Iranian Journal of
Mathematical Chemistry

IJMC

DOI: 10.22052/IJMC.2023.252986.1722
Vol. 14, No. 3, 2023, pp. 135-143
Research Paper

Introducing Two Transformations in Fullerene Graphs, Star and Semi-Star

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Keywords:
Fullerene graphs,
Perfect star packing,
Perfect pseudo matching,
Fullerene transformations

AMS Subject Classification (2020):

05C10; 05C90; 05C92

Article History:

Received: 21 May 2023

Accepted: 2 June 2023

Abstract

A perfect star packing in a given graph G can be defined as a spanning subgraph of G, wherein each component is isomorphic to the star graph $K_{1,3}$. A perfect star packing of a fullerene graph G is of type $P0$ if all the centers of stars lie on hexagons of G. Many fullerene graphs arise from smaller fullerene graphs by applying some transformations. In this paper, we introduce two transformations for fullerene graphs that have the perfect star packing of type $P0$ and examine some characteristics of the graphs obtained from this transformation.

1 Introduction

Fullerene graphs are cubic, 3-connected, planar graphs with only pentagonal and hexagonal faces. Euler's formula dictates that the quantity of pentagonal facets incorporated into each fullerene graph is invariably twelve. In a paper by Grünbaum and Motzkin [1] the authors show that fullerene graphs with n vertices exist for all even $n \geq 24$ and for $n = 20$. According to the findings of Klein and Liu [2], it has been demonstrated that fullerene graphs, possessing isolated pentagons, can be constructed with a vertex count of $n = 60$ and for each even value of $n \geq 70$. For a systematic introduction to fullerene graphs, we refer the reader to the monograph [3] and for some symmetry-related aspects to [4].
A set of edges of graph G is a matching if no two edges of it have a vertex in common. A matching in a graph G is perfect if each vertex of G is incident with an edge from said matching. Within the domain of chemistry, the perfect matchings are commonly referred to as Kekulé structures. In other words, a perfect matching is a spanning subgraph whose all components are isomorphic to K_2. Suppose that G and L are two graphs. A spanning subgraph of G whose all components are isomorphic to L is called perfect L −packing in G. If L is the star graph $K_{1,3}$, is called a perfect star packing.
In [5], the authors investigate which fullerene graphs allowed perfect star packings, and a type

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Academic Editor: Tomislav Doslic

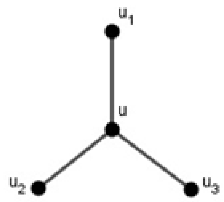

Figure 1: u is in the center of one star.

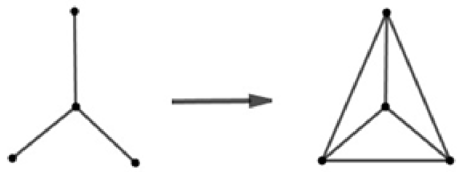

Figure 2: Connecting vertices adjacent to the central vertex of a star.

of perfect star packing was defined, which is called $P0$.
As noted, many fullerene graphs arise from smaller fullerene graphs by applying some transformations. The number of vertices in the resulting graph is usually a multiple of the number of vertices of starting graph. The best-known such transformation is the leapfrog transformation which can be thought of as a truncation of the dual. The number of vertices is tripled by this operation.
Two other transformations, commonly referred to as Chamfer and Capra, yield fullerene graphs possessing four and seven times as many vertices as the initial graph, respectively. To find more information on these transformations, we refer the reader to [4, 6–9]. This paper delves into the examination of fullerene graphs with a perfect star packing of $P0$ type and introduces two transformations to these fullerene graphs.

2 Star transformation

Let G be a fullerene graph. A perfect star packing in G is of type $P0$ if no center of a star is on a pentagon of G. The findings reported in [5], demonstrate that the perfect star packings of type $P0$ are absent in smaller fullerenes, and restricted solely to fullerenes containing isolated pentagons.

Theorem 2.1. (See [10]). *The smallest fullerene having a perfect star packing of type $P0$ is the icosahedral isomer of C_{80}.*

Using the definition of this packing, we introduce a transformation for fullerene graphs. We call this transformation, the star transformation.

Let G be a fullerene graph with a perfect star packing of type $P0$. Let u be in the center of one star, as shown in Figure 1.

We connect each pair of vertices u_1, u_2, and u_3. We do this for all stars in the perfect star packing in G. (See Figure 2).

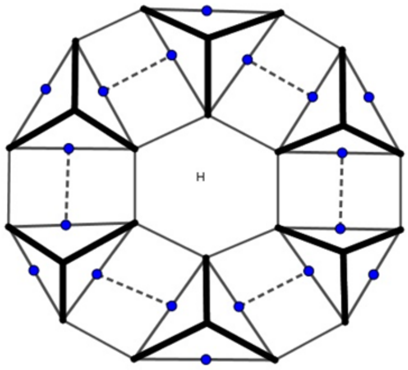

Figure 3: How the star transformation works.

By continuing this process, we will have $\frac{n}{4}$ subgraphs as shown in Figure 2 right. We subdivide each new edge by one vertex and connect each of these added vertices to its corresponding vertex in the same hexagon. (Dashed lines in Figure 3.)

As we know, in a perfect star packing of type $P0$, the center of each star is shared by three hexagons. We consider hexagons that none of whose vertices is the central vertex of the star, (Like the hexagon in Figure 3). Into each of these hexagons and also in all pentagons, inscribe a polygon with the same number of sides. Join each vertex of the initial fullerene to three vertices of the newly inscribed polygons. (See Figure 4)

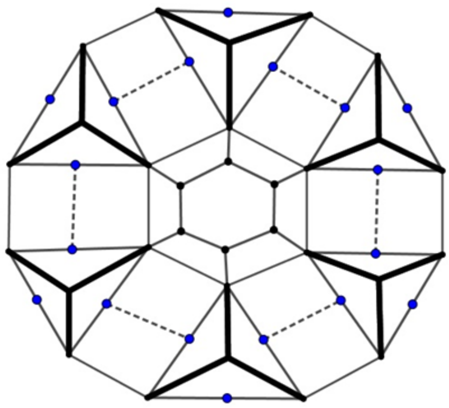

Figure 4: Connecting each vertex of the original fullerene with vertices of newly polygons.

Finally, erase the center of stars in G. The faces of the resulting graphs have only pentagons and hexagons. The number of pentagons in it is exactly twelve and the resulting graph is planar, cubic, and 3-connected, hence it is a fullerene graph.

Proposition 2.2. *The number of vertices of the fullerene graph created by the star transformation is* $\frac{9}{4}$ *times the number of vertices in the original fullerene.*

Table 1: The number of vertices via star transformation

	$n(G)$	$n(F)$, Star Transformation
1	80	180
2	96	216
3	104	234
4	112	252
5	120	270
6	128	288
7	136	306
8	144	324
9	152	342
10	160	360

Proof. Let F be a fullerene graph arising from the fullerene graph G via the star transformation. Suppose u is in the center of one star in G and u_1, u_2 and u_3 are vertices connected to u. Via the Star transformation, Vertices $v_1, v_2, \ldots, v_6$ are created. So, the four vertices u, u_1, u_2, and u_4 become nine vertices u_1, u_2, u_3, v_1, v_2, v_3, v_4, v_5 and v_6. (See Figure 5) Therefore, the number of vertices of F is $\frac{9}{4}$ times the number of vertices of G. Finally, because G has a perfect star packing, the number of its vertices is a multiple of four, and therefore $\frac{9}{4}$ $n(G)$ is always an integer. ∎

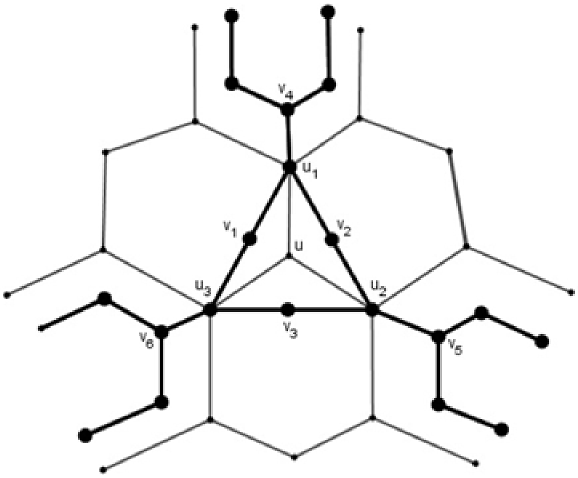

Figure 5: Added vertices in star transformation.

Figure 6 illustrates the effect of the star transformation on a graph comprising 80 vertices, and the resulting graph consisting of 180 vertices.

From paper [5], we know that the smallest fullerene graph having a perfect star packing of type $P0$ has 80 vertices. Table 1, shows the number of vertices of some fullerene graphs F arising from a fullerene graph G via the star transformation. From paper [11], we know that only fullerene graphs with $8n$ vertices have a perfect star packing. Therefore, in the above

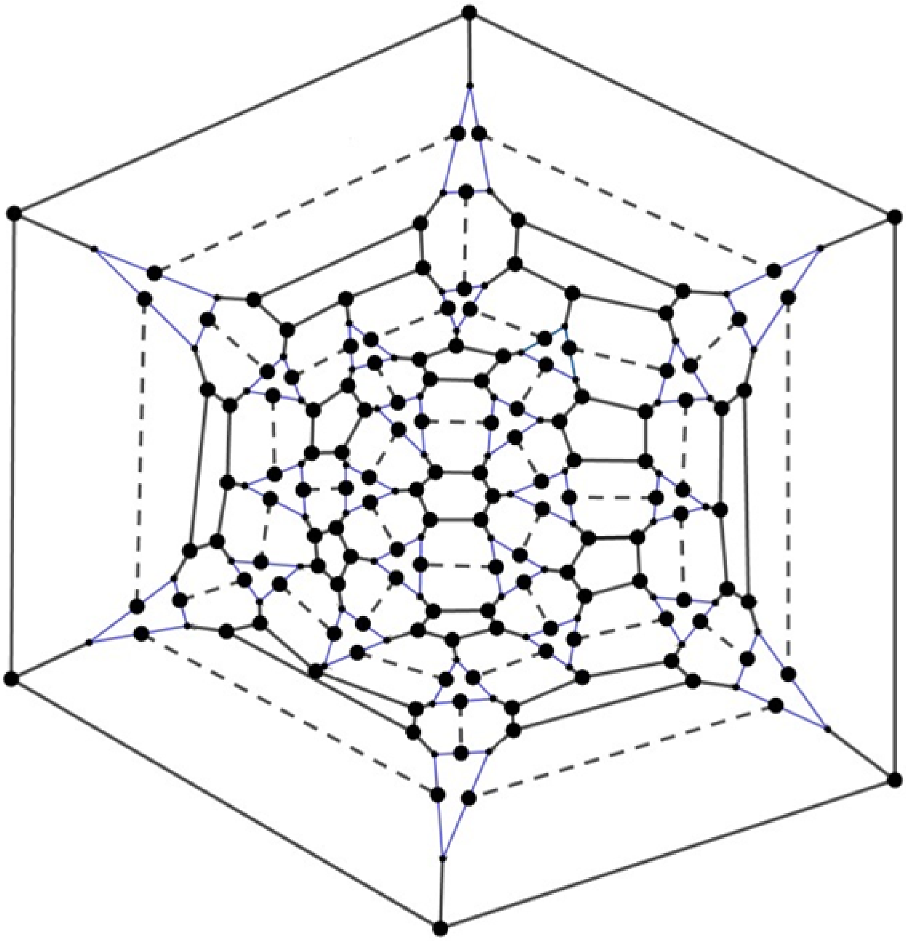

Figure 6: A graph of 180 vertices created by star transformation.

table, only fullerene graphs of rows 2, 6, and 10 can have perfect star packing. Therefore, as an open problem, it is possible to check which of the graphs with $8n$ vertices arising from star transformation have star packing.

Theorem 2.3. *Let G be a fullerene graph on n vertices, $n \geq 80$, F be a fullerene graph arising from G via the star transformation. Then F has a perfect $\{C_5,\ C_6\}$-packing.*

Proof. According to how the star transformation works, all vertices of F can be covered by a collection of disjoint cycles of sizes 5 and 6. ■

A perfect pseudo matching in a graph, denoted as G, involves a spanning subgraph of G whose components are isomorphic either to $K_{1,3}$ or K_2. For more on the problem of pseudo matching and the size of a perfect pseudo matching in fullerene graphs, see [5, 10].

Theorem 2.4. *Let G be a fullerene graph on n vertices and F be a fullerene graph arising from G via the star transformation. Then F has a perfect pseudo matching with two components isomorphic to star graph $K_{1,3}$.*

Proof. Consider a hexagon adjacent to five pentagons in F. We cover the vertices of this hexagon as shown in Figure 7. Around this hexagon, there are six other hexagons, which are covered in the form of Figure 7. Other vertices can be covered by the components isomorphic to K_2. ■

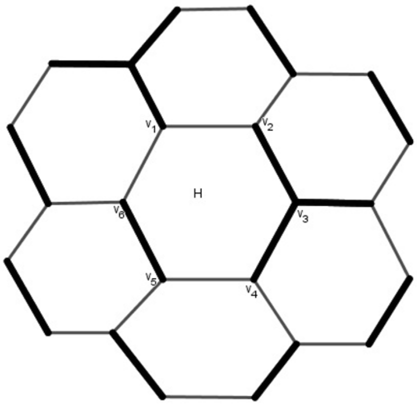

Figure 7: Packing of hexagon H.

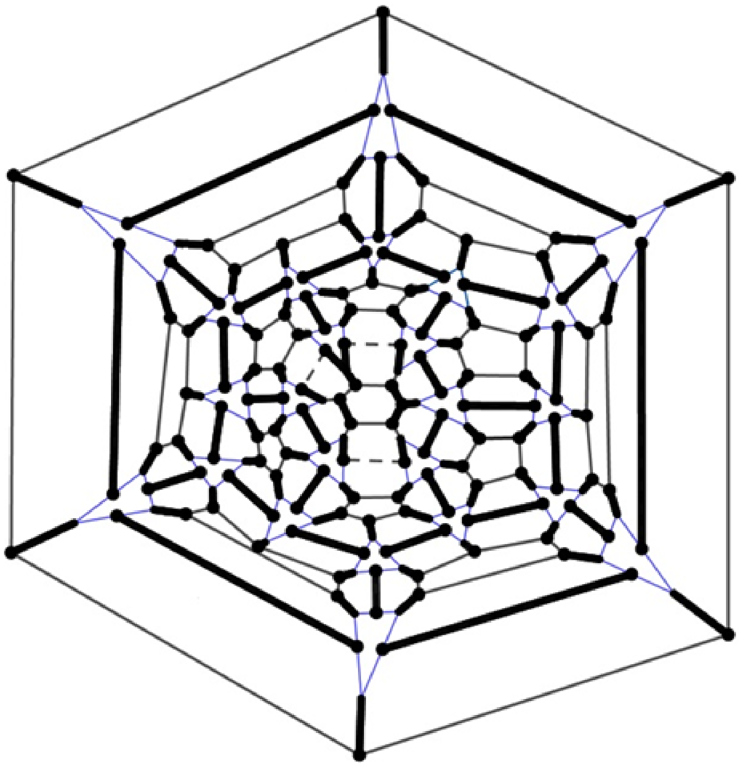

Figure 8: Packaging with two stars (Thick lines).

An example of this type of packaging is shown in Figure 8. From the recently established fact we know that all fullerene graphs are Hamiltonian [12].

Theorem 2.5. *Let G be a fullerene graph with a perfect star packing of type $P0$ and F arises from G via the star transformation. Then there is a perfect packing of $P_3 = K_{1,2}$ and P_9 in G.*

Proof. Considering that the number of vertices of F is a multiple of 9 and according to the Hamiltonian of fullerene graphs, the Hamiltonian path in F can be considered as follows.

$$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}\cdots v_{9n-2}v_{9n-1}v_{9n}$$

According to the number of vertices, we divide the above path into paths of length 8. Therefore, a packing of P_9 is created. By a similar argument, the above Hamiltonian path can be divided into paths that are isomorphic to P_3. ■

3 Semi-star transformation

Let G be a fullerene graph on n vertices ($n \geq 80$) with a perfect star packing of type $P0$ arising from a smaller fullerene via the star transformation. Then G has an even number of the star graph $K_{1,3}$. All the vertices of graph G can be covered by stars by packing as shown in Figure 9.

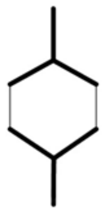

Figure 9: Packing a hexagon by two stars.

Subdivide each edge of the stars by one vertex. As shown in Figure 10, we connect two vertices in all hexagons. By doing this process, we arrive at a fullerene graph F which has twice

Figure 10: Connecting two vertices in a hexagon (Dash line).

the number of hexagons.

According to the way F is created, the number of vertices of F is equal to

$$n + \frac{3n}{4} = \frac{7n}{4}.$$

Let me call this transformation the semi-star transformation. The effect of semi-star transformation on a fullerene graph on 80 vertices with a perfect star packing of type $P0$ is shown in Figure 11.

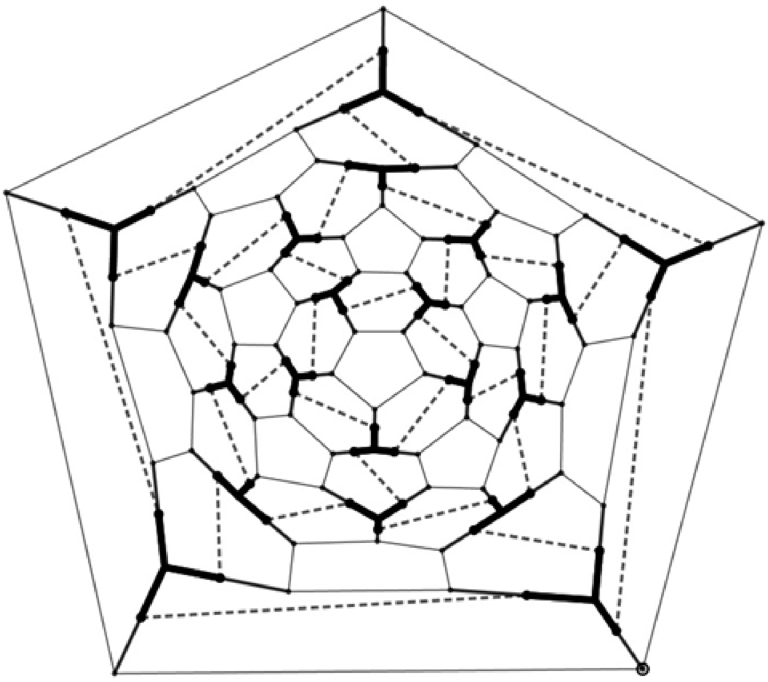

Figure 11: Semi-star transformation in $C_{80}(I_h)$.

Theorem 3.1. *Let G be a fullerene graph on n vertices with a perfect star packing of type $P0$ arising from a smaller fullerene via the chamfer transformation. If F arises from G via semi-star transformation, then F has a perfect packing of subdivided star, $S(K_{1,3})$.*

Proof. Consider an arbitrary star in the perfect packing star of G. According to the act of the star transformation, if we create a vertex on each edge of the stars, a graph will be created as shown in Figure 12. which is a subdivided star that covers all vertices of F. ■

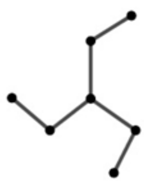

Figure 12: Subdivided star $S(K_{1,3})$.

One of the subjects of interest is whether the graphs created by star transformation have star packing or not.

Conflicts of Interest. The author declares that he has no conflicts of interest regarding the publication of this article.

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