b-1}\frac{y(y+a)}{(x+y+a)^2} = \frac{bx^{b-1}y^b(y+a)}{(x+y+a)^{b+1}} \geq 0,$$

since $b \geq 0$, $y + a \geq 0$ and $x, y, x+y+a > 0$.

The function $f(x,y) = (x+a)(y+a)$ for $a > -2$ has property Q, since for $x, y \geq 2$:

$$\text{(i)} \quad (x+a)(y+a) > 0 \quad \text{and} \quad \text{(ii)} \quad \frac{\partial f(x,y)}{\partial x} = y + a > 0.$$

■

Let us present a few functions, which are special cases of $(xy)^a(x+y)^b$ for $a \geq 0$ and $b \geq -a$.

Corollary 2.3. *The functions $xy(x+y)$, $(xy)^2(x+y)^2$, $\frac{xy}{x+y}$, $(xy)^a$ and $(x+y)^a$ for $a \geq 0$ have property Q.*

Proof. By Lemma 2.2, $(xy)^a(x+y)^b$ has property Q for $a \geq 0$ and $b \geq -a$.

- If $a = 1$ and $b = 1$, we get $xy(x+y)$.
- If $a = 2$ and $b = 2$, we get $(xy)^2(x+y)^2$.
- If $a = 1$ and $b = -1$, we get $\frac{xy}{x+y}$.
- If $b = 0$, we get $(xy)^a$ for $a \geq 0$.

- If $a = 0$, we get $(x + y)^b$ for $b \geq 0$.

■

The first two conditions of Definitions 2.1 and 2.4 are almost equal. In Definition 2.1 we consider $f(x, y)$ for $x, y \geq 2$. In Definition 2.4 we consider $f(x, y)$ for $x, y \geq 1$. Moreover, in Definition 2.4 we have a new third condition.

Definition 2.4. A symmetric function $f(x, y)$ of two variables x and y having property P is any function satisfying the following conditions:

(i) $f(x, y) > 0$ for $x, y \geq 1$,

(ii) $f(x_1, y_1) \leq f(x_2, y_2)$ for $1 \leq x_1 \leq x_2$ and $1 \leq y_1 \leq y_2$,

(iii) $g(x_1, y_1) = f(x_1 + c, y_1 + c') - f(x_1, y_1) \leq f(x_2 + c, y_2 + c') - f(x_2, y_2) = g(x_2, y_2)$ for $1 \leq x_1 \leq x_2$, $1 \leq y_1 \leq y_2$ and $c, c' \geq 0$.

Since we have an additional condition in Definition 2.4, there are functions that have property Q, but not property P.

Lemma 2.5. *The following functions of two variables x and y have property P:*

- $(xy)^a(x + y)^b$ *and* $(x^a + y^a)^b$ *for* $a, b \geq 1$,
- $(x + y)^a$ *and* $(xy)^a$ *for* $a \geq 1$,
- $(x + a)(y + a)$ *for* $a > -1$.

Proof. Let $f(x, y) = (x + a)(y + a)$ where $a > -1$. Let $x, y \geq 1$.

(i) We have $(x + a)(y + a) > 0$.

(ii) We get $\frac{\partial f(x,y)}{\partial x} = y + a > 0$. Similarly, $\frac{\partial f(x,y)}{\partial y} > 0$.

(iii) For

$$g(x, y) = f(x + c, y + c') - f(x, y) = (x + c + a)(y + c' + a) - (x + a)(y + a),$$

we have

$$\frac{\partial g(x, y)}{\partial x} = (y + c' + a) - (y + a) = c' \geq 0.$$

The function $f(x, y)$ is symmetric, thus $g(x, y)$ is symmetric. Therefore $\frac{\partial g(x,y)}{\partial y} \geq 0$. Thus, for $1 \leq x_1 \leq x_2$, $1 \leq y_1 \leq y_2$ and $c, c' \geq 0$, we have $g(x_1, y_1) = f(x_1 + c, y_1 + c') - f(x_1, y_1) \leq f(x_2 + c, y_2 + c') - f(x_2, y_2) = g(x_2, y_2)$.

Hence, $f(x, y) = (x + a)(y + a)$ has property P for $a > -1$.

Conditions (i) and (ii) of Definition 2.4 for the functions $(x^a + y^a)^b$ and $(xy)^a(x + y)^b$ (containing special cases $(x + y)^a$ and $(xy)^a$) are proved in Lemma 2.2. Condition (iii) for $(x + y)^a$, $(xy)^a$ and $(xy)^a(x + y)^b$, where $a, b \geq 1$, was proved in [20]. It remains to show that $(x^a + y^a)^b$ satisfies condition (iii).

Let $f(x, y) = (x^a + y^a)^b$ where $a, b \geq 1$. We consider

$$g(x, y) = f(x + c, y + c') - f(x, y) = ([x + c]^a + [y + c']^a)^b - (x^a + y^a)^b.$$

We obtain

$$\frac{\partial g(x,y)}{\partial x} = ab([x+c]^a + [y+c']^a)^{b-1}[x+c]^{a-1} - ab(x^a+y^a)^{b-1}x^{a-1} \ge 0,$$

since for $a, b \ge 1$, we have $[x+c]^{a-1} \ge x^{a-1}$, $[x+c]^a \ge x^a$, $[y+c']^a \ge y^a$ and $([x+c]^a + [y+c']^a)^{b-1} \ge (x^a+y^a)^{b-1}$. Similarly, $\frac{\partial g(x,y)}{\partial y} \ge 0$. So condition (iii) of Definition 2.4 is satisfied by the function $(x^a+y^a)^b$. Hence, $(x^a+y^a)^b$ has property P for $a, b \ge 1$. ■

Let us compare I_f of two graphs that differ only by one edge. We use Lemma 2.6 in the proofs of Theorems 3.1 and 4.2.

Lemma 2.6. *Let G be a connected/2-connected graph containing two non-adjacent vertices v_1 and v_2. Then for a function $f(x,y)$ satisfying conditions (i) and (ii) of Definition 2.4/Definition 2.1, we get $I_f(G) < I_f(G + v_1v_2)$.*

Proof. For connected graphs and a function with slightly different condition (ii) in Definition 2.4, Lemma 2.6 was proved in [20]. The proof for our function introduced in Definition 2.4 is identical. Let us consider Lemma 2.6 for 2-connected graphs. Note that in Definition 2.4, we use $f(x,y)$ for $x, y \ge 1$, but in Definition 2.1, we use $f(x,y)$ for $x, y \ge 2$.

If G is 2-connected, then also $G + v_1v_2$ is 2-connected. 2-connected graphs do not contain vertices of degree 1, therefore for any vertex $v \in V(G)$, we get $d_{G+v_1v_2}(v) \ge d_G(v) \ge 2$. Then, similarly as in [20], it can be easily shown that $I_f(G) < I_f(G + v_1v_2)$. ■

3 Upper bound for graphs with given connectivity

For two graphs G_1 and G_2, the union $G_1 \cup G_2$ and the join $G_1 + G_2$ have the vertex set $V(G_1) \cup V(G_2)$. The edge set of $G_1 \cup G_2$ is $E(G_1) \cup E(G_2)$. The edge set of $G_1 + G_2$ consists of $E(G_1)$, $E(G_2)$, and every vertex of G_1 is adjacent to every vertex of G_2. Let us denote the complete graph of order n by K_n. Note that $1 \le \kappa \le n-2$ for the connectivity κ of any connected graph of order n except for K_n.

Theorem 3.1. *Let G be any graph with n vertices and connectivity κ, where $1 \le \kappa \le n-2$. If f has property P, then*

$$I_f(G) \le \binom{n-\kappa-1}{2} f(n-2, n-2) + \binom{\kappa}{2} f(n-1, n-1) + \kappa(n-\kappa-1) f(n-1, n-2) + \kappa f(n-1, \kappa).$$

with equality if and only if G is $(K_{n-\kappa-1} \cup K_1) + K_\kappa$.

Proof. Among graphs with n vertices and connectivity κ, let G' be any graph with the largest I_f. Thus there is a set $S \subset V(G')$ with κ vertices, such that $G' - S$ is disconnected. So, it is possible to divide the vertices in $V(G') \setminus S$ into two sets S_1 and S_2, such that no vertex in S_1 is adjacent to a vertex in S_2. The function f has property P, thus by Lemma 2.6, I_f increases with the addition of edges. So any two vertices in S_1 are adjacent, any two vertices in S_2 are adjacent and every vertex of S has degree $n-1$ in G'. Let $|S_1| = n_1$ and $|S_2| = n_2$. Without loss of generality, we can assume that $n_1 \ge n_2 \ge 1$. We obtain $n_1 + n_2 = n - \kappa$, so G' is $(K_{n_1} \cup K_{n_2}) + K_\kappa$. Let us prove by contradiction that $n_2 = 1$.

Suppose that $n_2 \ge 2$ (where $n_1 \ge n_2$). Let us compare I_f of $G' = (K_{n_1} \cup K_{n_2}) + K_\kappa$ and $G'' = (K_{n_1+1} \cup K_{n_2-1}) + K_\kappa$. For every $z \in S$, we have

$$d_{G'}(z) = d_{G''}(z) = n-1.$$

In G', we have

$$d_{G'}(v)=\kappa+n_1-1 \ \text{ and } \ d_{G'}(v')=\kappa+n_2-1,$$

for every $v\in V(K_{n_1})$ and every $v'\in V(K_{n_2})$. In G'', we have

$$d_{G''}(w)=\kappa+n_1 \ \text{ and } \ d_{G''}(w')=\kappa+n_2-2,$$

for every $w\in V(K_{n_1+1})$ and every $w'\in V(K_{n_2-1})$. We obtain

$$\begin{aligned}
&I_f(G'')-I_f(G')\\
&=\kappa(n_1+1)\,f(n-1,n_1+\kappa)-\kappa n_1\,f(n-1,n_1+\kappa-1)\\
&+\kappa(n_2-1)\,f(n-1,n_2+\kappa-2)-\kappa n_2\,f(n-1,n_2+\kappa-1)\\
&+\binom{n_1+1}{2}f(n_1+\kappa,n_1+\kappa)-\binom{n_1}{2}f(n_1+\kappa-1,n_1+\kappa-1)\\
&+\binom{n_2-1}{2}f(n_2+\kappa-2,n_2+\kappa-2)-\binom{n_2}{2}f(n_2+\kappa-1,n_2+\kappa-1)\\
&=\kappa\,f(n-1,n_1+\kappa)-\kappa\,f(n-1,n_2+\kappa-2)\\
&+\kappa n_1[f(n-1,n_1+\kappa)-f(n-1,n_1+\kappa-1)]\\
&-\kappa n_2[f(n-1,n_2+\kappa-1)-f(n-1,n_2+\kappa-2)]\\
&+\left[\frac{n_1(n_1-1)}{2}+n_1\right]f(n_1+\kappa,n_1+\kappa)-\frac{n_1(n_1-1)}{2}f(n_1+\kappa-1,n_1+\kappa-1)\\
&+\left[\frac{n_2(n_2-1)}{2}-(n_2-1)\right]f(n_2+\kappa-2,n_2+\kappa-2)\\
&-\frac{n_2(n_2-1)}{2}f(n_2+\kappa-1,n_2+\kappa-1)\\
&=\kappa[f(n-1,n_1+\kappa)-f(n-1,n_2+\kappa-2)]\\
&+\kappa(n_1-n_2)[f(n-1,n_1+\kappa)-f(n-1,n_1+\kappa-1)]\\
&+\kappa n_2[f(n-1,n_1+\kappa)-f(n-1,n_1+\kappa-1)]\\
&-\kappa n_2[f(n-1,n_2+\kappa-1)-f(n-1,n_2+\kappa-2)]\\
&+\frac{n_1(n_1-1)-n_2(n_2-1)}{2}[f(n_1+\kappa,n_1+\kappa)-f(n_1+\kappa-1,n_1+\kappa-1)]\\
&+\frac{n_2(n_2-1)}{2}[f(n_1+\kappa,n_1+\kappa)-f(n_1+\kappa-1,n_1+\kappa-1)]\\
&-\frac{n_2(n_2-1)}{2}[f(n_2+\kappa-1,n_2+\kappa-1)-f(n_2+\kappa-2,n_2+\kappa-2)]\\
&+(n_2-1)[f(n_1+\kappa,n_1+\kappa)-f(n_2+\kappa-2,n_2+\kappa-2)]\\
&+(n_1-n_2+1)f(n_1+\kappa,n_1+\kappa).
\end{aligned}$$

Since $n_1\ge n_2\ge 2$, $\kappa\ge 1$ and the function f has property P, from part (ii) of Definition 2.1, we obtain

$$f(n-1,n_1+\kappa)\ge f(n-1,n_2+\kappa-2),\ f(n-1,n_1+\kappa)\ge f(n-1,n_1+\kappa-1),$$

$$f(n_1+\kappa,n_1+\kappa)\ge f(n_1+\kappa-1,n_1+\kappa-1),\ f(n_1+\kappa,n_1+\kappa)\ge f(n_2+\kappa-2,n_2+\kappa-2).$$

By Definition 2.1 (i), we have $f(n_1+\kappa,n_1+\kappa)>0$. By Definition 2.1 (iii), we have

$$f(n-1,n_1+\kappa)-f(n-1,n_1+\kappa-1)\ge f(n-1,n_2+\kappa-1)-f(n-1,n_2+\kappa-2),$$

and

$$f(n_1 + \kappa, n_1 + \kappa) - f(n_1 + \kappa - 1, n_1 + \kappa - 1)$$
$$\geq f(n_2 + \kappa - 1, n_2 + \kappa - 1) - f(n_2 + \kappa - 2, n_2 + \kappa - 2).$$

Thus $I_f(G'') - I_f(G') > 0$, so $I_f(G'') > I_f(G')$, which means that G' does not have the largest I_f. We have a contradiction.

Thus $n_2 = 1$. Then $n_1 = n - \kappa - 1$, so G' is $(K_{n-\kappa-1} \cup K_1) + K_\kappa$ and

$$I_f((K_{n-\kappa-1} \cup K_1) + K_\kappa) = \binom{n-\kappa-1}{2} f(n-2, n-2) + \binom{\kappa}{2} f(n-1, n-1)$$
$$+ \kappa(n-\kappa-1) f(n-1, n-2) + \kappa f(n-1, \kappa).$$

■

4 Lower bound for 2-connected graphs

A proper ear decomposition of G is a decomposition of G into a sequence of ears $P_0, P_1, \ldots, P_k$, where $k \geq 1$, P_0 is a cycle and P_i for $1 \leq i \leq k$ is a path whose terminal vertices are in $V(P_0) \cup \cdots \cup V(P_{i-1})$ and internal vertices (if any) are not in $V(P_0) \cup \cdots \cup V(P_{i-1})$. Whitney [27] gave a well-known characterization of 2-connected graphs.

Lemma 4.1. *A graph is 2-connected if and only if it has a proper ear decomposition.*

We use Lemma 4.1 to obtain a lower bound on I_f for 2-connected graphs.

Theorem 4.2. *Let G be any 2-connected graph with n vertices, where $n \geq 3$. If f has property Q, then*

$$I_f(G) \geq n f(2, 2),$$

with equality if and only if G is the cycle C_n.

Proof. For $n = 3$, we have only one 2-connected graph which is C_3, so Theorem 4.2 holds for $n = 3$. We prove Theorem 4.2 by induction on n. Assume that $n \geq 4$ and for any graph G of order $m < n$, we have $I_f(G) \geq m f(2, 2)$ with equality if and only if G is C_m.

Let H be a graph with the smallest I_f among 2-connected graphs with n vertices except for C_n. From Lemma 4.1, we know that H has a proper ear decomposition $P_0, P_1, \ldots, P_k$. Since H is not a cycle, we have $k \geq 1$. Let u and v be the terminal vertices of P_k. So $u, v \in V(P_0) \cup \cdots \cup V(P_{k-1})$. Let r be the number of internal vertices of P_k. We have $r \geq 0$. Let H' be obtained from H by the removal of all r internal vertices of P_k and all $r + 1$ edges of P_k. Then H' is a 2-connected graph containing the ears $P_0, P_1, \ldots, P_{k-1}$. The order of H' is $n - r$. We consider the cases $r = 0$ and $r \geq 1$.

Case 1: $r = 0$.

Then P_k contains only one edge uv. We have $V(H') = V(H)$ and $E(H') = E(H) \setminus \{uv\}$. So, the order of H' is n. By Lemma 2.6, $I_f(H') < I_f(H)$.

If $k \geq 2$, then H' is not C_n, but the inequality $I_f(H') < I_f(H)$ contradicts the fact that H is a graph with the smallest I_f among 2-connected graphs with n vertices except for C_n.

If $k = 1$, then H' is P_0 which is C_n, so $I_f(C_n) < I_f(H)$ which means that C_n is the 2-connected graph of order n with the smallest I_f. Hence, the proof of the case $r = 0$ is complete.

Case 2: $r \geq 1$.

If $uv \in E(H)$, then $uv \in E(P_i)$ for some $i \in \{0, 1, \dots, k-1\}$. Let us construct P'_i with $V(P'_i) = V(P_i) \cup V(P_k)$ and $E(P'_i) = E(P_i) \cup E(P_k) \setminus \{uv\}$. Let P'_k contain only one edge uv. Clearly, when we replace P_i and P_k in $P_0, P_1, \dots, P_k$ by P'_i and P'_k, we again obtain a proper ear decomposition, where P'_k contains only one edge and such situation was solved in Case 1.

Therefore, we can assume that $uv \notin E(H)$. Let $N_{H'}(u) = \{u_1, \dots, u_s\}$ and $N_{H'}(v) = \{v_1, \dots, v_t\}$. Note that $s, t \geq 2$. By the induction hypothesis, we have $I_f(H') \geq (n-r)f(2,2)$. Thus

$$\begin{aligned} I_f(H) &= I_f(H') + (r-1)f(2,2) + f(d_H(u),2) + f(d_H(v),2) \\ &+ \sum_{i=1}^{s}[f(d_H(u), d_H(u_i)) - f(d_H(u)-1, d_H(u_i))] \\ &+ \sum_{i=1}^{t}[f(d_H(v), d_H(v_i)) - f(d_H(v)-1, d_H(v_i))] \\ &\geq (n-1)f(2,2) + f(d_H(u),2) + f(d_H(v),2) \\ &+ \sum_{i=1}^{s}[f(d_H(u), d_H(u_i)) - f(d_H(u)-1, d_H(u_i))] \\ &+ \sum_{i=1}^{t}[f(d_H(v), d_H(v_i)) - f(d_H(v)-1, d_H(v_i))]. \end{aligned}$$

Since $d_H(u) \geq 3$, $d_H(v) \geq 3$ and the function f has property Q, from part (ii) of Definition 2.1, we obtain

$$f(d_H(u),2) \geq f(2,2),\ \ f(d_H(v),2) \geq f(2,2),$$

$$f(d_H(u), d_H(u_i)) \geq f(d_H(u)-1, d_H(u_i)) \text{ and } f(d_H(v), d_H(v_i)) \geq f(d_H(v)-1, d_H(v_i)).$$

By Definition 2.1 (i), we have $f(2,2) > 0$. Thus

$$I_f(H) \geq (n+1)f(2,2) > nf(2,2) = I_f(C_n),$$

which means that C_n is the 2-connected graph of order n with the smallest I_f. ■

5 Conclusion

In Theorem 3.1, we presented a bound on I_f, where f is a function having property P introduced in Definition 2.4. In Lemma 2.5, we obtained several functions having property P. Hence, by Theorem 3.1 and Lemma 2.5, we get Corollary 5.1.

Corollary 5.1. *Among graphs with n vertices and connectivity κ, where $1 \leq \kappa \leq n-2$, $(K_{n-\kappa-1} \cup K_1) + K_\kappa$ is the unique graph with the maximum*

- *$M_{a,b}$ and $SO_{a,b}$ for $a, b \geq 1$,*
- *χ_a and R_a for $a \geq 1$,*
- *GRM_a for $a > -1$.*

So, Corollary 5.1 holds also for the following special cases of χ_a, R_a, $M_{a,b}$ and $SO_{a,b}$: first Zagreb index χ_1, first hyper-Zagreb index χ_2, second Zagreb index R_1, second hyper-Zagreb index R_2, second Gourava index $M_{1,1}$, second hyper-Gourava index $M_{2,2}$ and forgotten index $SO_{2,1}$.

By Theorem 4.2 and Lemma 2.2, we obtain Corollary 5.2.

Corollary 5.2. *Among 2-connected graphs with n vertices, where $n \geq 3$, the cycle C_n is the unique graph with the minimum*

- $M_{a,b}$ *for* $a \geq 0$, $b \geq -a$,
- $SO_{a,b}$ *for* $a, b \geq 0$ *or* $a, b \leq 0$,
- $AZI_{a,b}$ *for* $a \geq -2$, $b \geq 0$,
- χ_a *and* R_a *for* $a \geq 0$,
- GRM_a *for* $a > -2$.

All the indices covered by Corollary 5.1 are covered also by Corollary 5.2. However, Corollary 5.2 holds for a larger number of indices. The following indices are special cases of general indices presented in Corollary 5.2, but not special cases of general indices given in Corollary 5.1: inverse sum indeg index $M_{1,-1}$, Sombor index $SO_{2,\frac{1}{2}}$, augmented Zagreb index $AZI_{-2,3}$, reciprocal sum-connectivity index $\chi_{\frac{1}{2}}$, reciprocal Randić index $R_{\frac{1}{2}}$ and reduced second Zagreb index GRM_{-1}.

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