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# *Resolving Topological Indices of Graphs*

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**ABSTRACT**

Topological indices are graph invariants most suitable for underlined structures of chemical compounds. Most of the topological indices are defined on the well-known graph concepts such as degree of a vertex, distances, eccentricity of a vertex, etc. In this paper, new type of degree of a vertex is defined with the aid of resolving property of the graph as the minimum cardinality of a resolving set containing that vertex. The mathematical properties of this newly defined degree is established with the help of standard graphs and an attempt to analyse its applicability in chemical compounds are carried by taking silicate structures.



## 1. INTRODUCTION

Throughout this paper, $G(V,E)$ is a simple, undirected, finite connected graph, with vertex set $V$ and edge set $E$, respectively. The degree of a vertex $a$ in $G$ is denoted by $deg_G(a)$ and

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$d(a,b)$ is the distance between the vertices $a$ and $b$ in $G$. We use standard terminology of graph theory. For the terms not defined here we refer to the book [2].

Topological indices (TI) are the numerical invariants of a molecular graph and are very useful for predicting physico-chemical properties of chemical compounds. A great variety of such indices are studied and used in theoretical chemistry. The family of Zagreb indices [3] is one of the oldest degree based TIs given as $M_1(G) = \sum_{a\in V} d_G(a)^2 = \sum_{ab\in E}[d_G(a) + d_G(b)]$ and $M_2(G) = \sum_{ab\in E} d_G(a) \cdot d_G(b)$. For related work on indices, we refer to [5,7,8,9,11].

The concept of metric dimension, based on the resolving set was first studied by Slater [12] and independently by Harary and Melter [4]. A subset $S \subseteq V$ is said to be a *resolving set* or *locating set* if for every pair of vertices $a, b \in V - S$, there exists a vertex $x \in S$ such that $d(x,a) \neq d(x,b)$. The minimum cardinality of a resolving set of $G$ is called *metric dimension* of $G$, denoted by $\beta(G)$, and all resolving sets of cardinality $\beta(G)$ is called *metric basis* of $G$. For the related work on resolving sets and metric dimension we refer to [1,10,13,14].

Recently in 2021, a new degree of a vertex called the domination degree of $v$, was introduced in [5] based on domination sets having certain property. In [5], authors have studied some basic properties of domination degree function and obtained exact values for domination Zagreb indices of some families of graphs.

In this paper, we define the resolving degree of a vertex $a$ denoted by $d_\beta(a)$, as the minimum cardinality of a resolving set of $G$ containing the vertex $a$. That is, $d_\beta(a) = \min\{|S_a|: S_a$ is a minimal resolving set of $G$ containing the vertex $a\}$. If the resolving degree of every vertex is $r$ then the graph is said to be resolving regular graph with resolving regularity $r$. Let $\Delta_\beta(G)$ and $\delta_\beta(G)$ be the maximum and minimum value of resolving degree of a vertex in $G$. Then, as the metric dimension of a graph $G$ is at most one less than its order, for every $a \in V$ it follows that;

$$1 \leq \delta_\beta(G) \leq d_\beta(a) \leq \Delta_\beta(G) \leq n - 1 \qquad (1)$$

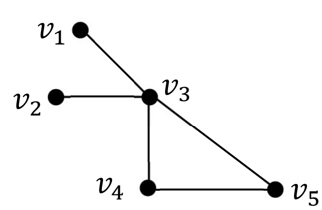



**Figure 1.** A graph $G$ with $\delta_\beta(G) = 2, \Delta_\beta(G) = 3$ .

**Example 1.** For the graph $G$ of Figure 1, the sets $S_1 = \{v_1, v_4\}$, $S_2 = \{v_1, v_5\}$, $S_3 = \{v_2, v_4\}$, $S_4 = \{v_2, v_5\}$ are the only resolving sets of minimum cardinalities (such sets are called metric basis). Thus, $d_\beta(v_1) = |S_1| = |S_2| = 2$. $d_\beta(v_2) = |S_3| = |S_4| = 2$, $d_\beta(v_4) = |S_1| = |S_3| = 2$ and $d_\beta(v_5) = |S_2| = |S_4| = 2$. But, the vertex $v_3$ is not in any metric basis. Now consider the set $S_5 = \{v_1, v_3, v_4\}$ which is a super set of $S_1$ and hence a resolving set of $G$ (note that resolving property is super hereditary) of minimum cardinality containing the vertex $v_3$. Therefore, $d_\beta(v_3) = 3$.

## 2 BASIC RESULTS ON RESOLVING DEGREE

We recall the following result of Khuller et al. [6] and Harary et al. [4] for immediate reference.

**Theorem 2.1.** [6] *A connected graph $G$ of order $n$ has dimension 1 if and only if $G \cong P_n$.*

**Theorem 2.2.** [4] *A connected graph $G$ of order $n(\geq 2)$ has dimension $n-1$ if and only if $G \cong K_n$.*

From the proof of Theorem 2.1, it is clear that only a singleton set containing one of the pendent vertices of the path $P_n$ is its resolving set of minimum cardinalities. We record this important fact in the form of the following lemma.

**Lemma 2.3.** For any vertex $a \in V(G)$ in a graph $G$, $d_\beta(a) = 1$ if and only if $a$ is a pendent vertex and $G \cong P_n$.

**Lemma 2.4.** *A non-trivial graph $G$ is resolving regular graph of resolving regularity 1 if and only if $G \cong P_2$.*

**Proof.** Follows directly from Lemma 2.3, and Theorem 2.1. ■

**Corollary 2.5.** *A path $P_n$ is resolving regular if and only if $n = 2$.*

**Proof.** Since none of the singleton subset containing an internal vertex of a path $P_n$ is a resolving set, $d_\beta(v) \geq 2$ for every internal vertex and $d_\beta(u) = 1$ for an end vertex (by Lemma 2.3), the graph $P_n$ is not resolving regular for any $n \geq 3$. Hence the result follow by Lemma 2.4. ■

**Lemma 2.6.** *If $G$ is a resolving regular graph of resolving regularity $k$, then $\Delta(G) \leq 3^{k-1}$.*

**Proof.** Let $\Delta(G) = l$ and $v$ be a vertex in $G$ of degree $l$. Let $a_1, a_2, \dots, a_l$ be the neighbours of $v$ in $G$. Let $S$ be a minimum resolving set of $G$ containing $v$. Then, $|S| = d_\beta(v) = k$. Let $d(x, v) = \alpha$. Then, as $|d(x, y) - d(x, z)| \leq 1$ for every adjacent vertex $y$ and $z$ in $G$ for any $x \in S$, it follows that $d(x, a_i) \in \{\alpha, \alpha+1, \alpha-1\}$ for every $1 \leq i \leq l$. So, $x \in S$ will resolve at most 3 neighbours of $v$. Therefore, the element in $S - \{v\}$ collectively resolve at most $3^{|S|-1}$ neighbours of $v$. Hence $|N(v)| \leq 3^{|S|-1}$ implies that $\Delta(G) = \deg(v) = \leq 3^{k-1}$. ■

**Theorem 2.7.** *A graph $G$ is resolving regular of resolving regularity 2 then one of the following holds.*

(i) *$G$ is a cycle.*

(ii) *$\Delta(G) = 3$, every vertex $v$ of degree 3 lies in an odd cycle $C$ of $G$ such that every chord of $C$ is parallel to an edge incident with $v$.*

**Proof.** Let $v_0, v_1, \dots, v_{n-1}$ be the vertices of $C_n$ in order. Then, for each $0 \le i \le n-1$, no pair $u, v \in \{v_0, v_1, \dots, v_{n-1}\}$ which are equidistant from both the vertices $v_i$ and $v_{i+1(\text{mod } n)}$ and hence, $S = \{v_i, v_{i+1}\}$ resolves $C_n$. So, $d_\beta(v_i) = 2$ for every $0 \le i \le n-1$. This shows that $C_n$ is a resolving regular graph of regularity 2.

Let us assume that $G$ is resolving regular graph of regularity 2. If $G$ is not a cycle, then $G$ is a path or $\Delta(G) \ge 3$. But, by Lemma 2.3 and Corollary 2.5, $G$ can not be a path. By Lemma 2.6, $\Delta(G) \le 3$. Therefore, $\Delta(G) = 3$. Let us now consider the graph $G$ with $\Delta(G) = 3$. Let $v$ be a vertex of degree 3 in $G$ and $a_1, a_2, a_3$ be the neighbours of $v$. Let $S = \{v, x\}$ be a resolving set of $G$ and $d(x, v) = \alpha$. Then, as in the proof of Lemma 2.6, $\{d(x, a_1), d(x, a_2), d(x, a_3)\} \subseteq \{\alpha, \alpha + 1, \alpha - 1\}$ and $d(x, a_i) \ne d(x, a_j)$ for $1 \le i < j \le 3$ (Since $d(v, a_i) = 1$ for every $1 \le i \le 3$). Thus, $\{d(x, a_1), d(x, a_2), d(x, a_3)\} = \{\alpha, \alpha + 1, \alpha - 1\}$. Without loss of generality, let $d(x, a_1) = \alpha$ and $d(x, a_2) = \alpha - 1$. Then, v and $a_1$ lie in different shortest paths from $x$. Let $P_1$ and $P_2$ shortest paths from $x$ to $v$ and $x$ to $a_1$, respectively. Then, $P_1$ and $P_2$ together with the edge $va_1$ is an odd cycle $C$ of $G$, and contains the vertex $v$ and edge $va_2$. Without loss of generality, we assume that $C$ is the smallest odd cycle containing $v$ (else we can choose new $x$ which is antipodal to both $v$ and $a_1$, in the smallest odd cycle induced by the vertices of $C$). If possible, let $v_iv_j$ be a chord of this cycle $C$. Then clearly, both $v_i$ and $v_j$ cannot lie in the same path. Let $d(x, v_j) = k$. Then $d(x, v_i) \in \{k, k+1, k = 1\}$. We first see that $(x, v_i) \ne k \pm 1$. In fact, if $d(x, v_i) \ne k \pm 1$, then the length of the cycle $C': v - a_1 - \dots - v_i - v_j - \dots - a_2 - v$ is equal to $1 + (\alpha - (k \pm 1)) + 1 + ((\alpha - 1) - k) + 1 = 2(\alpha - k) + 1$ is odd, a contradiction to the fact the $C$ is the smallest odd cycle containing $v$. Therefore, $d(x, v_i) = k$. But then, $d(a_1, v_i) = d(v, v_j) = \alpha - k$ and hence, $v_iv_j$ is parallel to $va_1$. Hence the theorem. ∎

**Remark 2.8.** The converse of the above theorem need not be true in general. For example, $K_4$ satisfies the condition, but $K_4$ is resolving regular of resolving regularity 3.

**Lemma 2.9.** *A resolving regular graph $G$ of resolving regularity 2 cannot have a sub graph isomorphic to $K_4$.*

**Proof.** If possible, suppose that $G$ has a sub graph $H$ isomorphic to $K_4$ and $V(H) = \{v_1, v_2, v_3, v_4\}$. Let $x$ be any vertex of $G$ and $d(x, v_1) = a$. Then $\{d(x, v_2), d(x, v_3), d(x, v_4) \in \{a, a-1, a+1\}$. But $|\{d(x, v_2), d(x, v_3), d(x, v_4)\} \cap \{a - 1,\ a +$

$1\}| = 1$ and hence $|\{d(x, v_2),\ d(x, v_3), d(x, v_4)\}| = 2$. So, no two-element set will resolve the vertices in $H$, a contradiction to the fact that resolving regularity of $G$ is 2. ■

**Lemma 2.10.** *A non-trivial graph $G$ is resolving regular graph of resolving regularity $n - 1$ if and only if $G \cong K_n$.*

**Proof.** Follows directly by Theorem 2.2 and vertex transitivity of $K_n$. ■

**Lemma 2.11.** *For every vertex $v$ of a connected graph $G$,*

$$\beta(G) \leq d_\beta(v) \leq \beta(G) + 1,$$

*and $d_\beta(v) = \beta(G)$ if and only if there is a metric basis containing $v$.*

**Proof.** Let $S$ be a metric basis of $G$. Then $|S| = \beta(G)$ and for each $v \in S$, $S$ is the smallest resolving set containing $v$. Also, for each $v \notin S$ by the property of super hereditary of resolving sets, the set $S \cup \{v\}$ is a resolving set containing $v$. Thus, for $d_\beta(v) = |S| = \beta(G)$ for all $v \in S$ and $d_\beta(v) \leq |S| + 1 = \beta(G) + 1$ for all $v \in V(G) - S$. ■

## 3 RESOLVING DEGREE BASED TOPOLOGICAL INDICES

Using the notion of newly defined $d_\beta(a)$, we now formally introduce new TIs called the resolving topological indices as follows:

**Definition 3.1**. *For a connected non-trivial graph $G(V, E)$, the first resolving Zagreb index are defined as*

$$_\beta M_1(G) = \sum_{a \in V} d_\beta(a)^2 \tag{2}$$

$$_\beta M_1^*(G) = \sum_{ab \in E}\left[d_\beta(a) + d_\beta(b)\right] \tag{3}$$

**Definition 3.2.** *For a connected non-trivial graph $G(V, E)$, the second resolving Zagreb index are defined as*

$$_\beta M_2(G) = \sum_{ab \in E}\left[d_\beta(a) \cdot\ d_\beta(b)\right] \tag{4}$$

**Remark 3.3.** In view of Lemma 2.11, the Equations (2), (3) and (4) can be written as

$$_\beta M_1(G) = \xi\left(\beta(G)\right)^2 + (|V(G)| - \xi)(\beta(G) + 1)^2 \tag{5}$$

$$_\beta M_1^*(G) = 2|E(G)|\beta(G) + (\eta_1 + 2\eta_2) \tag{6}$$

$$_\beta M_2(G) = |E(G)|\left(\beta(G)\right)^2 + (\eta_1 + 2\eta_2)\beta(G) + \eta_2 \tag{7}$$

where

$$\xi = |\{v: d_\beta(v) = \beta(G)\}|,$$

$$\eta_1 = |\{e = uv \in E(G): d_\beta(u) = \beta(G), d_\beta(v) = \beta(G) + 1\}|,$$

$$\eta_2 = |\{e = uv \in E(G): d_\beta(u) = d_\beta(v) = \beta(G) + 1\}|.$$

**Note.** The indices ${}_{\beta}M_1(G)$ and ${}_{\beta}M_1^*(G)$ are not identical. In fact, for the graph of Figure 1,

$${}_{\beta}M_1(G) = 4(2)^2 + 1(3)^2 = 25,$$
$${}_{\beta}M_1^*(G) = [2+2] + 4[2+3] = 24,$$
$${}_{\beta}M_2(G) = [2\times 2] + 4[2\times 3] = 28.$$

However, if $d_{\beta}(a) = deg(a)$ for all $a \in V$, then ${}_{\beta}M_1(G) = {}_{\beta}M_1^*(G)$. The following proposition shows the existence of such a graph.

**Proposition 3.4.** *For a connected graph $G$ of order $n$, ${}_{\beta}M_1(G) = {}_{\beta}M_1^*(G)$ and ${}_{\beta}M_2(G) = 1$ if and only if $G \cong P_2$.*

From Lemma 2.4, Lemma 2.10 and Equations (2) and (3), it is clear that for every non-trivial connected graph $G$ the following hold.

$$2 \le {}_{\beta}M_1(G),\ {}_{\beta}M_1^*(G) \le n(n-1)^2 \tag{8}$$

**Remark 3.5**. The equality in Equation (8) holds only for the complete graphs. In fact ${}_{\beta}M_1(G) \ge 2$ and the equality hold only if $G \cong K_2$ and ${}_{\beta}M_1(G) \le n(n-1)^2$ holds only for $K_n$.

We now obtain improved bounds for most general cases.

**Theorem 3.6.** *For any non-trivial connected graph $G$* of order $n$,

$$2(2n-3) \le {}_{\beta}M_1(G) \le n + (n-2)\beta(G)^2 + (2n-1)\beta(G). \tag{9}$$
$$2(2n-3) \le {}_{\beta}M_1^*(G) \le \big((n-1)\beta(G) + n\big)\Delta(G) \tag{10}$$
$$8(n-2) \le 2\,{}_{\beta}M_2(G) \le (n\beta(G)^2 + (2n-1)\beta(G) + n)\Delta(G). \tag{11}$$

**Proof.** Let $S$ be a metric basis of $G$. Then, by Lemma 2.11, $d_{\beta}(v) = \beta(G)$ for all $v \in S$ and $d_{\beta}(v) \le \beta(G) + 1$ for all $v \in \bar{S} = V(G) - S$.

Now substituting these in Equation (2), we get

$$\begin{aligned} {}_{\beta}M_1(G) &= \textstyle\sum_{a\in V} d_{\beta}(a)^2 \\ &= \textstyle\sum_{a\in S} d_{\beta}(a)^2 + \sum_{a\in \bar{S}} d_{\beta}(a)^2 \\ &\le \textstyle\sum_{a\in S} \beta(G)^2 + \sum_{a\in \bar{S}} (\beta(G)+1)^2 \\ &\le |S|\beta(G)^2 + |V(G) - S|(\beta(G)+1)^2 \\ \textit{i.e}\quad {}_{\beta}M_1(G) &\le \beta(G)\beta(G)^2 + (n - \beta(G))(\beta(G)+1)^2. \end{aligned}$$

Simplifying this we get Equation (9). Similarly substituting the above degrees in (3) and (4), yields

$$\begin{aligned} {}_{\beta}M_1^*(G) &= \textstyle\sum_{ab\in E}\big[d_{\beta}(a) + d_{\beta}(b)\big] = \sum_{a\in V} \deg(a)\, d_{\beta}(a) \le \Delta(G) \sum_{a\in V} d_{\beta}(a) \\ &= \Delta(G)\big(\textstyle\sum_{a\in S} d_{\beta}(a) + \sum_{a\in \bar{S}} d_{\beta}(a)\big) \\ &\le \Delta(G)\big(|S|\beta(G) + |V - S|(\beta(G)+1)\big) \\ &= \Delta(G)\big(\beta(G)\beta(G) + (n-\beta(G))(\beta(G)+1)\big) \\ &= \Delta(G)\big((n-1)\beta(G) + n\big), \text{ which is Equation (10).} \end{aligned}$$

$$
\begin{aligned}
{}_{\beta}M_2(G) &= \textstyle\sum_{ab\in E}\left[d_\beta(a)\cdot\ d_\beta(b)\right] = \frac{1}{2}\sum_{a\in V}\left[d_\beta(a)\sum_{b\in N(a)} d_\beta(b)\right]\\
&= \tfrac{1}{2}\left[\textstyle\sum_{a\in S}\left(d_\beta(a)\sum_{b\in N(a)} d_\beta(b)\right) + \sum_{a\in \bar{S}}\left(d_\beta(a)\sum_{b\in N(a)} d_\beta(b)\right)\right]\\
&\le \tfrac{1}{2}\left[\textstyle\sum_{a\in S}\left(\beta(G)\sum_{b\in N(a)}(\beta(G)+1)\right) + \sum_{a\in V-S}\left[(\beta(G)+1)\sum_{b\in N(a)}(\beta(G)+1)\right]\right]\\
&= \tfrac{1}{2}\left[\textstyle\sum_{a\in S}\beta(G)[(\beta(G)+1)\,deg(a)] + \sum_{a\in V-S}(\beta(G)+1)[(\beta(G)+1)\,deg(a)]\right]\\
&\le \tfrac{1}{2}\Delta(G)\left[\textstyle\sum_{a\in S}\beta(G)(\beta(G)+1) + \sum_{a\in V-S}(\beta(G)+1)^2\right]\\
&= \tfrac{1}{2}\Delta(G)\left[|S|\beta(G)(\beta(G)+1) + |V-S|(\beta(G)+1)^2\right]\\
&= \tfrac{1}{2}\Delta(G)\left[\beta(G)\beta(G)(\beta(G)+1) + (n-\beta(G))(\beta(G)+1)^2\right]\\
&= \tfrac{1}{2}\Delta(G)\left[(n-1)\beta(G)^2 + (2n-1)\beta(G) + n\right],
\end{aligned}
$$

which proves Equation (11).

Lower bounds follows from Lemma 2.3 (since a connected graph on $n$ vertices has at most two vertices of resolving degree 1 and at least $n-1$ edges). ■

The lower bound in Theorem 3.6 is tight and is justified by the following theorem.

**Theorem 3.7.** *For a path graph* $P_n$, ${}_{\beta}M_1(P_n) = {}_{\beta}M_1^*(P_n) = 2(2n-3)$ *for* $n \ge 2$ *and* ${}_{\beta}M_2(P_n) = 4(n-2)$ *for* $n \ge 3$.

**Proof.** In a path $P_n$, $d_\beta(v) = 1$ only for two of its end vertices and hence $d_\beta(u) = 2$ for all the remaining $n-2$ internal vertices (by Lemma 2.11). Thus, substituting these in (2), we get ${}_{\beta}M_1(G) = \sum_{a\in V} d_\beta(a)^2 = (1)^2 \times 2 + (2)^2 \times (n-2) = 4n-6$.

Also, for each of the 2 pendent edges the resolving degree of its end vertices are 1 and 2, and for each of the other $n-3$ non-pendent edges the resolving degree of each of its end vertices is 2. Substituting these in (3) and (4), we get

$$
\begin{aligned}
{}_{\beta}M_1^*(G) &= \textstyle\sum_{ab\in E}\left[d_\beta(a) + d_\beta(b)\right]\\
&= [1+2]\times 2 + [2+2]\times(n-3) = 4n-6, \text{ and}\\
{}_{\beta}M_2(G) &= \textstyle\sum_{ab\in E}\left[d_\beta(a)\cdot d_\beta(b)\right]\\
&= [1\times 2]\times 2 + [2\times 2]\times(n-3) = 4n-8.
\end{aligned}
$$

Hence the theorem. ■

The following lemma is a direct consequence of definition of resolving regularity of a graph.

**Lemma 3.8.** *If* $G$ *is a graph with* $n, m$ *and* $r$ *as its order, size and resolving regularity respectively then,* ${}_{\beta}M_1(G) = nr^2$, ${}_{\beta}M_1^*(G) = 2mr$ *and* ${}_{\beta}M_2(G) = mr^2$. *In particular, if* $r = 2$ *and* $G$ *is uni-cyclic, then* ${}_{\beta}M_1(G) = 4n$, ${}_{\beta}M_1^*(G) = {}_{\beta}M_2(G) = 4m$.

**Corollary 3.9.** For a cycle graph $C_n$ ($n \ge 3$),

$$
{}_{\beta}M_1(C_n) = {}_{\beta}M_1^*(C_n) = {}_{\beta}M_2(C_n) = 4n.
$$

**Proof.** The proof follows from Theorem 2.7 and Lemma 3.8. ■

**Corollary 3.10.** For a complete graph $K_n$, ${}_{\beta}M_1(K_n) = {}_{\beta}M_1^*(K_n) = n(n-1)^2$ and ${}_{\beta}M_2(K_n) = \frac{n}{2}(n-1)^3$ for $n \geq 2$.

**Proof.** Follows by Theorem 2.9 and Lemma 3.8. ■

**Remark 3.11.** For any non-trivial graph $G$,

1. $3 \notin \{ {}_{\beta}M_1(G),\ {}_{\beta}M_1^*(G),\ {}_{\beta}M_2(G)\}$,
2. $4 \notin \{ {}_{\beta}M_1(G),\ {}_{\beta}M_1^*(G)\}$, and
3. ${}_{\beta}M_2(G) = 4$ if and only if $G \cong P_3$.

**Theorem 3.12.** For the star graph $K_{1,n}$, ${}_{\beta}M_1(K_{1,n}) = n(n^2-n+1)$, ${}_{\beta}M_1^*(K_{1,n}) = n(2n-1)$ and ${}_{\beta}M_2(K_{1,n}) = n^2(n-1)$ for $n \geq 2$.

**Proof.** Let $u$ be the central vertex and $v_i$ be the pendent vertices of $K_{1,n}$. As each of the pendent vertices $v_i$ is at distance 2 from remaining all other pendent vertices, $S_{v_i}$ contains $n-1$ pendent vertices including $v_i$ and $S_u = S_{v_i} \cup \{u\}$. Thus, $d_\beta(v_i) = n-1$ and $d_\beta(u) = n$. Hence, ${}_{\beta}M_1(K_{1,n}) = n(n-1)^2 + n^2 = n(n^2-n+1)$, ${}_{\beta}M_1^*(K_{,n}) = n((n-1)+n) = n(2n-1)$ and ${}_{\beta}M_2(K_{1,n}) = n((n-1)\times n) = n^2(n-1)$. Hence the theorem. ■

**Theorem 3.13.** [9] *If $S$ is a metric basis and $v_0$ be the central vertex of the wheel $W_{1,n}$, then $v_0 \notin S$ for every $n \geq 4$.*

**Theorem 3.14.** [9] *For a wheel $W_{1,n}$, $n \geq 3$,*

$$\beta(G) = \begin{cases} \lceil (2n-2)/5 \rceil & if\ n \neq 3,6 \\ 3 & if\ n = 3,6 \end{cases}.$$

**Theorem 3.15.** *For a wheel $W_{1,n}$, $n \geq 3$,*

(i). $${}_{\beta}M_1(W_{1,n}) = \begin{cases} \left(\left\lceil \frac{2n-2}{5} \right\rceil + 1\right)^2 + n\left(\left\lceil \frac{2n-2}{5} \right\rceil\right)^2 & if\ n \neq 3,6 \\ 36 & if\ n = 3 \\ 70 & if\ n = 6 \end{cases},$$

(ii). $${}_{\beta}M_1^*(W_{1,n}) = \begin{cases} \left(4\left\lceil \frac{2n-2}{5} \right\rceil + 1\right) n & if\ n \neq 3,6 \\ 36 & if\ n = 3 \\ 78 & if\ n = 6 \end{cases},$$

(iii). $${}_{\beta}M_2(W_{1,n}) = \begin{cases} n\left\lceil \frac{2n-2}{5} \right\rceil \left(2\left\lceil \frac{2n-2}{5} \right\rceil + 1\right) & if\ n \neq 3,6 \\ 54 & if\ n = 3 \\ 126 & if\ n = 6 \end{cases}.$$

**Proof.** From Theorem 3.13, Theorem 3.14, and Lemma 2.11, it follows (by symmetry) that for every rim vertex $v_i$ $(1 \le i \le n)$ of the wheel graph

$$d_\beta(v_i) = \begin{cases} \lceil (2n-2)/5 \rceil & \text{if } n \ne 3,6 \\ 3 & \text{if } n = 3,6 \end{cases}.$$

Also, for the central vertex $v_0$,

$$d_\beta(v_0) = \begin{cases} \lceil (2n-2)/5 \rceil + 1 & \text{if } n \ne 3,6 \\ 3 & \text{if } n = 3 \\ 4 & \text{if } n = 6 \end{cases}.$$

Therefore, substituting these in Equation (2), gives

$$_\beta M_1(W_{1,n}) = \sum_{a\in V} d_\beta(a)^2 = [d_\beta(v_0)]^2 + \sum_{i=1}^{n} [d_\beta(v_i)]^2$$

$$= \begin{cases} \left(\left\lceil \frac{2n-2}{5} \right\rceil + 1\right)^2 + n\left(\left\lceil \frac{2n-2}{5} \right\rceil\right)^2 & \text{if } n \ne 3,6 \\ 9(n+1) & \text{if } n = 3 \\ 9n+16 & \text{if } n = 6 \end{cases}.$$

Substituting the above resolving degrees in (3) and (4), gives

$$_\beta M_1^*(W_{1,n}) = \sum_{ab\in E} \left[d_\beta(a) + d_\beta(b)\right]$$

$$= \sum_{i=1}^{n} \left[d_\beta(v_0) + d_\beta(v_i)\right]$$

$$+ \sum_{i=1}^{n-1} \left[d_\beta(v_i) + d_\beta(v_{i+1})\right] + \left[d_\beta(v_n) + d_\beta(v_1)\right]$$

$$= \begin{cases} n\left(\left\lceil \frac{2n-2}{5} \right\rceil + 1 + \left\lceil \frac{2n-2}{5} \right\rceil\right) + 2n\left\lceil \frac{2n-2}{5} \right\rceil & \text{if } n \ne 3,6 \\ 6n + 6n & \text{if } n = 3 \\ 7n + 6n & \text{if } n = 6 \end{cases}$$

$$= \begin{cases} \left(4\left\lceil \frac{2n-2}{5} \right\rceil + 1\right) n & \text{if } n \ne 3,6 \\ 12n & \text{if } n = 3 \\ 13n & \text{if } n = 6 \end{cases}.$$

$$_\beta M_2(W_{1,n}) = \sum_{ab\in E} \left[d_\beta(a) \cdot d_\beta(b)\right]$$

$$= \sum_{i=1}^{n} \left[d_\beta(v_0) \times d_\beta(v_i)\right]$$

$$+ \sum_{i=1}^{n-1} \left[d_\beta(v_i) \times d_\beta(v_{i+1})\right] + \left[d_\beta(v_n) \times d_\beta(v_1)\right]$$

$$= \begin{cases} n\left(\left(\left\lceil \frac{2n-2}{5} \right\rceil + 1\right) \cdot \left\lceil \frac{2n-2}{5} \right\rceil\right) + n\left(\left\lceil \frac{2n-2}{5} \right\rceil\right)^2 & \text{if } n \ne 3,6 \\ 9n + 9n & \text{if } n = 3 \\ 12n + 9n & \text{if } n = 6 \end{cases}$$

$$= \begin{cases} n\left\lceil \frac{2n-2}{5} \right\rceil \left(2\left\lceil \frac{2n-2}{5} \right\rceil + 1\right) & \text{if } n \ne 3,6 \\ 18n & \text{if } n = 3 \\ 21n & \text{if } n = 6 \end{cases}.$$

Hence the theorem. ■

The *corona product* of two graph $G_1$ and $G_2$, denoted by $G_1 \odot G_2$ is defined as the graph obtained from $G_1$ and $G_2$ by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ and joining by an edge each vertex from the $i^{th}$-copy of $G_2$ with the $i^{th}$-vertex of $G_1$.

**Theorem 3.16.** *For a comb graph $P_n \odot K_1$ $(n \geq 1)$,*

(i). $${}_{\beta}M_1(P_n \odot K_1) = \begin{cases} 2(2n-3), & if\ n = 1,2 \\ 29, & if\ n = 3 \\ 6(3n-5), & if\ n \geq 4 \end{cases},$$

(ii). $${}_{\beta}M_1^*(P_n \odot K_1) = \begin{cases} 2(2n-3), & if\ n = 1,2 \\ 23, & if\ n = 3 \\ 2(6n-7), & if\ n \geq 4 \end{cases},$$

(iii). $${}_{\beta}M_2(P_n \odot K_1) = \begin{cases} 4(n-2), & if\ n = 1,2 \\ 26, & if\ n = 3 \\ 18n-31, & if\ n \geq 4 \end{cases}.$$

**Proof.** Let $G = P_n \odot K_1$ with $V(G) = \{v_i, u_i : 1 \leq i \leq n\}$ and $(G) = \{v_i v_{i+1}, v_i u_i, v_n u_n : 1 \leq i \leq n-1\}$. When $n = 1$, 2, $G$ is a path and hence, the result follows by Theorem 3.7. For $n \geq 3$, it is easy to see that the set $\{u_i, v_n\}$ for any $1 \leq i \leq 2$ is a metric basis. Hence, by symmetry $d_\beta(v) = \beta(G) = 2$, for all the vertices $v \in \{v_1, v_n, u_1, u_2, u_{n-1}, u_n\}$. Further, as $G$ has no odd cycles, for each of $v_i$, $2 \leq i \leq n-1$, $d_\beta(v_i) = 3$ (by Theorem 2.7 and Lemma 2.11). Finally, for each $u_i$, $3 \leq i \leq n-3$, $d(u_i, v_{i+2}) = d(u_i, u_{i+1}) = 3$, and (similarly $d(u_i, v_{i-2}) = d(u_i, u_{i-1}) = 3$) $d(x, v_{i+2}) = d(x, u_{i+1})$ for every $x \in \{v_i, v_j, u_j : 1 \leq j \leq i-1\}$ (similarly $d(x, v_{i-2}) = d(x, u_{i-1})$ for every $x \in \{v_i, v_j, u_j : i+1 \leq j \leq n\}$). Hence, $d_\beta(u_i) > 2$ implies that $d_\beta(u_i) = 3$, for every $3 \leq i \leq n-3$. Thus, for the graph $G$, the parameters

$$\xi = \begin{cases} 5, & if\ n = 3 \\ 6, & if\ n \geq 4 \end{cases}, \eta_1 = \begin{cases} 3, & if\ n = 3 \\ 4, & if\ n \geq 4 \end{cases} \text{ and } \eta_2 = \begin{cases} 0, & if\ n = 3 \\ 2n-7, & if\ n \geq 4 \end{cases}.$$

Substituting these in Equations (5), (6) and (7), gives

$$\begin{aligned} {}_{\beta}M_1(G) &= \xi\beta(G)^2 + (|V(G)| - \xi)(\beta(G)+1)^2 \\ &= \begin{cases} 5(2)^2 + (6-5)(2+1)^2, & if\ n = 3 \\ 6(2)^2 + (2n-6)(2+1)^2, & if\ n \geq 4 \end{cases} \\ &= \begin{cases} 29, & if\ n = 3. \\ 6(3n-5), & if\ n \geq 4. \end{cases} \end{aligned}$$

$$\begin{aligned} {}_{\beta}M_1^*(G) &= 2|E(G)|\beta(G) + (\eta_1 + 2\eta_2) \\ &= \begin{cases} 2(2n-1)2 + (3 + 2(0)), & if\ n = 3 \\ 2(2n-1)2 + (4 + 2(2n-7)), & if\ n \geq 4 \end{cases} \\ &= \begin{cases} 23, & if\ n = 3 \\ 2(6n-7), & if\ n \geq 4 \end{cases}. \end{aligned}$$

$$\begin{aligned} {}_{\beta}M_2(G) &= |E(G)|\big(\beta(G)\big)^2 + (\eta_1 + 2\eta_2)\beta(G) + \eta_2 \\ &= \begin{cases} (2n-1)(2)^2 + \big(3 + 2(0)\big)(2) + 0, & if\ n = 3 \\ (2n-1)(2)^2 + \big(4 + 2(2n-7)\big)(2) + 2n - 7, & if\ n \geq 4 \end{cases} \\ &= \begin{cases} 26, & if\ n = 3 \\ 18n-31, & if\ n \geq 4 \end{cases}. \end{aligned}$$

Hence the theorem. ∎

**Theorem 3.17.** *For the sunlet graph* $S_n = C_n \odot K_1$ $(n \geq 3)$,

(i). $_{\beta}M_1(S_n) = \begin{cases} 8n, & if\ n\ is\ odd \\ 18n, & if\ n\ is\ even \end{cases}$,

(ii). $_{\beta}M_1^*(S_n) = \begin{cases} 8n, & if\ n\ is\ odd \\ 12n, & if\ n\ is\ even \end{cases}$,

(iii). $_{\beta}M_2(S_n) = \begin{cases} 8n, & if\ n\ is\ odd \\ 18n, & if\ n\ is\ even \end{cases}$.

**Proof.** Since $S_n$ is a 3-regular uni-cyclic graph, by Lemma 2.7 and its proof, it is clear that $d_\beta(v) \geq 3$ for every vertex of degree 3 ($i.e$ for the vertices of $C_n$ in $S_n$) whenever *n* is even. Further, for each pendent $u$ vertex the set $\{u, w\}$ is a resolving set only if $\{u', w\}$ is a resolving set where $u'$ is the support of $u$. So, as $d_\beta(u') \geq 3$ (support is in $C_n$), get $d_\beta(v) = 3$, for every vertex $v$ of $S_n$, whenever $n$ is even. To prove the reverse inequality, let $V(S_n) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(S_n) = \{v_i v_{i+1(mod\ n)}, v_i u_i : 0 \leq i \leq n-1\}$. Consider the sets $S_{v_i} = \left\{v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}-1}\right\}$ and $S_{u_i} = \left\{u_i, u_{i+\frac{n}{2}}, u_{i+\frac{n}{2}-1}\right\}$. The sets $S_{v_i}$ and $S_{u_i}$ are resolving sets of $S_n$. Hence, by symmetry, we conclude that $d_\beta(v) = 3$, for all each vertex $v$ of the graph $S_n$ whenever $n$ is even.

When *n* is odd, the sets $S_{v_i} = \{v_i, v_{i+\lfloor n/2 \rfloor}\}$ and $S_{u_i} = \{u_i, u_{i+\lfloor n/2 \rfloor}\}$ are clearly metric basis. Hence, by symmetry, we conclude that $d_\beta(v) = 2$, for all each vertex *v* of the graph $S_n$ whenever *n* is odd.

Substituting these in Equations (2), (3) and (4) gives, $_{\beta}M_1(S_n) = {}_{\beta}M_1^*(S_n) = {}_{\beta}M_2(S_n) = mr^2 = 2n(2)^2 = 8n$ for all odd *n*, $_{\beta}M_1(S_n) = (2n)3^2 = 18n$, $_{\beta}M_1^*(S_n) = 2mr = 2(2n)3 = 12n$, and $_{\beta}M_2(S_n) = mr^2 = 2n(3)^2 = 18n$ for all odd *n*, proving the theorem. ■

**Theorem 3.18.** *For any connected graph* $G$ *of order* n *and size* m, *and any integer* $p \geq 2$,

(i). $_{\beta}M_1(G \odot pK_1) = \begin{cases} p^5 - 2(n-1)p^3 + (n-1)^2 p, & if\ n = 1 \\ n^3p^3 - n^3p^2 - n^2(n-1)p + n(n-1)^2, & if\ n \geq 2 \end{cases}$,

(ii). $_{\beta}M_1^*(G \odot pK_1) = \begin{cases} 2p^2 - p, & if\ n = 1 \\ 2n^2p^2 + (2n(m-n)+1)p - 2(mn-1), & if\ n \geq 2 \end{cases}$,

(iii). $_{\beta}M_2(G \odot pK_1) = \begin{cases} p^3 - p^2, & if\ n = 1 \\ n^2p^3 + (m-n)np^2 - m(m-1), & if\ n \geq 2 \end{cases}$.

**Proof.** Let $G_1 = G \odot pK_1$. When $n = 1$, the result follows by Theorem 3.12. Let $n \geq 2$. The graph $G_1$ contains $p$ pendent vertices attached to each vertex of $G$ and hence to resolve $G$, each metric basis shall contains exactly $p-1$ pendent vertices out of $p$ pendent vertices at every vertex $v$ of $G$ (since these are the only vertices will resolve the vertex $v$ as well as all the pendent vertices at $v$). Thus, $\beta(G_1) = (p-1)|V(G)| = n(p-1)$, also (in $G_1$) for each pendent vertex there is a metric basis containing it and no metric basis contains any vertex of $G$. Hence, $d_\beta(v) = \beta(G_1) + 1$ for every non-pendent vertex $v$ of $G_1$ and $d_\beta(u) = \beta(G_1)$ for every pendent vertex $u$ of $G_1$. Thus, for the graph $G$, the parameters $\xi = \eta_1 = p|V(G)| = np$, and $\eta_2 = |E(G)| = m$. Substituting these in Equations (5), (6) and (7), gives

$$\begin{aligned}{}_\beta M_1(G_1) &= np\big(n(p-1)\big)^2 + (n(p+1) - np)(n(p-1)+1)^2\\ &= n^3p^3 - n^3p^2 - n^2(n-1)p + n(n-1)^2.\\ {}_\beta M_1^*(G_1) &= 2(m+np)\big(n(p-1)\big) + (np+2m)\\ &= 2n^2p^2 + (2n(m-n)+1)p - 2(mn-1).\\ {}_\beta M_2(G_1) &= (m+np)\big(n(p-1)\big)^2 + (np+2m)n(p-1) + m\\ &= n^2p^3 + (m-n)np^2 - m(m-1).\end{aligned}$$

Hence the theorem. ■

## 4. SPECIAL CLASSES OF TREES

In this section, the newly defined indices are studied for certain classes of trees. Consider a tree $T$ which is not a path. Let $u_1, u_2, \ldots, u_k$ $(k \geq 3)$ be the end vertices of $T$. Let $P_i$ denotes the maximal induced path of $T$ containing the vertex $u_i$ and exactly one vertex of degree more than 2 in $T$, for $1 \leq i \leq k$. Let $\tau_i$ denote the size of the maximal induced path $P_i$. For, each vertex $v \in V(T)$, let $\gamma(v) = |\{P_i : v \in V(P_i)\}|$. Then, clearly $\beta(G) = \sum_{\gamma(v)>0}(\gamma(v) - 1)$. Further, for every $1 \leq i \leq k$, each of the $\tau_i$ vertices of $P_i$ with $\gamma(v) = 1$ lies in a metric basis containing it if and only if $\gamma(u) \geq 2$ for some $u \in V(P_i)$.

### 4.1 SUBDIVIDED TREE

The subdivision of a graph $G$, denoted by $S(G)$, is a graph obtained from $G$ by inserting a vertex on each edge of $G$.

**Theorem 4.1**. *For any tree $T$ on n vertices which is not a path,*

(i). ${}_\beta M_1(S(T)) = 2{}_\beta M_1(T) - (\beta(T)+1)^2$,

(ii). ${}_\beta M_1^*(S(T)) = {}_\beta M_1^*(T) + 2(n-1)\beta(T) + 2\eta_2$,

(iii). ${}_\beta M_2(S(T)) = 2{}_\beta M_2(T) - \eta_1\beta(T)$,

*where* $\xi = |\{v \in V(T) : d_\beta(v) = \beta(T)\}|$, $\eta_1 = \big|\{uv \in E(T) : |d_\beta(u) - d_\beta(v)| = 1\}\big|$ *and* $\eta_2 = \big|\{uv \in E(T) : d_\beta(u) = d_\beta(v) = \beta(T)+1\}\big|$.

**Proof.** Let $\xi$, $\eta_1, \eta_2$ be the value of the parameters the given tree T on $n$ vertices. Let $\xi'$, $\eta_1', \eta_2'$ be the same parameters corresponding to the graph $S(T)$. Then, by the definition of $S(T)$, it is easy to see that $\xi' = 2\xi$, $n' = \big|V\big(S(T)\big)\big| = 2n-1$, $\eta_1' = \eta_1$, $\eta_2' = 2\eta_2$, $\big|E\big(S(T)\big)\big| = 2(n-1)$, $\beta\big(S(T)\big) = \beta(T)$. Now, substituting these in (5), (6), and (7), we get

$$
\begin{aligned}
{}_{\beta}M_1(S(T)) &= \xi'\beta(S(T))^2 + \left(|V(S(T))| - \xi'\right)(\beta(S(T)) + 1)^2 \\
&= 2\xi\beta(T)^2 + (2n-1-2\xi)(\beta(T)+1)^2 \\
&= 2(\xi\beta(T)^2 + (n-\xi)(\beta(T)+1)^2) - (\beta(T)+1)^2 \\
&= 2{}_{\beta}M_1(T) - (\beta(T)+1)^2. \\
{}_{\beta}M_1^*(S(T)) &= 2|E(S(T))|\beta(S(T)) + (\eta_1' + 2\eta_2') \\
&= 2 \times 2(n-1)\beta(T) + (\eta_1 + 2(2\eta_2)) \\
&= 4(n-1)\beta(T) + (\eta_1 + 4\eta_2) \\
&= {}_{\beta}M_1^*(T) + 2(n-1)\beta(T) + 2\eta_2. \\
{}_{\beta}M_2(S(T)) &= |E(S(T))|\left(\beta(S(T))\right)^2 + (\eta_1' + 2\eta_2')\beta(S(T)) + \eta_2' \\
&= (2n-2)\beta(T)^2 + (\eta_1 + 4\eta_2)\beta(T) + 2\eta_2 \\
&= 2[(n-1)\beta(T)^2 + (\eta_1 + 2\eta_2)\beta(T) + \eta_2] - \eta_1\beta(T) \\
&= 2{}_{\beta}M_2(T) - \eta_1\beta(T).
\end{aligned}
$$

Hence the theorem. ■

## 4.2 Broom Graph

A *Broom Graph* $B_{n,r}$ is a graph of $n$ vertices, which have a path $P$ with $r$ vertices and $(n-r)$ pendant vertices, all of these being adjacent to one of the end vertex of $P$.

**Theorem 4.2.** *For a Broom Graph* $B_{n,r}$, $n \geq r \geq 1$.

$$
{}_{\beta}M_1(B_{n,r}) = \begin{cases} (n-1)(n^2-3n+3), & if\ r = 1,2 \\ 2(2n-3), & if\ n-r = 0,1\ and\ r \geq 3, \\ (n-r)(n^2-nr+2)+1, & if\ n-r \geq 2,\ and\ r \geq 3 \end{cases}
$$

$$
{}_{\beta}M_1^*(B_{n,r}) = \begin{cases} (n-1)(2n-3), & if\ r = 1,2 \\ 2(2n-3), & if\ n-r = 0,1\ and\ r \geq 3, \\ (2n-1)(n-r)+1, & if\ n-r \geq 2,\ and\ r \geq 3 \end{cases}
$$

$$
{}_{\beta}M_2(B_{n,r}) = \begin{cases} (n-1)^2(n-2), & if\ r = 1,2 \\ 4(n-2), & if\ n-r = 0,1\ and\ r \geq 3. \\ (n-r)(n^2-nr+1), & if\ n-r \geq 2,\ and\ r \geq 3 \end{cases}
$$

**Proof.** Let $G = B_{n,r}$ with $V(G) = \{v_i, : 1 \leq i \leq n\}$ and $(G) = \{v_i v_{i+1} : 1 \leq i \leq r-1\} \cup \{v_r v_j : 1 \leq j \leq n-r\}$. When $r = 1$, 2 the graph $G \cong K_{1,n-1}$, so the result follows by Theorem 3.12. If $n = r, r+1$ , then $G \cong P_n$, so the result follows by Theorem 3.8. For $n \geq r+2 \geq 4$, it is easy to see that $u_1 = v_1$, $u_2 = v_{r+1}$, $u_3 = v_{r+2}$,..., $u_{n-r+1} = v_{n-r}$ are the end vertices of $G$, so $k = n-r+1, \tau_1 = r-1, \tau_i = 1$, for all $2 \leq i \leq k$. Further, the vertex $v_r$ lies in every $P_i$ path and $r(v_r) = k > 1$. Hence, every vertex in $P_i$, except $v_r$ lies in a metric basis for every $1 \leq i \leq k$. Therefore, $\xi = n-1$, $\eta_1 = k = n-r+1$, $\eta_2 = 0$, $\beta(G) = \sum_{\gamma(v)>0}(\gamma(v)-1) = \gamma(v_r) - 1 = k-1 = n-r$. Hence, Equations (5), (6) and (7), gives

$$\begin{aligned} {}_{\beta}M_1(G) &= (n-1)(n-r)^2 + \big(n-(n-1)\big)(n-r+1)^2 \\ &= (n^2-nr+2)(n-r)+1. \\ {}_{\beta}M_1^*(G) &= 2(n-1)(n-r) + \big(n-r+1+2(0)\big) \\ &= (2n-1)(n-r)+1. \\ {}_{\beta}M_2(G) &= (n-1)(n-r)^2 + (n-r+1+0)(n-r)+0 \\ &= (n^2-nr+1)(n-r). \end{aligned}$$

Hence the theorem. ■

### 4.3. DOUBLE STAR GRAPH

A *double star* $DS_{r,s}$ is a graph obtained by adding an edge between a central vertex $v_0$ and a central vertex $u_0$ of the star graphs $K_{1,r}$ and $K_{1,s}$, respectively.

**Theorem 4.3.** *For the given $r, s \in \mathbb{Z}^+$ with $r \le s$, and the double star $DS_{r,s}$,*

(i). $${}_{\beta}M_1\big(DS_{r,s}\big) = \begin{cases} 10, & if\ r=s=1 \\ 25, & if\ r=1, s=2 \\ (r+s)^3-2(r+s)^2+2, & if\ r,s\ge 2 \end{cases},$$

(ii). $${}_{\beta}M_1^*\big(DS_{r,s}\big) = \begin{cases} 10, & if\ r=s=1 \\ 19, & if\ r=1, s=2 \\ 2(r+s)^2-(r+s)-2, & if\ r,s\ge 2 \end{cases},$$

(iii). $${}_{\beta}M_2\big(DS_{r,s}\big) = \begin{cases} 8, & if\ r=s=1 \\ 22, & if\ r=1, s=2 \\ (r+s)^3-2(r+s)^2+1, & if\ r,s\ge 2 \end{cases}.$$

**Proof.** Let $G = DS_{r,s}$. When $s=1$, the graph $G$ is isomorphic to $P_4$ and hence the result follows by Theorem 3.8. When $r=1$ and $s=2$, it is easy to see that for every vertex $v$ of degree at most 2, the set $S=\{u,v\}$ is a metric basis, where $u$ is a pendent vertex not adjacent to $v$. Also, for the vertex of degree 3, at least two of its neighbours are equidistant from every other vertex. Thus, $d_\beta(v)=2$ for the vertices of degree at most 2 and $d_\beta(v)=3$ for the vertex of degree 3. Hence,

$${}_{\beta}M_1(G) = \textstyle\sum_{a\in V} d_\beta(a)^2 = 4\times 2^2 + 1\times 3^2 = 25,$$

$${}_{\beta}M_1^*(G) = \textstyle\sum_{ab\in E}\big[d_\beta(a)+d_\beta(b)\big] = 1\times[2+2]+3\times[2+3] = 19,$$

$${}_{\beta}M_2(G) = \textstyle\sum_{ab\in E}\big[d_\beta(a)\cdot d_\beta(b)\big] = 1\times[2\times 2]+3\times[2\times 3] = 22.$$

Let us now consider the case $r,s\ge 2$. Let $v_1, v_2, \dots, v_r$ be the end vertices of $K_{1,r}$ and $u_1, u_2, \dots, u_s$ be the end vertices of $K_{1,s}$. Then these are the only end vertices of $G$, so $k = r+s$. Also, $\tau_i = 1$, for all $1\le i\le r$, $v_0$ is common to all $P_i's$ for $i=1,2,\dots,r$, and $u_0$ is common to all $P_j's$ for $j=1,2,\dots,s$. Therefore, $v$ lies in a metric basis for every $v\in V(G)-\{v_0,u_0\}$; $\gamma(v_0)=r$ and $\gamma(u_0)=s$. So, $\xi = |V(G)|-2 = r+s$, $\eta_1 = k = r+s$, $\eta_2 = 1$

(the edge $v_0u_0$), $\beta(G) = \sum_{\gamma(v)>0}(\gamma(v) - 1) = \gamma(v_0) - 1 + \gamma(u_0) - 1 = (r - 1) + (s - 1) = r + s - 2$. Hence, Equations (5), (6) and (7), gives

$$\begin{aligned} {}_{\beta}M_1(G) &= (r+s)(r+s-2)^2 + ((r+s+2)-(r+s))(r+s-2+1)^2 \\ &= (r+s)(r+s-2)^2 + 2(r+s-2+1)^2 \\ &= (r+s)^3 - 2(r+s)^2 + 2, \\ {}_{\beta}M_1^*(G) &= 2(r+s+1)(r+s-2) + (r+s+2(2)) \\ &= 2(r+s)^2 - (r+s) - 2, \\ {}_{\beta}M_2(G) &= (r+s+1)(r+s-2)^2 + \big(r+s+2(2)\big)(r+s-2) + 2 \\ &= (r+s)^3 - 2(r+s)^2 + 1. \end{aligned}$$

Hence the theorem. ∎

### 4.4. Generalized Broom Graph

A *generalized broom graph,* denoted by $B_{n,r}(v)$, is a graph on $n$ vertices obtained by attaching $n - r \geq 2$ pendant vertices to an internal vertex $v$ of the path $P_r$.

**Note.** If $v$ is an end vertex of $P_r$, then $B_{n,r}(v) = B_{n,r}$. Therefore, we consider a non-pendent vertex $v$ in the following theorem.

**Theorem 4.4.** *For the given $n, r \in \mathbb{Z}^+$ and the generalized broom graph $B_{n,r}(v)$,*

(i). ${}_{\beta}M_1\big(B_{n,r}(v)\big) = n(n-r+1)^2 + 2(n-r+1) + 1$,

(ii). ${}_{\beta}M_1^*\big(B_{n,r}(v)\big) = (2n-1)(n-r+1) + 1$,

(iii). ${}_{\beta}M_2\big(B_{n,r}(v)\big) = n(n-r+1)^2 + (n-r+1)$.

**Proof.** Let $v_1, v_2, \dots, v_r$ be the vertices of the path $P_r$ in order. Let $= B_{n,r}(v_i)$, $2 \leq i \leq r - 1$. Then, $G$ has $n - r + 2$ pendent vertices and only one vertex namely $v_i$ is of degree more than 2 which is adjacent to two vertices of pat. Thus, similar to the broom graph, $\beta(G) = n - r + 1$, $\xi = (n - r) + (r - 1) = n - 1$, $\eta_1 = n - r + 2$, $\eta_2 = 0$. Hence, Equations (5), (6) and (7), gives

$$\begin{aligned} {}_{\beta}M_1(G) &= (n-1)(n-r+1)^2 + \big(n-(n-1)\big)(n-r+2)^2 \\ &= n(n-r+1)^2 + 2(n-r+1) + 1, \\ {}_{\beta}M_1^*(G) &= 2(n-1)(n-r+1) + \big(n-r+2+2(0)\big) \\ &= (2n-1)(n-r+1) + 1, \\ {}_{\beta}M_2(G) &= (n-1)(n-r+1)^2 + (n-r+2+2(0))(n-r+1) + 0 \\ &= n(n-r+1)^2 + (n-r+1). \end{aligned}$$

Hence the theorem. ∎

### 4.5 Double Broom Graph

A *double broom graph* denoted by $B_{n,r_1,r_2}$ is a graph on $n + r_1 + r_2$ vertices obtained from $P_n$ by attaching $r_1$ pendent vertices to one of its end vertices and attaching $r_2$ pendent vertices to another end vertex.

**Theorem 4.5.** *For the given* $n, r_1, r_2, \in \mathbb{Z}^+$ *with* $r_1, r_2 \geq 2$,

(i). ${}_{\beta}M_1(B_{n,r_1,r_2}) = (r_1 + r_2)^3 + (n-1)(r_1 + r_2)^2 + 2(2-n)(r_1 + r_2) + n,$

(ii). ${}_{\beta}M_1^*(B_{n,r_1,r_2}) = 2[(r_1 + r_2)^2 + (n-1)(r_1 + r_2) + 1],$

(iii). ${}_{\beta}M_2(B_{n,r_1,r_2}) = 2(r_1 + r_2)^3 + (2n-9)(r_1 + r_2)^2 - 6(n-2)(r_1 + r_2) + 5(n-1).$

**Proof.** Let $G = B_{n,r_1,r_2}$. Since $r_1, r_2 \geq 2$, each pendent vertex of $G$ are in some metric basis. Also, no vertex of $P_n$ is in any metric basis (since deletion of such vertex from a resolving set still resolve the vertices of $G$). Thus, similar to the broom graph, $\beta(G) = r_1 + r_2 - 2$, $\xi = r_1 + r_2$, $\eta_1 = r_1 + r_2$, $\eta_2 = n - 1$ (for the edges of $P_n$). Hence, Equations (5), (6) and (7), gives

$$\begin{aligned} {}_{\beta}M_1(G) &= (r_1 + r_2)(r_1 + r_2 - 2)^2 + \big(n + r_1 + r_2 - (r_1 + r_2)\big)(r_1 + r_2 - 1)^2 \\ &= (r_1 + r_2)(r_1 + r_2 - 2)^2 + (n)(r_1 + r_2 - 1)^2, \\ {}_{\beta}M_1^*(G) &= 2(n + r_1 + r_2 - 2)(r_1 + r_2 - 1) + \big(r_1 + r_2 + 2(n-1)\big) \\ &= 2[(r_1 + r_2)^2 + (n-1)(r_1 + r_2) + 1], \\ {}_{\beta}M_2(G) &= 2(n + r_1 + r_2 - 1)(r_1 + r_2 - 2)^2 + (r_1 + r_2 + 2n - 2)(r_1 + r_2 - 2) + n - 1 \\ &= 2(r_1 + r_2)^3 + (2n-9)(r_1 + r_2)^2 - 6(n-2)(r_1 + r_2) + 5(n-1). \end{aligned}$$

Hence the theorem. ■

### 4.6. THORN STAR GRAPH

A *Thorn Star*, denoted $S_{k,t}$ is a graph obtained from a star $K_{1,k}$ by attaching $t - 1$ terminal vertices to each of the star arms.

**Note.** For the case $k = 1$, the graph $S_{k,t}$ is isomorphic to double star. Hence, we consider the case $k \geq 2$.

**Theorem 4.6.** *For the given* $k.t, \in \mathbb{Z}^+$ *with* $k, t \geq 2$,

(i). $${}_{\beta}M_1(S_{k,t}) = \begin{cases} 2k^3 - 3k^2 + 2k, & if\ t = 2 \\ k(t-2)[(t-2)tk^2 + tk + 2] + k + 1, & if\ t \geq 3 \end{cases},$$

(ii). $${}_{\beta}M_1^*(S_{k,t}) = \begin{cases} k(4k+1), & if\ t = 2 \\ 2t(t-2)k^2 + (t+1)k, & if\ t \geq 3 \end{cases},$$

(iii). $${}_{\beta}M_2(S_{k,t}) = \begin{cases} k(k-1)(2k+1), & if\ t = 1 \\ k[2t(t-2)^2k^2 + (t+1)(t-2)k + 1], & if\ t \geq 3 \end{cases}.$$

**Proof.** Let $G = S_{k,t}$. When $t = 2$, the graph is subdivision of $K_{1,k}$ and hence the result follows by Theorem 4.1 and Theorem 3.12.

$$
\begin{aligned}
{}_{\beta}M_1(S_{k,t}) &= 2_{\beta}M_1(K_{1,k}) - \left(\beta(K_{1,k}) + 1\right)^2 \\
&= 2k(k^2 - k + 1) - (k - 1 + 1)^2 \\
&= 2k^3 - 3k^2 + 2k. \\
{}_{\beta}M_1^*(S_{k,t}) &= {}_{\beta}M_1^*(K_{1,k}) + 2(n-1)\beta(K_{1,k}) + 2\eta_2 \\
&= k(2k-1) + 2(k+1-1)(k-1) + 2(0) = 4k^2 + k. \\
{}_{\beta}M_2(S_{k,t}) &= 2_{\beta}M_2(K_{1,k}) - \eta_1\beta(K_{1,k}) \\
&= 2k^2(k-1) + k(k-1) = k(k-1)(2k+1).
\end{aligned}
$$

For $t \geq 3$, the graph $G$ has $k(t-1)$ end vertices and only for these end vertices $d_{\beta}(v) = \beta(G) = k(t-2)$. Thus, $\xi = k(t-1)$. Further, for only $t-1$ pendent edges attached at each of the $k$ end vertices of $K_{1,k}$, the difference in their resolving degree is 1 and hence $\eta_1 = k(t-1)$. Finally, for the $k$ edges incident with the central of $K_{1,k}$ in $G$ degree of both of its end vertices is $\beta(G) + 1$ and hence $\eta_2 = k$. Hence, Equations (5), (6) and (7), gives

$$
\begin{aligned}
{}_{\beta}M_1(G) &= k(t-1)[k(t-2)]^2 + [(kt+1) - k(t-1)][(k(t-2)+1]^2 \\
&= k(t-2)[(t-2)tk^2 + tk + 2] + k + 1, \\
{}_{\beta}M_1^*(G) &= 2(kt)k(t-2) + k(t-1) + 2(k) \\
&= 2t(t-2)k^2 + (t+1)k\ , \\
{}_{\beta}M_2(G) &= 2(kt)k^2(t-2)^2 + [k(t-1) + 2k]k(t-2) + k \\
&= \ k[2t(t-2)^2k^2 + (t+1)(t-2)k + 1].
\end{aligned}
$$

Hence the theorem. ■

## 5. Silicate Structure

In this section, the newly defined indices are applied for the study of different silicate structures. Silicates are abundantly available minerals which are very essential in a vast array of industries, the main ones being the glass, foundries, construction, ceramics, and the chemical industry. For rock forming minerals, silicates serve as the building blocks. All silicates contain $SiO_4$ tetrahedron structure, which is shown in Figure 2, in which red vertices represent oxygen ions and grey one is the silicon ion. Molecular graph of $SiO_4$ is as shown in Figure 3, where oxygen atom represents the vertices of the graph. When *n* tetrahedrons augmented linearly with other, then a single-row silicate chain is obtained and is denoted as *SL*(*n*). Figure 5 shows the molecular graph *SL*(5) which contains 5 silicates tetrahedrons.


**Figure 2.** $SiO_4$ molecule and its 2-dimensional representations.

### 5.1 SILICATE MOLECULAR GRAPH

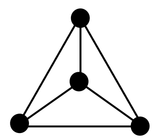


**Figure 3.** Molecular graph of $SiO_4$.

Silicate molecular graph is a complete graph $K_4$ which is a 3-*resolving regular graph*. Hence $M_1(G) = 4(3)^2 = 36$, $M_1^*(G) = 6(3+3) = 36$ and $M_2(G) = 6(3 \times 3) = 54$.

### 5.2 SILICATE PAIR

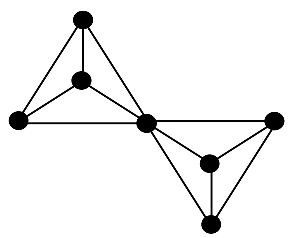


**Figure 4.** Silicate Pair $(Si_2O_7)^{6-}$known as *Pyrosilicate*.

For the Silicate pair graph $G$, every metric basis $S$ of $G$ shall contains exactly two vertices of degree 3 from each copy $K_4$ in $G$. Thus, it is easy to see that $\beta(G) = |S| = 2 \times 2 = 4$, $d_\beta(v) = \beta(G) = 4$ for every $v$ with $\mathrm{d_G}(v) = 3$,

$$\xi = |\{v: d_\beta(v) = 3\}| = 6,$$

$$\eta_1 = \left|\{uv \in E(G): |d_\beta(u) - d_\beta(v)\right| = 1\}\Big| = 6,$$

$$\eta_2 = \left|\{uv \in E(G): d_\beta(u) = d_\beta(v) = \beta(G) + 1\}\right| = 0.$$

Therefore, Equations (5), (6) and (7) gives,

$$_\beta M_1(G) = 6(16) + (7-6)(25) = 121,$$
$$_\beta M_1^*(G) = 2(12)(4) + (6+0) = 102,$$
$$_\beta M_2(G) = (12)(16) + (6+0)(4) + 0 = 216.$$

### 5.3 Silicate Rings (6-fold)

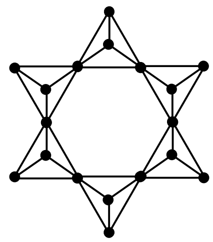


**Figure 5.** Most general Silicate Ring $(Si_6O_{18})^{12-}$ (*Cyclosilicate*).

For the Silicate ring graph $G$, every metric basis $S$ of $G$ shall contains exactly one vertices of degree 3 from each copy $K_4$ in $G$. Thus, it is easy to see that $\beta(G) = |S| = 1 \times 6 = 6$, $d_\beta(v) = \beta(G) = 6$ for every $v$ with $d_G(v) = 3$,

$$\xi = |\{v: \mathrm{d}_G(v) = 3\}| = 12,$$

$$\eta_1 = |\{uv \in E(G): d_G(u) \neq d_G(v)\}| = 4 \times 6 = 24,$$

$$\eta_2 = |\{uv \in E(G): d_G(u), d_G(v) > 3\}| = 6.$$

Therefore, Equations (5), (6) and (7) gives,

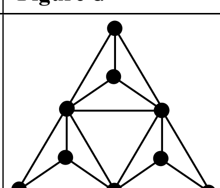
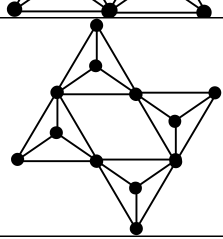
$${}_\beta M_1(G) = 12(36) + (18 - 12)(49) = 726,$$

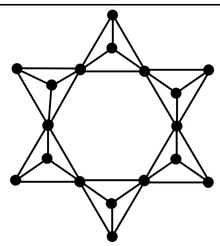
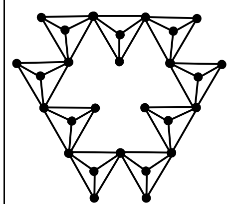
$${}_\beta M_1^*(G) = 2|E(G)|\beta(G) + (\eta_1 + 2\eta_2) = 2(36)(6) + (24 + 12) = 468,$$

$${}_\beta M_2(G) = (36)(36) + (24 + 12)(6) + 6 = 1518.$$

Similarly for other known Silicate rings having $k$-member, the corresponding invariants are computed and are shown in the Table 1.

**Table 1.** The graph invariants and topological indices of $k$-member Silicate rings.

| $k$ | Figure $G$ | Formula | $\|V(G)\|$ | $\|E(G)\|$ | $\beta(G)$ | $\xi$ | $\eta_1$ | $\eta_2$ | ${}_\beta M_1(G)$ | ${}_\beta M_1^*(G)$ | $M_2(G)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **3** |  | $(Si_3O_9)^{6-}$ | 9 | 18 | 3 | 6 | 12 | 3 | 102 | 126 | 219 |
| **4** |  | $(Si_4O_{12})^{8-}$ | 12 | 24 | 4 | 8 | 16 | 4 | 228 | 216 | 484 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **6** |  | $(Si_6O_{18})^{12-}$ | 18 | 36 | 6 | 12 | 24 | 6 | 726 | 468 | 1518 |
| **9** |  | $(Si_9O_{27})^{18-}$ | 27 | 54 | 9 | 18 | 36 | 9 | 2358 | 1026 | 4869 |

### 5.4 SILICATE CHAIN

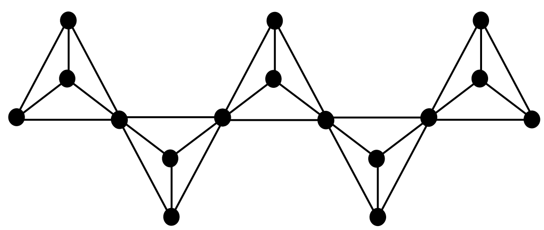


**Figure 6.** The graph $\mathrm{SL}(5)$ of Silicate (single) chains $(\mathrm{SiO}_3^{2-})_5$ (Inosilicates) of period 5 and length 5.

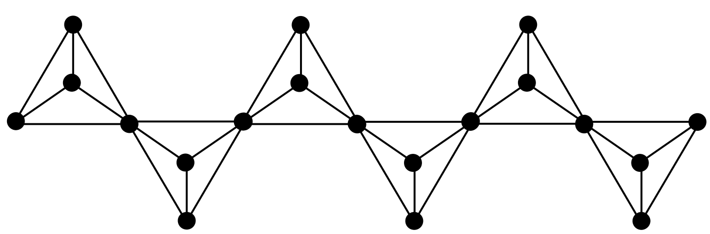


**Figure 7.** The graph $SL(6)$ of Silicate (single) chains $(SiO_3^{2-})_6$ (*Inosilicates*) of period 2 and length 6.

The number of blocks ($K_4$ copies) in $SL(n)$ is $n$ and the number of vertices is $3n+1$.

**Theorem 5.1.** For a graph $= SL(n)$ $(n \geq 3)$,

(i). ${}_{\beta}M_1(G) = 3n^3 + 15n^2 + 19n - 1$, (ii). $_{\beta}M_1^*(G) = 12n^2 + 30n - 6,$

(iii). $_{\beta}M_2(G) = 6n^3 + 30n^2 + 31n - 14.$

**Proof.** The graph $G$ is connected, contains $n$ copies (blocks of 4 oxygen atom) of $K_4$ such that two of them have at most one vertex (cut vertex) in common and the sub graph induced by the cut vertices is a path $P_{n-1}$. The vertices of each copy of $K_{1,n}$ which is not common to any other copy are their private vertices. Each pair of private vertices of a copy of $K_{1,4}$ are equidistant from every other vertex in $G$. Let $p_i$ denotes the number of private vertices of $i^{th}$-copy of $K_{1,4}$ in $SL(n)$. Thus, $p_i = 3$ if $i = 1, n$; $p_i = 2$ for $2 \leq i \leq n$, and every resolving set shall include $p_i - 1$ private vertices of each copy of $K_{1,4}$. Hence, we conclude that $S = \{p_{i_1}, p_{i_2}, p_{j_1}, p_{n_1}, p_{n_2} : j = 2, 3, \ldots, n-1\}$ is a metric basis for $G$, where $p_{l_m}$denotes $m^{th}$ private vertex of $l^{th}$-copy of $K_{1,4}$ in $G$. Therefore, we conclude, due to vertex transitivity of private vertices with each copy, that $d_\beta(v) = \beta(G) = 2 + (n-2) + 2 = n + 2$ only for (each) private vertex $v$ and hence $\xi =$ number of private vertices$= 3 + 2(n-2) + 3 = 2n + 2$, $\eta_1 = |\{uv \in E(G) : u$ is a private vertex and $v$ is an adjacent cut vertex $G\}| = (3) + (n-2)(4) + (3) = 4n - 2$ and $\eta_2 = |\{uv \in E(G) : u$ and $v$ are cut vertices of $G\}| = n - 2$. Finally, $|V(G)| = 3n + 1$ and $|E(G)| = 6n$. Substituting these in (5), (6) and (7), gives

$$_{\beta}M_1(G) = (2n+2)(n+2)^2 + (3n + 1 - 2n - 2)(n+3)^2$$
$$= 3n^3 + 15n^2 + 19n - 1,$$
$$_{\beta}M_1^*(G) = 2(6n)(n+2) + (4n - 2 + 2(n-2)) = 12n^2 + 30n - 6,$$
$$_{\beta}M_2(G) = (6n)(n+2)^2 + (4n - 2 + 2(n-2))(n+2) + n - 2$$
$$= 6n^3 + 30n^2 + 31n - 14.$$

Hence the theorem. ■

### 5.5 Silicate Double Chain: $\mathbf{SL(2, n)}$

There are three types of double chains.

### 5.2.1 Type-I: $\mathbf{SL_1(2, 2k+1)}$

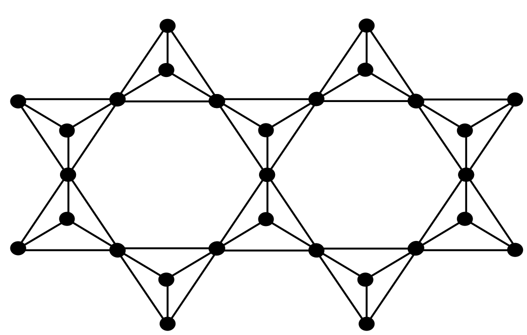


**Figure 8.** Silicate 5-periodic double chain $SL_1(2,5)$.

The graph $G = SL_1(2,2k+1)$ is connected, contains $4k+2$ copies $G_{1,1}, G_{1,2}, \dots, G_{1,2k+1}, G_{2,1}, G_{2,2}, \dots, G_{2,2k+1}$ (clique of 4 oxygen atom) isomorphic to $K_4$ such that $G_{i,j}$ and $G_{k,l}$ have a vertex in common if and only if either (i) $i = k$ and $|l - j| = 1$, or (ii) $|i - k| = 1$, and $j = l$ = odd integer. The number of private vertices (not common to any other copy of $K_4$) in the sub graph $G_{i,j}$ is denoted as $p_{i,j}$. Two private vertices in a copy $G_{i,j}$ are at equidistant from every other vertex in $G$. Therefore, no pair of two private vertices of the same copy lie in $\bar{S}$ for any metric basis $S$ of $G$.

**Theorem 5.2.** For the graph $G = SL_1(2, 2k+1)$, $k \in \mathbb{Z}^+$,

$${}_{\beta}M_1(G) = 44k^3 + 232k^2 + 347k + 103,$$
$${}_{\beta}M_1^*(G) = 96k^2 + 276k + 96,$$
$${}_{\beta}M_2(G) = 96k^3 + 504k^2 + 734k + 184.$$

**Proof.** The number of private vertices $p_{ij}$ of the graph $G$ for $i = 1, 2$ is given by

$$p_{ij} = \begin{cases} 2 & \text{for } j = 1, 2k+1 \\ 2, & \text{for } 2 \le j \le 2k \text{ and } j \text{ is even} \\ 1, & \text{for } 3 \le j \le 2k-1 \text{ and } j \text{ is odd} \end{cases}.$$

Therefore, every resolving set shall include $p_{i,j} - 1$ private vertices of each copy $G_{i,j}$ of $K_{1,4}$. Hence, we conclude that metric basis of S of $G$ contains exactly one private vertex from each $G_{i,j}$ where $i \in \{1,2\}$ and $j \in \{1, 2k+1\} \cup \{2l : l \in \mathbb{Z}^+, 1 \le l \le k\}$.

Hence; $\beta(G) = \sum_{i=1}^{2} \sum_{p_{i,j}>1} (p_{i,j} - 1) = 2 \sum_{p_{1,j}>1} (p_{i,j} - 1) = |S| = 2[(2-1) + (2-1) + k(2-1)] = 2(k+2)$ and

$$\xi = \sum_{i=1}^{2} \sum_{p_{i,j}>1} p_{i,j} = 2 \sum_{p_{1,j}>1} p_{i,j} = 2[2 + 2 + k(2)] = 4(k+2),$$
$$\eta_1 = \left|\{uv \in E(G) : |d_\beta(u) - d_\beta(v)| = 1\}\right| = 2(4 + 4 + k(4)) = 8(k+2),$$
$$\eta_2 = \left|\{uv \in E(G) : d_\beta(u) = d_\beta(v) = \beta(G) + 1\}\right|$$
$$= 2(1 + 1 + k(1) + (k-1)(6)) = 2(7k - 4).$$

Finally, $|V(G)| = 11k + 7$ and $|E(G)| = 12(2k+1)$. Substituting these in (5), (6) and (7), gives

$${}_{\beta}M_1(G) = \xi\,(\beta(G))^2 + (|V(G)| - \xi)(\beta(G) + 1)^2$$
$$= 4(k+2)4(k+2)^2 + (11k + 7 - 4(k+2))(2k+5)^2$$
$$= 44k^3 + 232k^2 + 347k + 103,$$
$${}_{\beta}M_1^*(G) = 2|E(G)|\beta(G) + (\eta_1 + 2\eta_2)$$
$$= 2(12(2k+1))2(k+2) + (8(K+2) + 4(7k-4))$$
$$= 96k^2 + 276k + 96,$$
$${}_{\beta}M_2(G) = |E(G)|(\beta(G))^2 + (\eta_1 + 2\eta_2)\beta(G) + \eta_2$$
$$= 12(2k+1)2(k+2)^2 + (8(K+2) + 4(7k-4))2(k+2) + 2(7k-4)$$
$$= 96k^3 + 504k^2 + 734k + 184.$$

Hence the theorem. ∎

**5.2.3 TYPE- II: $\mathbf{SL_2(2, 2k+1)}$**

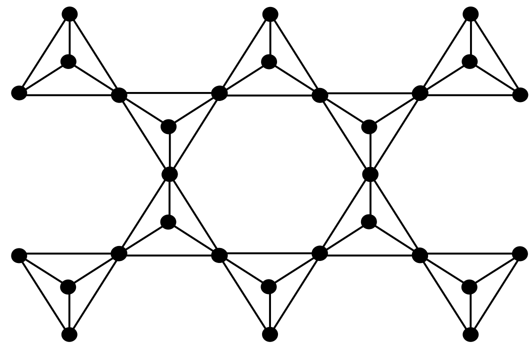


**Figure 9.** Silicate double chain $SL_2(2,5)$.

The graph $G = SL_1(2,2k+1)$ is connected, contains $4k+2$ copies $G_{1,1}, G_{1,2}, \dots, G_{1,2k+1}, G_{2,1}, G_{2,2}, \dots, G_{2,2k+1}$ (clique of 4 oxygen atom) isomorphic to $K_4$ such that $G_{i,j}$ and $G_{k,l}$ have a vertex in common if and only if either (i) $i = k$ and $|l - j| = 1$, or (ii) $|i - k| = 1$, and $j = l =$ even integer.

**Theorem 5.3.** For the graph $G = SL_2(2, 2k+1)$, $k \in \mathbb{Z}^+$,

$${}_{\beta}M_1(G) = 44k^3 + 324k^2 + 679k + 288,$$
$${}_{\beta}M_1^*(G) = 96k^2 + 372k + 144,$$
$${}_{\beta}M_2(G) = 96k^3 + 696k^2 + 1382k + 430.$$

**Proof.** The number of private vertices $p_{ij}$ of the graph $G$ for $i = 1, 2$ is given by

$$p_{ij} = \begin{cases} 3 & \text{for } j = 1, 2k+1 \\ 1, & \text{for } 2 \le j \le 2k \text{ and } j \text{ is even} \\ 2, & \text{for } 3 \le j \le 2k-1 \text{ and } j \text{ is odd} \end{cases}.$$

Therefore, similarly in the proof of previous theorem, we get $\beta(G) = |S| = 2[2 + 2 + (k-1)(1)] = 2k + 6$. and;

$$\xi = 2(3 + 3 + (k-1)(2)) = 2(2k+4),$$
$$\eta_1 = 2(3 + 3 + (k-1)(4)) = 2(4k+2),$$
$$\eta_2 = 2\big((k-1)(1) + (k)(6)\big) = 2(7k-1).$$

Also, $|V(G)| = 11k + 8$ and $|E(G)| = 12(2k+1)$. Substituting these in (5), (6) and (7), gives

$${}_{\beta}M_1(G) = 2(2k+4)(2k+6)^2 + (11k + 7 - 2(2k+4))(2k+7)^2$$
$$= 44k^3 + 324k^2 + 679k + 288,$$
$${}_{\beta}M_1^*(G) = 2(12(2k+1))(2k+6) + (8k + 4 + 28k - 4)$$
$$= 96k^2 + 372k + 144,$$
$${}_{\beta}M_2(G) = (12(2k+1))(2k+6)^2 + (8k + 4 + 28k - 4)(2k+6) + 2(7k-1)$$
$$= 96k^3 + 696k^2 + 1382k + 430.$$

Hence the theorem. ■

#### 5.2.4 TYPE – III: $\mathbf{SL(2, 2k)}$ SILICATE TORI

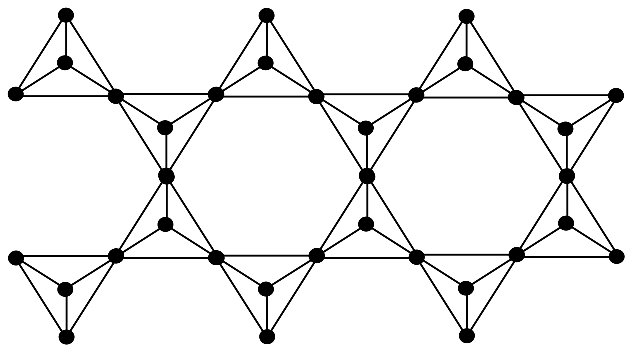


**Figure 10.** Silicate chain $SL(2,6)$.

The graph $G = SL(2,2k)$ is connected, contains 4k copies $G_{1,1}$, $G_{1,2}, \dots, G_{1,2k}$, $G_{2,1}$, $G_{2,2}$, $\dots, G_{2,2k}$ (clique of 4 oxygen atom) isomorphic to $K_4$ such that $G_{i,j}$ and $G_{k,l}$ have a vertex in common if and only if either (i) $i = k$ and $|l - j| = 1$, or (ii) $|i - k| = 1$, and $j = l$ =even integer (or equivalently $j = l =$ odd as the graphs are isomorphic).

**Theorem 5.4.** For the graph $G = SL(2, 2k)$, $k \in \mathbb{Z}^+$,

$$_{\beta}M_1(G) = 44k^3 + 212k^2 + 255k - 4,$$
$$_{\beta}M_1^*(G) = 96k^2 + 228k - 18,$$
$$_{\beta}M_2(G) = 96k^3 + 456k^2 + 506k - 84.$$

**Proof.** The number of private vertices $p_{ij}$ for $i = 1,2$ are

$$p_{ij} = \begin{cases} 3 & \text{for } j = 1 \\ 2 & \text{for } j = 2k \\ 2 & \text{for } 3 \le j \le 2\text{k} - 1 \text{ and } j \text{ is odd} \\ 1 & \text{for } 2 \le j \le 2k - 2 \text{ and } j \text{ is even} \end{cases}.$$

Therefore, $\beta(G) = 2[2 + 1 + (k - 1)(1)] = 2(k + 2)$ and

$$\xi = 2[3 + 2 + (k - 1)(2)] = 2(2k + 3),$$
$$\eta_1 = 2[3 + 4 + (k - 1)(4)] = 2(4k + 3),$$
$$\eta_2 = 2(0 + (k - 1)(6) + (k - 1)(1) + 1) = 2(7k - 6).$$

Also, $|V(G)| = 11k + 2$ and $|E(G)| = 24k$. Substituting these in (5), (6) and (7), gives

$$_{\beta}M_1(G) = 2(2k + 3)(2k + 4)^2 + (11k + 2 - 2(2k + 3))(2k + 5)^2$$
$$= 44k^3 + 212k^2 + 255k - 4,$$
$$_{\beta}M_1^*(G) = 2(24k)(2k + 4) + (2(4k + 3) + 2(14k - 12))$$

$$= 96k^2 + 228k - 18,$$

$$_{\beta}M_2(G) = (24k)(2k+4)^2 + (2(4k+3) + 2(14k-12))(2k+4) + 14k - 12$$

$$= 96k^3 + 456k^2 + 506k - 84.$$

Hence the theorem. ∎

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