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Research Paper

Entire Sombor Index of Graphs

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Abstract

Let $G = (V,\ E)$ be a simple graph with vertex set V and edge set E. The Sombor index of the graph G is a degree-based topological index, defined as

$$SO(G) = \sum_{uv \in E} \sqrt{d\,(u)^2 + d(v)^2},$$

in which $d(x)$ is the degree of the vertex x. In this paper, we introduce a new topological index called the entire Sombor index of a graph which is defined as the sum of the terms $\sqrt{d\,(x)^2 + d(y)^2}$ where x is either adjacent or incident to y and $x,\ y \in V \cup E$. We obtain exact values of this new topological index in some graph families. Some important properties of this index are obtained.

1 Introduction

Gutman defined a new vertex degree-based topological index, named the Sombor index, and defined for a graph G as follows

$$SO(G) = \sum_{uv \in E} \sqrt{d\,(u)^2 + d(v)^2},$$

where $d(u)$ and $d(v)$ denote the degree of vertices u and v in G , respectively [1]. Other versions of the Sombor index such as reduced Sombor index, average Sombor index, general Sombor index, modified Sombor index, delta Sombor index and reverse Sombor index are introduced and studied in [1–9].

In molecular structures there exist relations between the atoms of a molecule and between atoms and bonds. Therefore, in some topological indices such as entire Zagreb indices [10], entire Randic index [11] and entire forgotten topological index [12] are considered into account the

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relations between the edges and vertices in addition to the relations between vertices. motivated by these topological indices and the Sombor index, we introduce a new topological index named the entire Sombor index. We investigate and publish some fundamental properties of it. Also, we compute the entire Sombor index for some graph families. Finally, we obtain the sharp bounds for the entire Somber index of a graph G.

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\}$. The set $N_G(u) = \{v \in V | uv \in E\}$ is called the neighborhood of vertex $u \in V$ in graph G. The number of edges incident to vertex u in G is denoted $deg_G(u) = d(u)$. The isolated vertex and pendant vertex are the vertices with degrees 0 and 1 in graph G, respectively. The minimum degree and the maximum degree of G are denoted by δ and Δ, respectively. The edge degree $d(e)$ of the edge $e = uv$ is defined as $d(e) = d(u) + d(v) - 2$. We denote the vertex x incident to the edge y in G by $x \sim y$.

The first and second Zagreb indices are two of the most useful topological graph indices, denoted by $M_1(G)$ and $M_2(G)$ and define as [13]

$$M_1\left(G\right) = \sum_{u \in V} d(u)^2, \quad M_2\left(G\right) = \sum_{uv \in V} d\left(u\right) d\left(v\right).$$

In [14], the reformulated first Zagreb index is defined as $RM_1\left(G\right) = \sum_{e \in E} d(e)^2$. Furtula and Gutman introduced in [15] the forgotten topological index and defined as

$$F\left(G\right) = \sum_{u \in V} d(u)^3 = \sum_{uv \in E} \left(d(u)^2 + d(v)^2\right),$$

and the reformulated forgotten index is defined as $EF\left(G\right) = \sum_{e \in E} d(e)^3$[16]. Recall that reformulating a topological index of graph is related to computing this index ofthe line graph of G. The line graph $L(G)$ of G is the graph that each vertex of it represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G. Throughout this paper, K_n , C_n and P_n denote a complete graph, the cycle and the path of order n , respectively.

2 Entire Sombor index for certain graphs

In this section, we propose a new topological index called the entire Sombor index. We obtain the exact values of the entire Sombor index for certain graphs.

Definition 2.1. For a graph $G = (V, E)$, the entire Sombor index is defined by

$$SO^{\varepsilon}\left(G\right) = \sum_{\{x,\ y\} \in B(G)} \sqrt{d\left(x\right)^2 + d(y)^2}, \tag{1}$$

where $B(G)$ is the set of all subsets of two members $\{x, y\} \subseteq V(G) \cup E(G)$ such that x and y are adjacent or incident to each other.

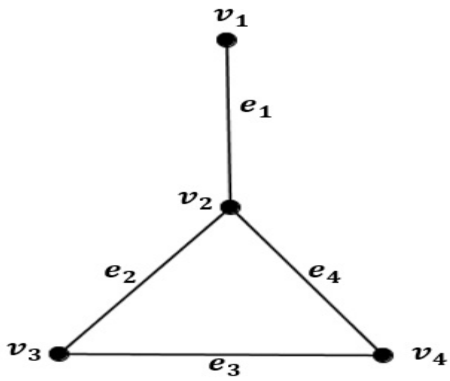

Figure 1: The graph G with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{e_1, e_2, e_3, e_4\}$.

Example 2.2. Let G be graph shown in Figure 1. We compute the entire Sombor index of G.

$$\begin{aligned}
SO^{\varepsilon}(G) &= \sum_{\{x,\ y\}\in B(G)} \sqrt{d\left(x\right)^2 + d(y)^2} \\
&= \sum_{xy\in E(G)} \sqrt{d\left(x\right)^2 + d(y)^2} + \sum_{xy\in E(L(G))} \sqrt{d\left(x\right)^2 + d(y)^2} + \sum_{x\sim y} \sqrt{d\left(x\right)^2 + d(y)^2} \\
&= \sqrt{d\left(v_1\right)^2 + d(v_2)^2} + \sqrt{d\left(v_2\right)^2 + d(v_3)^2} + \sqrt{d\left(v_2\right)^2 + d(v_4)^2} + \sqrt{d\left(v_3\right)^2 + d(v_4)^2} \\
&= \sqrt{d\left(e_1\right)^2 + d(e_2)^2} + \sqrt{d\left(e_1\right)^2 + d(e_4)^2} + \sqrt{d\left(e_2\right)^2 + d(e_4)^2} + \sqrt{d\left(e_2\right)^2 + d(e_3)^2} \\
&= \sqrt{d\left(e_3\right)^2 + d(e_4)^2} + \sqrt{d\left(v_1\right)^2 + d(e_1)^2} + \sqrt{d\left(v_2\right)^2 + d(e_1)^2} + \sqrt{d\left(v_2\right)^2 + d(e_4)^2} \\
&= \sqrt{d\left(v_2\right)^2 + d(e_2)^2} + \sqrt{d\left(v_3\right)^2 + d(e_3)^2} + \sqrt{d\left(v_3\right)^2 + d(e_2)^2} + \sqrt{d\left(v_4\right)^2 + d(e_3)^2} \\
&= \sqrt{d\left(v_4\right)^2 + d(e_4)^2} = \sqrt{10} + 9\sqrt{13} + 15\sqrt{2} + \sqrt{5}.
\end{aligned}$$

Observation 2.3. According to Definition 2.1 and the definition of the Sombor index, the entire Sombor index can be expressed in terms of the Sombor index of G and the Sombor index of the line graph G as follows

$$\begin{aligned}
SO^{\varepsilon}(G) &= \sum_{\{x,\ y\}\in B(G)} \sqrt{d\left(x\right)^2 + d(y)^2} \\
&= \sum_{xy\in E(G)} \sqrt{d\left(x\right)^2 + d(y)^2} + \sum_{xy\in E(L(G))} \sqrt{d\left(x\right)^2 + d(y)^2} + \sum_{x\sim y} \sqrt{d\left(x\right)^2 + d(y)^2} \\
&= SO\left(G\right) + SO\left(L\left(G\right)\right) + \sum_{u\in V(G)} \sum_{v\in N(u)} \sqrt{d\left(u\right)^2 + d(uv)^2}.
\end{aligned}$$

Therefore, the expression (1) is equivalent to

$$SO^{\varepsilon}\left(G\right) = SO\left(G\right) + SO\left(L\left(G\right)\right) + \sum_{u\in V(G)} \sum_{v\in N(u)} \sqrt{d\left(u\right)^2 + d(uv)^2}. \tag{2}$$

Now we compute the entire Sombor index for some families of graphs. First, we recall some results that are used in this paper.

Lemma 2.4. *[17]*

i. *If G is k-regular graph of order n, then* $SO\left(L\left(G\right)\right)=\sqrt{2}nk(k-1)^{2}$.

ii. *If C_n is a cycle of order n, then* $SO\left(L\left(C_n\right)\right)=2\sqrt{2}n$.

iii. *If K_n is a complete graph of order n, then* $SO\left(L\left(K_n\right)\right)=\sqrt{2}n\left(n-1\right)\left(n-2\right)^{2}$.

iv. *If $K_{p,\ q}$ is a complete bipartite graph with $p+q$ vertices and pq edges, then* $SO\left(L\left(K_{p,q}\right)\right)=\frac{\sqrt{2}}{2}pq(p+q-2)^{2}$.

Proposition 2.5. *Let G be a k-regular graph of order n. Then*

$$SO^{\varepsilon}\left(G\right)=\frac{nk}{2}\left(k\sqrt{2}+2\sqrt{2}(k-1)^{2}+2\sqrt{k^{2}+4(k-1)^{2}}\right).$$

Proof. Let G be a k-regular graph of order n and $m=\frac{nk}{2}$ edges. Since the line graph $L(G)$ of G is a $2(k-1)$-regular graph with m vertices and $m'=\frac{nk}{2}(k-1)$. Using the expression (2) and Lemma 2.4 (i) we have

$$\begin{aligned}
SO^{\varepsilon}\left(G\right) &= SO\left(G\right)+SO\left(L\left(G\right)\right)+\sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d\left(u\right)^{2}+d\left(uv\right)^{2}}\\
&= mk\sqrt{2}+nk\sqrt{2}(k-1)^{2}+nk\sqrt{k^{2}+(2k-2)^{2}}\\
&= k\left(\frac{nk}{2}\right)\sqrt{2}+nk\sqrt{2}(k-1)^{2}+nk\sqrt{k^{2}+(2k-2)^{2}}\\
&= \frac{nk}{2}\left(k\sqrt{2}+2\sqrt{2}(k-1)^{2}+2\sqrt{k^{2}+(2k-2)^{2}}\right).
\end{aligned}$$

■

Proposition 2.6. *For a complete bipartite graph $K_{p,q}$*

$$SO^{\varepsilon}\left(K_{p,q}\right)=pq\left(\sqrt{p^{2}+q^{2}}+\frac{\sqrt{2}}{2}(p+q-2)^{2}+\sqrt{q^{2}+(p+q-2)^{2}}+\sqrt{p^{2}+(p+q-2)^{2}}\right).$$

Proof. Using the expression (2) and Lemma 2.4 (iv), we have

$$\begin{aligned}
SO^{\varepsilon}\left(K_{p,q}\right) &= SO\left(K_{p,q}\right)+SO\left(L\left(K_{p,q}\right)\right)+\sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d\left(u\right)^{2}+d(uv)^{2}}\\
&= pq\sqrt{p^{2}+q^{2}}+\frac{\sqrt{2}}{2}pq(p+q-2)^{2}+pq\left(\sqrt{q^{2}+(p+q-2)^{2}}+\sqrt{p^{2}+(p+q-2)^{2}}\right)\\
&= pq\left(\sqrt{p^{2}+q^{2}}+\frac{\sqrt{2}}{2}(p+q-2)^{2}+\sqrt{q^{2}+(p+q-2)^{2}}+\sqrt{p^{2}+(p+q-2)^{2}}\right).
\end{aligned}$$

■

The following results are obtained directly from Proposition 2.5.

Corollary 2.7. *For star graph S_n of order $n \geq 3$,*

$$SO^{\varepsilon}(S_n) = (n-1)\left(\sqrt{(n-1)^2+1} + \frac{\sqrt{2}}{2}(n-2)^2 + \sqrt{(n-1)^2+(n-2)^2} + \sqrt{1+(n-2)^2}\right).$$

Proposition 2.8. *Let K_n, C_n and P_n be the complete graph, the cycle graph and the path graph of order n, respectively.*

i. $SO^{\varepsilon}(K_n) = \frac{n(n-1)}{2}\left(\sqrt{2}(n-1) + 2\sqrt{2}(n-2)^2 + 2\sqrt{5(n-2)^2+(2n-3)}\right)$.

ii. $SO^{\varepsilon}(C_n) = 8\sqrt{2}n$.

iii. $SO^{\varepsilon}(P_n) = 6\sqrt{5} + 8(n-3)\sqrt{2}$.

Proof. i. By applying Lemma 2.4 (*iii*), we get

$$\begin{aligned} SO^{\varepsilon}(K_n) &= SO(K_n) + SO(L(K_n)) + \sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2} \\ &= \frac{n(n-1)^2}{2}\sqrt{2} + \sqrt{2}n(n-1)(n-2)^2 + n(n-1)\sqrt{(n-1)^2+4(n-2)^2} \\ &= \frac{n(n-1)}{2}\left(\sqrt{2}(n-1) + 2\sqrt{2}(n-2)^2 + 2\sqrt{(n-1)^2+4(n-2)^2}\right). \end{aligned}$$

ii. Since the line graph C_n is the cycle C_n, using Lemma 2.4 (*ii*) we get

$$\begin{aligned} SO^{\varepsilon}(C_n) &= SO(C_n) + SO(L(C_n)) + \sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2} \\ &= 2\sqrt{2}n + 2\sqrt{2}n + 2n\sqrt{2^2+2^2} = 8\sqrt{2}n. \end{aligned}$$

iii. Since the line graph P_n is the path P_{n-1}, we get

$$\begin{aligned} SO^{\varepsilon}(P_n) &= SO(P_n) + SO(L(P_n)) + \sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2} \\ &= 2\sqrt{5} + (2n-6)\sqrt{2} + 2\sqrt{5} + (2n-8)\sqrt{2} + \sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2}. \end{aligned}$$

Since d$(v_1) = d(v_n) = d(e_1) = d(e_{n-1}) = 1$ and $d(v_i) = d(e_j) = 2$ for $2 \leq i \leq n-1$ and $2 \leq j \leq n-2$, we get

$$\begin{aligned} \sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2} &= 2\sqrt{2} + 2\sqrt{5} + \sqrt{d(v_2)^2+d(e_2)^2} + \sqrt{d(v_{n-1})^2+d(e_{n-2})^2} \\ &+ 2\sum_{u\in V(P_n)\setminus\{v_1,\, v_2,\, v_{n-1},\, v_n\}}\sqrt{2^2+2^2} \\ &= 2\sqrt{2} + 2\sqrt{5} + 4\sqrt{2} + 4(n-4)\sqrt{2} \\ &= 6\sqrt{2} + 2\sqrt{5} + 4(n-4)\sqrt{2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} SO^{\varepsilon}(P_n) &= 2\sqrt{5}+(2n-6)\sqrt{2}+2\sqrt{5}+(2n-8)\sqrt{2}+\sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d(u)^2+d(uv)^2} \\ &= 4\sqrt{5}+2(2n-8)\sqrt{2}+2\sqrt{2}+6\sqrt{2}+2\sqrt{5}+4(n-4)\sqrt{2} \\ &= 6\sqrt{5}+8\sqrt{2}+4(n-4)\sqrt{2}+4(n-4)\sqrt{2} \\ &= 6\sqrt{5}+8(n-3)\sqrt{2}. \end{aligned}$$

■

3 Properties of the entire Sombor index of graphs

In this section, we investigate some mathematical properties of the entire Sombor index of a graph. We first study a graph G for the removal of any arbitrary edge or vertex. To do this, we need the following known results.

Lemma 3.1. *[1] If G is a connected graph with n vertices, then $SO(P_n) \leq SO(G) \leq SO(K_n)$, with equality if and only if $G \cong P_n$ and $G \cong K_n$.*

Lemma 3.2. *[1] If T is a tree with n vertices, then $SO(P_n) \leq SO(T) \leq SO(S_n)$, with equalities if and only if $T \cong P_n$ and $T \cong S_n$.*

Lemma 3.3. *[18] For any graph G with $m \geq 1$ edges, $SO(G) \leq \sqrt{mF(G)}$.*

At first, we investigate the effects on $SO(G)$ when $SO^{\varepsilon}(G)$ is changed by removing a vertex and an edge of G.

Theorem 3.4. *Let $G=(V,E)$ be a graph with the minimum degree $\delta \geq 1$. For any arbitrary edge $e=uv \in E$*

$$SO^{\varepsilon}(G-e) < \begin{cases} SO^{\varepsilon}(G)-\sqrt{2}(\alpha+2) & if\ \delta \geq 2, \\ SO^{\varepsilon}(G)-\sqrt{2} & if\ \delta = 1, \end{cases}$$

where $\alpha = 4\delta^2-5\delta+2$.

Proof. We consider graph $G-e$ obtained from deleting edge $e=uv$ of G with the entire Sombor index $SO^{\varepsilon}(G-e)$. So, we add the edge $e=uv$ to the graph $G-e$. Suppose that $\delta \geq 1$ and without loss of generality, we suppose that $d(u) \geq d(v)$ and we have $d(e)=d(u)+d(v)-2 \geq 2\delta-2$ for any $e \in E$. We study two cases.

Case 1. Suppose that $\delta \geq 2$. In this case, for the edge $e=xy$, $d(e) \geq \{d(x), d(y)\}$. Therefore, using the expression (2) we get

$$\begin{aligned}
SO^{\varepsilon}(G) &> SO^{\varepsilon}(G-e)+\sqrt{d(u)^2+d(v)^2}+\sqrt{d(u)^2+d(e)^2}+\sqrt{d(v)^2+d(e)^2} \\
&+ \sum_{\substack{i=1 \\ e_i \ adjacent \ to \ e}}^{d(e))} \sqrt{d(e)^2+d(e_i)^2} \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}d(v)+\sqrt{2}d(u)+\sqrt{2}d(v)+\sum_{\substack{i=1 \\ e_i \ adjacent \ to \ e}}^{d(e)} \sqrt{d(e)^2+d(e_i)^2} \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}\delta+\sqrt{2}\delta+\sqrt{2}\delta+d(e)\sqrt{2}(2\delta-2) \\
&\geq SO^{\varepsilon}(G-e)+3\sqrt{2}\delta+\sqrt{2}(2\delta-2)^2.
\end{aligned}$$

By rearranging, we get

$$SO^{\varepsilon}(G-e) < SO^{\varepsilon}(G)-\sqrt{2}(4\delta^2-5\delta+4).$$

By putting $\alpha = 4\delta^2-5\delta+2$, the result holds.

Case 2. If $\delta=1$, then for any edge $e = xy \in E$ we have $d(e) \geq d(x)-1$ and $d(e) \geq d(y)-1$. Therefore, we have

$$\begin{aligned}
SO^{\varepsilon}(G) &> SO^{\varepsilon}(G-e)+\sqrt{d(u)^2+d(v)^2}+\sqrt{d(u)^2+d(e)^2}+\sqrt{d(v)^2+d(e)^2} \\
&+ \sum_{\substack{i=1 \\ e_i \ adjacent \ to \ e}}^{d(e)} \sqrt{d(e)^2+d(e_i)^2} \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}d(v)+\sqrt{d(u)^2+(d(u)-1)^2}+\sqrt{d(v)^2+(d(v)-1)^2} \\
&+ d(e)\sqrt{2(2\delta-2)^2} \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}\delta+\sqrt{(d(u)-1)^2+(d(u)-1)^2}+\sqrt{(d(v)-1)^2+(d(v)-1)^2} \\
&+ 4\sqrt{2}(\delta-1)^2 \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}\delta+\sqrt{2}(\delta-1)+\sqrt{2}(\delta-1)+4\sqrt{2}(\delta-1)^2 \\
&\geq SO^{\varepsilon}(G-e)+\sqrt{2}\delta+2\sqrt{2}(\delta-1)+4\sqrt{2}(\delta-1)^2 \\
&= SO^{\varepsilon}(G-e)+\sqrt{2}\left(4\delta^2-5\delta+2\right).
\end{aligned}$$

By rearranging, we get

$$SO^{\varepsilon}(G-e) < SO^{\varepsilon}(G)-\sqrt{2}(4\delta^2-5\delta+2).$$

By putting $\delta=1$, the result is complete.

■

Theorem 3.5. *Let $G=(V,E)$ be a graph with the minimum degree $\delta \geq 1$. For any arbitrary vertex $u \in V$,*

$$SO^{\varepsilon}(G-u) < \begin{cases} SO^{\varepsilon}(G)-2\sqrt{2}\delta\alpha & if \ \delta \geq 2, \\ SO^{\varepsilon}(G)-\sqrt{2} & if \ \delta = 1, \end{cases}$$

where $\alpha = 2\delta - 1$.

Proof. We consider graph $G-u$ obtained from removing vertex u and all related edges of G with the entire Sombor index $SO^{\varepsilon}(G-u)$. Now, we add the vertex u and its related edges to the graph $G-u$. Similar to the proof of Theorem 3.4, we have

Case 1 If $\delta \geq 2$, then

$$\begin{aligned}
SO^{\varepsilon}(G) &> SO^{\varepsilon}(G-u) + \sum_{ux_i \in E} \sqrt{d(u)^2 + d(x_i)^2} + \sum_{\substack{i=1 \\ u \ \ incident \ \ to \ e_i}}^{d(u)} \sqrt{d(u)^2 + d(e_i)^2} \\
&+ \sum_{i=1}^{d(u)} \sum_{\substack{e_i \ \ adjacent \ \ to \ \ e \\ u \ \ incident \ \ to \ \ e_i}} \sqrt{d(e)^2 + d(e_i)^2} \\
&\geq SO^{\epsilon}(G-u) + \sqrt{2}\delta d(u) + d(u)\left(\sqrt{2}\delta d(u)\right) + d(u)\left(2\sqrt{2}(\delta-1)\right) \\
&\geq SO^{\varepsilon}(G-u) + \sqrt{2}\delta^2 + \sqrt{2}\delta^2 + 2\sqrt{2}\delta(\delta-1) \\
&\geq SO^{\varepsilon}(G-u) + 2\sqrt{2}\delta^2 + 2\sqrt{2}\delta(\delta-1).
\end{aligned}$$

By rearranging, we get

$$SO^{\varepsilon}(G-u) < SO^{\varepsilon}(G) - 2\sqrt{2}\delta(2\delta-1).$$

By putting $\alpha = 2\delta - 1$, the result holds.

Case 2 If $\delta = 1$, then

$$\begin{aligned}
SO^{\varepsilon}(G) &> SO^{\varepsilon}(G-u) + \sum_{ux_i \in E} \sqrt{d(u)^2 + d(x_i)^2} + \sum_{\substack{i=1 \\ u \ \ incident \ \ to \ e_i}}^{d(u)} \sqrt{d(u)^2 + d(e_i)^2} \\
&+ \sum_{i=1}^{d(u)} \sum_{\substack{e_i \ \ adjacent \ \ to \ \ e \\ u \ \ incident \ \ to \ \ e_i}} \sqrt{d(e)^2 + d(e_i)^2} \\
&\geq SO^{\epsilon}(G-u) + \sqrt{2}\delta d(u) + d(u)\sqrt{d(u)^2 + (d(u)-1)^2} + d(u)\left(\sqrt{2(2\delta-2)^2}\right) \\
&\geq SO^{\varepsilon}(G-u) + \sqrt{2}\delta^2 + d(u)\sqrt{2(d(u)-1)^2} + 2\sqrt{2}d(u)(\delta-1) \\
&\geq SO^{\varepsilon}(G-u) + \sqrt{2}\delta^2 + \sqrt{2}\delta(\delta-1) + 2\sqrt{2}\delta(\delta-1) \\
&\geq SO^{\varepsilon}(G-u) + \sqrt{2}\delta^2 + 3\sqrt{2}\delta(\delta-1).
\end{aligned}$$

By rearranging, we get

$$SO^{\varepsilon}(G-u) < SO^{\varepsilon}(G) - \sqrt{2}\delta(4\delta-3).$$

By putting $\delta = 1$, the result is complete.

■

We use a similar technique lower bound for the Sombor index of a graph in [1] to obtain the bounds for the entire Sombor index for a connected graph given in the following theorem. For the graph G, we define a set $\mathcal{O}(G)$ of different types of ordered pairs, initially equal to empty, and add its elements according to the following rules.
For a vertex u incident to edge e, we add an ordered pair of type $(deg(u), deg(e))$. For every pair of adjacent edges e_1 and e_2, we add an ordered pair of type $(deg(e_1), deg(e_2))$-edge.

Theorem 3.6. *For any connected graph G of order n*

$$SO^{\varepsilon}(P_n) \leq SO^{\varepsilon}(G) \leq SO^{\varepsilon}(K_n).$$

Equalities hold if and only if $G \cong P_n$ and $G \cong K_n$.

Proof. Let $G = (V, E)$ be a connected graph. The upper bound is obtained directly from the definition. For the lower bound, using Theorem 3.4 by deleting an edge from the graph G, $SO^{\varepsilon}(G)$ decreases. Therefore, the connected graph with the minimum entire Sombor index is a tree.
It can easily be checked that for $n = 2,\ 3$ the result holds. We suppose that $n \geq 4$. By the definition of the entire Sombor index in (2) we have

$$SO^{\varepsilon}(G) = SO(G) + SO(L(G)) + \sum_{u \in V(G)} \sum_{v \in N(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

Using Lemma 3.2, we have $SO(P_n) \leq SO(T)$ for any tree T of order n. We show that $SO(L(P_n)) \leq SO(L(T))$. Since $L(P_n) = P_{n-1}$ and the line graph of the tree T is aconnected graph of order $n-1$, thus by applying Lemma 3.1 we have $SO(L(P_n)) \leq SO(L(T))$. Therefore, it remains to prove

$$\sum_{u \in V(P_n)} \sum_{v \in N_{P_n}(u)} \sqrt{d(u)^2 + d(uv)^2} \leq \sum_{u \in V(T)} \sum_{v \in N_T(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

According to the proof of Proposition 2.8 (iii), $\mathcal{O}(P_n)$ includes two ordered pairs of type $(1,\ 1)$, two ordered pairs of type $(2,\ 1)$ and $2(n-3)$ ordered pairs of type $(2,\ 2)$. Therefore, we have

$$\begin{aligned} \sum_{u \in V(P_n)} \sum_{v \in N_{P_n}(u)} \sqrt{d(u)^2 + d(uv)^2} &= 2\sqrt{2} + 2\sqrt{5} + 2(n-3)\sqrt{8} \\ &= 2\sqrt{5} + 4n\sqrt{2} - 10\sqrt{2}. \end{aligned} \tag{3}$$

We consider the tree T' of order n with three pairs of type $(1,2)$-edge, three pairs of type $(2,3)$-edge and $n-7$ pairs of type $(2,2)$-edge. Therefore, $\mathcal{O}(T')$ includes 3 ordered pairs of type $(1,1)$, 3 ordered pairs of type $(2,1)$, 3 ordered pairs of type $(3,2)$, 3 ordered pairs of type $(3,3)$ and $2(n-7)$ ordered pairs of type $(2,2)$. Therefore,

$$\begin{aligned} \sum_{u \in V(T')} \sum_{v \in N_{T'}(u)} \sqrt{d(u)^2 + d(uv)^2} &= 3\sqrt{2} + 3\sqrt{5} + 3\sqrt{13} + 3\sqrt{18} + 2(n-7)\sqrt{8} \\ &= 3\sqrt{5} + 3\sqrt{13} + 4n\sqrt{2} - 16\sqrt{2}. \end{aligned} \tag{4}$$

Using the relations (3) and (4) and since $2\sqrt{5} + 4n\sqrt{2} - 10\sqrt{2} \leq 3\sqrt{5} + 3\sqrt{13} - 16\sqrt{2}$, we get

$$\sum_{u \in V(P_n)} \sum_{v \in N_{P_n}(u)} \sqrt{d(u)^2 + d(uv)^2} \leq \sum_{u \in V(T')} \sum_{v \in N_{T'}(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

By the above discussion, we obtain

$$\begin{aligned} SO^{\varepsilon}(P_n) &= SO(P_n) + SO(L(P_n)) + \sum_{u\in V(P_n)} \sum_{v\in N_{P_n}(u)} \sqrt{d(u)^2 + d(uv)^2} \\ &\leq SO(T') + SO(L(T')) + \sum_{u\in V(T')} \sum_{v\in N_{T'}(u)} \sqrt{d(u)^2 + d(uv)^2} \\ &= SO^{\varepsilon}\left(T'\right). \end{aligned}$$

By a similar technique, for any tree T with t ordered pairs of type $(1,\ 2)$-edge where $t \geq 4$, the result holds. For the tree with one or two $(1,\ 2)$-edge, it can easily be investigated that $SO^{\varepsilon}(P_n) \leq SO^{\varepsilon}(T)$.

■

Theorem 3.7. *For any tree T of order n,*

$$SO^{\varepsilon}(P_n) \leq SO^{\varepsilon}(T) \leq SO^{\varepsilon}(S_n).$$

Equalities hold if and only if $T \cong P_n$ and $T \cong S_n$.

Proof. The lower bound follows from Theorem 3.6. For the upper bound, we consider T a tree of order n and using the expression (2) we have

$$SO^{\varepsilon}(T) = SO(T) + SO(L(T)) + \sum_{u\in V(T)} \sum_{v\in N_T(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

Using Lemma 3.2, we have $SO(T) \leq SO(S_n)$ for any tree T of order n. Since $L(T)$ is a connected graph of order $n-1$ and $(S_n) = K_{n-1}$, by Lemma 3.1 we have $SO(L(T)) \leq SO(L(S_n))$. Therefore, it is sufficient to show that

$$\sum_{u\in V(T)} \sum_{v\in N_T(u)} \sqrt{d(u)^2 + d(uv)^2} \leq \sum_{u\in V(S_n)} \sum_{v\in N_{S_n}(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

$\mathcal{O}(S_n)$ includes $n-1$ ordered pairs of type $(1, n-2)$ and $(n-1)$ ordered pairs of type $(n-1, n-2)$. Therefore, we have

$$\sum_{u\in V(S_n)} \sum_{v\in N_{S_n}(u)} \sqrt{d(u)^2 + d(uv)^2} = (n-1)\sqrt{1 + (n-2)^2} + (n-1)\sqrt{(n-1)^2 + (n-2)^2}. \tag{5}$$

By using the proof of Theorem 3.6, and considering the tree T' of order n with 3 ordered pairs of type $(1,1)$, 3 ordered pairs of type $(2,1)$, 3 ordered pairs of type $(3,2)$ and $2(n-7)$ ordered pairs of type $(2,2)$. Note that the greatest values of the terms $\sqrt{x^2+y^2}$ of the ordered pair $(x,\ y)$ where $x = deg(u)$ and $y = deg(e)$ for a vertex u incident to an edge e is $(n-1, n-2)$ and for other cases, $(1,\ n-2)$ greatest values of $(2,\ 1)$, $(3,\ 2)$, $(3,\ 3)$ and $(2,\ 2)$. Therefore, using the relations (4) and (5), we get

$$\sum_{u\in V(T')} \sum_{v\in N_{T'}(u)} \sqrt{d(u)^2 + d(uv)^2} \leq \sum_{u\in V(S_n)} \sum_{v\in N_{S_n}(u)} \sqrt{d(u)^2 + d(uv)^2},$$

and consequently,

$$\begin{aligned} SO^{\varepsilon}(T') &= SO(T') + SO(L(T')) + \sum_{u\in V(T')}\sum_{v\in N_{T'}(u)} \sqrt{d(u)^2 + d(uv)^2} \\ &\le SO(S_n) + SO(L(S_n)) + \sum_{u\in V(S_n)}\sum_{v\in N_{S_n}(u)} \sqrt{d(u)^2 + d(uv)^2} \\ &= SO^{\varepsilon}(S_n). \end{aligned}$$

By the above discussion, for any tree T with t ordered pairs of type $(1,\ 2)$-edge where $t \ge 4$ or $t \le 2$, the result holds.

■

We obtain an upper bound of the entire Sombor index in terms of some topological indices in G. To do this, we need the following know inequality.

Cauchy-Schwarz inequality [19] For all sequences of real numbers a_i and b_i

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Theorem 3.8. *Let G be a connected graph of order n and size m whose vertices have degree d_i for $i = 1, 2, \ldots, n$. Then*

$$SO^{\varepsilon}(G) \le \sqrt{mF(G)} + \sqrt{\left(\frac{1}{2}M_1(G) - m\right)EF(G)} + \sqrt{2m(F(G) + 2RM_1(G))}.$$

The equality holds if and only if G is a regular graph.

Proof. For the connected graph G, by the definition of the entire Sombor index (2), we have

$$SO^{\varepsilon}(G) = SO(G) + SO(L(G)) + \sum_{u\in V(G)}\sum_{v\in N(u)} \sqrt{d(u)^2 + d(uv)^2}.$$

Using Lemma 3.3, we have

$$SO(G) \le \sqrt{mF(G)}. \tag{6}$$

Since the number of edges in line graph $L(G)$ is equal to $m' = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m = \frac{1}{2}M_1(G) - m$ and by applying Lemma 3.3, we get

$$SO(L(G)) \le \sqrt{m' F(L(G))} = \sqrt{\left(\frac{1}{2}M_1(G) - m\right)EF(G)}. \tag{7}$$

Therefore, it is sufficient to prove that

$$\sum_{u\in V(G)}\sum_{v\in N(u)} \sqrt{d(u)^2 + d(uv)^2} \le \sqrt{2m(F(G) + 2RM_1(G))}.$$

According to Cauchy-Schwarz inequality and put $a_i = 1$ and $b_i = \sqrt{d(u)^2 + d(uv)^2}$, we have

$$\begin{aligned}
\left(\sum_{u\sim uv}\sqrt{d(u)^2+d(uv)^2}\right)^2 &\leq \left(\sum_{u\sim uv}1\right)\left(\sum_{u\sim uv}\left(\sqrt{d(u)^2+d(uv)^2}\right)^2\right)\\
&= 2m\left(\sum_{u\sim uv}\left(d(u)^2+d(uv)^2\right)\right)\\
&= 2m\left(\sum_{u\in V(G)}\sum_{v\in N(u)}d(u)^2+\sum_{u\in V(G)}d(uv)^2\right)\\
&= 2m\left(\sum_{u\in V}d(u)^3+2\sum_{uv\in E}d(uv)^2\right)\\
&= 2m\left(F\left(G\right)+2RM_1\left(G\right)\right).
\end{aligned}$$

Therefore,

$$\sum_{u\in V(G)}\sum_{v\in N(u)}\sqrt{d\left(u\right)^2+d(uv)^2}\leq\sqrt{2m\left(F\left(G\right)+2RM_1\left(G\right)\right)}. \tag{8}$$

Therefore, using the relations (6), (7) and (8) in the expression (2) the result completes. The equalities hold if and only if G is a regular graph. ■

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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