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Kirchhoff Index and Kirchhoff Energy

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ABSTRACT

The Kirchhoff energy and Kirchhoff Laplacian energy for Kirchhoff matrix are examined in this paper. The Kirchhoff index with Kirchhoff Laplacian eigenvalues is defined and different inequalities including the distances, the vertices and the edges are obtained. Indeed, some bounds for the degree Kirchhoff index associated with its eigenvalues are found.

1. INTRODUCTION

Let G be a simple, connected (molecular) graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance d_{ij} is the number of edges of shortest path between vertices v_i and v_j in G. If G is a connected (molecular) graph then the minimum and maximum degree will be represented by δ and Δ, respectively. In this paper, some new bounds for molecular graphs are obtained by investigating Kirchhoff matrix, Kirchhoff index and Kirchhoff energy.

The resistance distance r_{ij} is described as between the v_i and v_j peaks at G. This distance is the resistor between the two related nodes of electronic nets; it is found by the

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principle of resistor electrical nets through Ohm and Kirchhoff laws. The Kirchhoff index is defined by the resistance distance matrix [3, 6] as $Kf(G) = \sum_{i<j} r_{ij}$.

The Kirchhoff index is used in many areas of chemistry. Some of those are molecular graphs of polycyclic structures, circulation graphs, distance-order graphs and Mobius stairs. The Kirchhoff's index brought a topological approach to the chain structure of molecule. With this index, it was shown that the macromolecule was related to the topological radius, mean square rotation radius and intrinsic viscosity. This contributed to the study of highly complex and branched polymers. The Kirchhoff index in its approaches has proved to be particularly helpful for examining the topological radius R_{top} of these molecules, where $R_{top} = Kf/n^2$ and n is the number of atoms $(non - hydrogen)$ in the polymer.

One of the aims of this paper is to give some bounds through structure exponents as vertices (number of atoms), edges (bonds), maximum peak degree (valence) for Kirchhoff index. In addition, specific connections for molecules will obtain by giving some definitions and limits about degree Kirchhoff index. The boundaries of an identifier are significant reports about a molecule (or its graph) when they determine the approximate area of an identifier associated with molecular constructional parameters. The second plan of this essay is to find important connections for Kirchhoff energy by the help of eigenvalues of Kirchhoff matrix. If Kirchhoff's energy is defined as the energy of the ability to build this system, it is determined by the eigenvalues of the Kirchhoff matrix. This energy can change the location, shape and content of the molecule.

The scheme of this paper is in the following: In Sections 1 and 2, some known statements and bounds are given. In the sequel, some results are obtained for the Kirchhoff energy using its eigenvalues. Indeed, the Kirchhoff index and the degree Kirchhoff index of graphs are observed as some parameters and some inequalities are formed by the help of defining relations.

2. PRELIMINARIES

Let $K = K(G)$ denote the square matrix of order n and let also k_{ij} denote (i, j)-entry. This symmetrical and zero-square matrix is called the Kirchhoff matrix [1]. Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of $K(G)$. These numbers are generally called Kirchhoff eigenvalues of G. The resistance distance can also be provided with the aid of eigenvalues and eigenvectors of the integrated and networked Laplacian matrix. Let Kirchhoff Laplacian matrix of a graph G be $KL(G) = KD(G) - KA(G)$, where $KD(G)$ is the Kirchhoff diagonal matrix and $KA(G)$ is the Kirchhoff adjacency matrix of a graph G [2, 18, 26]. The Kirchhoff Laplacian eigenvalues are $\rho_1^L \geq \rho_2^L \geq \cdots \geq \rho_{n-1}^L \geq \rho_n^L = 0$. As in [26], $t = t(G) = \frac{1}{n}\prod_{i=1}^{n-1} \rho_i^L$ is spanning tree and $\sum_{i=1}^{n} \rho_i^L = 2m$.

The normalized Laplacian matrices of G are identified as $NL(G) = D(G)^{\frac{-1}{2}} L(G) D(G)^{\frac{1}{2}}$ [4, 8]. Using this, the normalized Kirchhoff Laplacian matrix of G is indicated by $KNL(G) = (KD)^{\frac{-1}{2}} KL(KD)^{\frac{1}{2}}$. The eigenvalues of $KNL(G)$ are shown by $\rho'_0, \rho'_1, \dots, \rho'_{n-1}$, where $\rho'_0 \geq \rho'_1 \geq \cdots \geq \rho'_{n-2} \geq \rho'_{n-1} = 0$. Also, $t = \frac{\Delta}{2m} \prod_{i=0}^{n-2} \rho'_i$ in [5].

A Laplacian-energy-like invariant is described as $LEL = LEL(G) = \sum_{i=1}^{n-1} (\rho_i^L)^{1/2}$ in [19] ,see [14, 20, 24] for more details. The Kirchhoff index of a graph G can be defined by means of eigenvalues as $Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\rho_i^L}$, where $n \geq 2$. This equality is proved in [12, 13], see also [3, 6, 19, 22, 23]. The degree Kirchhoff index of G is explained as a new index $Kf'(G) = 2m \sum_{i=0}^{n-2} \frac{1}{\rho'_i}$ in [10] and [17].

The following results are important to verify the main results:

Lemma 2.1. [26] *Let G be a graph of order n and $\bar{G}$ its complement. If $Spec(G) = \{\rho_1^L, \rho_2^L, \dots, \rho_{n-1}^L, 0\}$, then $Spec(\bar{G}) = \{ n - \rho_1^L, n - \rho_2^L, \dots, n - \rho_{n-1}^L, 0\}$. Then, $\rho_1^L < n$ if and only if $\bar{G}$ is disconnected.*

Lemma 2.2. [21] *Let G be a connected graph with n. Then $\rho_1^L = \rho_2^L = \cdots = \rho_{n-1}^L$ if and only if $G \cong K_n$.*

Lemma 2.3. [11] *If $\alpha_1, \alpha_2, \dots, \alpha_s$ are all positive numbers then*

$$s\left[\frac{1}{s}\sum_{i=1}^{s} \alpha_i - \left(\prod_{i=1}^{s} \alpha_i\right)^{\frac{1}{s}}\right] \leq s\sum_{i=1}^{s} \alpha_i - \left(\sum_{i=1}^{s} \sqrt{\alpha_i}\right)^2. \qquad (1)$$

Lemma 2.4. [27] *If $\alpha_1, \alpha_2, \dots, \alpha_s \geq 0$, $p_1, p_2, \dots, p_s \geq 0$ and $\sum_{i=1}^{s} p_i = 1$ then*

$$\sum_{i=1}^{s} p_i \alpha_i - \prod_{i=1}^{s} \alpha_i^{p_i} \geq s\lambda \left(\frac{1}{s}\sum_{i=1}^{s} \alpha_i - \prod_{i=1}^{s} \alpha_i^{\frac{1}{s}}\right), \qquad (2)$$

where $\lambda = \min(p_1, p_2, \dots, p_s)$. Furthermore, equality in (2) holds if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_s$.

The notationsK_n, P_n $K_{1,n-1}$, $K_{p,q}$, $(p + q = n)$ show the complete, path, star, and complete bipartite graph on n vertices, respectively.

3. MAIN RESULTS

3.1 KIRCHHOFF ENERGY

Due to the apparent performance of the graph energy idea and the fast spread of mathematical theories in this concept, the energies built on the eigenvalues of the graph matrices are given one by one. The Kirchhoff energy is one of these. In this subsection, the

Kirchhoff energy is investigated by the help of its eigenvalues with organic compounds; degrees, edges and nodes. The Kirchhoff energy $KE(G)$ is represented by $KE = KE(G) = \sum_{i=1}^{n} |\rho_i|$ in this paper, see [15, 16].

As shown in [1], $\sum_{i=1}^{n} \rho_i = 0$ and $\sum_{i=1}^{n} (\rho_i)^2 = 2\sum_{1\leq i<j\leq n} (k_{ij})^2$, where k_{ij} is the (i,j)-th entry of the Kirchhoff matrix for $1 \leq i < j \leq n$. To simplify our argument, the symbol $\sum_{1\leq i<j\leq n}$ is denoted by $\sum$.

In [1] the following upper bounds for $KE(G)$ are established

$$KE(G) \leq \sqrt{2n\sum (k_{ij})^2}. \tag{3}$$

$$KE(G) \leq \frac{2}{n}\sum (k_{ij})^2 \sqrt{(n-1)[2\sum (k_{ij})^2 - (\frac{2}{n}\sum (k_{ij})^2)^2]}. \tag{4}$$

Also, the following lower bound for $KE(G)$ is given in [1];

$$KE(G) \geq \sqrt{2\sum (k_{ij})^2 + n(n-1)(\nabla)^{\frac{2}{n}}}, \tag{5}$$

where ∇ is the absolute value of the determinant of Kirchhoff matrix.

Corollary 3.1. *Let G be a connected graph with the maximum degree Δ, then*

$$KE \leq \Delta + \frac{\sqrt{2n\sum (k_{ij})^2}}{2}. \tag{6}$$

Equality holds if and only if $G \cong K_n$ or $G \cong K_{1,n}$.

Inequalities (6) and (3) are not comparable. Thus, if $G \cong K_n$ or $G \cong K_{1,n}$ the inequality (6) is stronger than (3). Since Laplacian spectrum of K_n is $(0, n, \dots, n)$, then $KE(G) = 2(n-1)$. Also, $2\sum (k_{ij})^2 = (n-1)^2 + (n-1)$. For example $n = 10$, $KE(G) = 30$ in Equation (3) and $KE(G) = 24$ in Equation (6).

Corollary 3.2. *Let G be a connected graph with $n \geq 2$ vertices, then*

$$KE \leq (n-1) + \frac{\sqrt{2n\sum (k_{ij})^2}}{2}. \tag{7}$$

Equality holds if and only if $G \cong K_n$ and $G \cong K_{1,n}$.

Remark 3.3. *Since* $\frac{2mn-2m}{n\Delta} + \frac{\sqrt{2n\sum (k_{ij})^2}}{2} = (n-1) + \frac{\sqrt{2n\sum (k_{ij})^2}}{2}$, (7) *is better than* (2).

Theorem 3.4. *Let G be a connected graph then*

$$KE \leq \frac{2\sum (k_{ij})^2 + n\delta^2}{2\delta}. \tag{8}$$

Equality holds if and only if $G \cong K_n$ and $G \cong K_{1,n}$.

Proof. Since $(|\rho_i| - \delta)^2 \geq 0$, $\sum_{i=1}^{n} |\rho_i|^2 - 2\sum_{i=1}^{n} |\rho_i|\delta + \sum_{i=1}^{n} \delta^2 \geq 0$. It is resulted that $2\sum (k_{ij})^2 + n\delta^2 \geq 2\delta KE$. Hence, $KE \leq \frac{2\sum (k_{ij})^2 + n\delta^2}{2\delta}$.

Inequalities (5) and (8) are incomparable. Hence, the inequality (8) is better than (4) if and only if $G \cong K_n$ or $G \cong K_{1,n}$. For example $n = 5$; $KE(G) \geq 7{,}404$ and $KE(G) \geq 12{,}5$ in (8). ■

Corollary 3.5. *Let G be a connected graph, $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Then*

$$KE \geq \frac{2\sum (k_{ij})^2 + n\Delta\delta}{\Delta + \delta}. \tag{9}$$

Equality holds if and only if $G \cong K_n$.

Proof. Since $\sqrt{\rho_i} \geq \sqrt{\delta}$ and $\sqrt{\Delta} \geq \sqrt{\rho_i}$ then $(\sqrt{\rho_i} - \sqrt{\delta})(\sqrt{\Delta} - \sqrt{\rho_i}) \geq 0$. Thus, $\sqrt{\rho_i}(\sqrt{\Delta} + \sqrt{\delta}) \geq |\rho_i| + \sqrt{\Delta\delta}$. Hence, $\sum_{i=1}^{n} |\rho_i|(\sqrt{\Delta} + \sqrt{\delta})^2 \geq \sum_{i=1}^{n} (\rho_i)^2 + 2\sqrt{\Delta\delta}\sum_{i=1}^{n} |\rho_i| + n\Delta\delta$. Therefore, $KE[(\sqrt{\Delta} + \sqrt{\delta})^2 - 2\sqrt{\Delta\delta}] \geq 2\sum (k_{ij})^2 + n\Delta\delta$ and the inequality is concluded by $KE \geq \frac{2\sum (k_{ij})^2 + n\Delta\delta}{\Delta + \delta}$. ■

Inequalities (9) and (5) are incomparable. Such as $G \cong K_n$, the inequality (9) is better than (5).

Lemma 3.6. *Let G be a connected graph with determinant ∇. Then*

$$\prod_{i=1}^{n} (1 + |\rho_i|) \geq (1 + \nabla^{\frac{1}{n}})^n. \tag{10}$$

Equality holds if and only if $G \cong K_n$.

Proof. Let $f(\alpha) = log(1 + e^{\alpha})$. Since $f(\alpha)$ is convex on $(-\infty, +\infty)$, it has $\sum_{i=1}^{n} log(1 + e^{\alpha_i}) \geq nlog(1 + \exp(\frac{1}{n}\sum_{i=1}^{n} \alpha_i))$. Replacing α_i by $log|\rho_i|$ $(i = 1,2,\ldots,n)$, it gets $log\prod_{i=1}^{n} (1 + |\rho_i|) \geq nlog(1 + (\prod_{i=1}^{n} |\rho_i|)^{\frac{1}{n}})$. So, $\prod_{i=1}^{n} (1 + |\rho_i|) \geq (1 + (\prod_{i=1}^{n} |\rho_i|)^{\frac{1}{n}})^n$. Hence, $\prod_{i=1}^{n} (1 + |\rho_i|) \geq (1 + \nabla^{\frac{1}{n}})^n$. Inequality holds in (10) if and only if all $|\rho_i|$'s are equal. ■

Theorem 3.7. *Let G be a connected graph with the quantitiy ∇. If $\rho_i > 0$ for $i = 1,2,\ldots,n$ then*

$$KE \geq n\nabla^{\frac{1}{n}},$$

if and only if $G \cong K_n$, $G \cong P_n$ and $G \cong K_{1,n-1}$.

Proof. Arithmetic-Geometric Mean Inequality shows that

$$\prod_{i=1}^{n} (1 + |\rho_i|) \leq (\sum_{i=1}^{n} \frac{1 + |\rho_i|}{n})^n = (1 + \frac{KE}{n})^n.$$

As in the Lemma 3.6, $(1 + \nabla^{\frac{1}{n}})^n \leq \prod_{i=1}^{n} (1 + |\rho_i|) \leq (1 + \frac{KE}{n})^n$. Hence, $KE \geq n\nabla^{\frac{1}{n}}$ if and only if $G \cong K_n$, $G \cong P_n$ and $G \cong K_{1,n-1}$. ■

3.2 KIRCHHOFF INDEX AND THE DEGREE KIRCHHOFF INDEX

In this subsection, different bounds are reported for the Kirchhoff index consisting its ordinary coefficients and the eigenvalues of Kirchhoff Laplacian matrix. The strategy of this subsection is to study extensively on Kirchhoff index. After that, it is to give some intimate relations for the degree Kirchhoff index of these graphs.

In [25], $\sum_{i=1}^{n} ({\rho'}_i)^2 = n + 2R_{-1}(G)$, where $R(G)$ is the general Randić index. For more information for $R_{-1}(G)$ see [5, 9].

Theorem 3.8. *Let G be a connected graph of order $n > 3$ and t spanning trees then*

$$Kf(G) \geq 1 + \frac{3n(n-2)}{2}\left(\frac{n^{\frac{2}{3(n-2)}}}{(nt(G))^{\frac{3n-4}{3(n-2)(n-1)}}} - \frac{1}{3(nt(G))^{\frac{1}{n-1}}}\right). \qquad (11)$$

with equality holding if and only if $G \cong K_n$.

Proof. As in Lemma 2.4, $a_i = \frac{1}{\rho_i^L}$, $i = 1,2,\ldots,n-1$, $p_1 = \frac{1}{3n-3}$ $p_i = \frac{3n-4}{3(n-2)(n-1)}$, $i = 2,3,\ldots,n-1$, it holds

$$\frac{1}{3n-3}\frac{1}{\rho_1^L} + \sum_{i=2}^{n-1}\frac{3n-4}{3(n^2-3n+2)}\frac{1}{\rho_i^L} - \left(\frac{1}{\rho_1^L}\right)^{\frac{1}{3n-3}}\prod_{i=2}^{n-1}\left(\frac{1}{\rho_i^L}\right)^{\frac{3n-4}{3(n^2-3n+2)}}$$
$$\geq \frac{1}{3}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{\rho_i^L} - \prod_{i=1}^{n-1}\left(\frac{1}{\rho_i^L}\right)^{\frac{1}{n-1}}\right).$$

Hence,

$$\frac{1}{3n-3}\frac{1}{\rho_1^L} + \frac{3n-4}{3(n^2-3n+2)}\left(\frac{Kf(G)}{n} - \frac{1}{\rho_1^L}\right) - \frac{\left(\frac{1}{\rho_1^L}\right)^{\frac{1}{3(n-1)}}}{\left(\frac{1}{\rho_1^L}\right)^{\frac{3n-4}{3(n^2-3n+2)}}}\prod_{i=1}^{n-1}\frac{1}{\rho_i^L}$$
$$\geq \frac{1}{3n-3}\frac{Kf(G)}{n} - \frac{1}{3}\prod_{i=1}^{n-1}\left(\frac{1}{\rho_1^L}\right)^{\frac{1}{n-1}}.$$

from which

$$\frac{-2}{3(n-2)}\frac{1}{\rho_1^L} + \frac{2}{3n(n-2)}Kf(G) \geq \frac{(\rho_1^L)^{\frac{2}{3n-6}}}{(nt(G))^{\frac{3n-4}{3(n^2-3n+2)}}} - \frac{1}{3(nt(G))^{\frac{1}{n-1}}}.$$

This implies that

$$Kf(G) \geq \frac{n}{\rho_1^L} + \frac{3n(n-2)}{2}\left(\frac{(\rho_1^L)^{\frac{2}{3(n-2)}}}{(nt(G))^{\frac{3n-4}{3(n^2-3n+2)}}} - \frac{1}{3(nt(G))^{\frac{1}{n-1}}}\right). \qquad (12)$$

By Lemma 2.1,i t can be demonstrated that

$$f(x) = \frac{n}{x} + \frac{3n(n-2)}{2}\frac{x^{\frac{2}{3n-6}}}{(nt(G))^{\frac{3n-4}{3(n^2-3n+2)}}}$$

is a decreasing function for $x \leq n$. Thus,

$$f(x) \geq f(n) = \frac{n}{n} + \frac{3n(n-2)}{2} \frac{n^{\frac{2}{3n-6}}}{(nt(G))^{\frac{3n-4}{3(n^2-3n+2)}}}.$$

Using this result, we obtain

$$Kf(G) \geq 1 + \frac{3n(n-2)}{2}\left(\frac{n^{\frac{2}{3(n-2)}}}{(nt(G))^{\frac{3n-4}{3(n^2-3n+2)}}} - \frac{1}{3(nt(G))^{\frac{1}{n-1}}}\right)$$

which completes the first part of the proof.

Assume that Equation (11) holds. Then all the above inequalities must be equalities. Then according to Lemma 2.2, it must be $\rho_1^L = \rho_2^L = \cdots = \rho_{n-1}^L$ and $G \cong K_n$. Conversely, equality in (11) for $G \cong K_n$ can be easily checked. ■

Corollary 3.9. *Let T be a tree of order $n > 3$ with the degree Δ, then*

$$Kf(G) \geq 1 + \frac{3n^{\frac{3n-4}{3(n-2)}}(n-2)}{2} - \frac{n^2-2n}{2}.$$

Proof. Since T is tree, $t = 1$. ■

Theorem 3.10. *Let G be a connected graph of order n then*

$$Kf(G) \leq \frac{n^2-n}{\Delta}, \tag{13}$$

with equality holding if and only if $G \cong K_n$.

Proof. It is known that $\frac{1}{\Delta} - \frac{1}{\rho_i^L} > 0$. If the sum is applied to both sides of the equation, this gets $\frac{n-1}{\Delta} > \frac{Kf(G)}{n}$. Hence, $Kf(G) \leq \frac{n^2-n}{\Delta}$ if and only if $G \cong K_n$. ■

Theorem 3.11. *Let G be a connected graph with $n \geq 2$ vertices, then*

$$Kf(G) \geq \frac{n}{n-2}\left(\frac{1}{LEL(G)^2} - \frac{n-1}{(nt(G))^{\frac{1}{n-1}}}\right).$$

Equality provides if and only if $G \cong K_n$ and $G \cong K_{1,n-1}$.

Proof. Let $n - 1 = \alpha$. Setting in (1),

$$\alpha\left[\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{\rho_i^L} - \left(\prod_{i=1}^{\alpha}\frac{1}{\rho_i^L}\right)^{\frac{1}{\alpha}}\right] \leq (\alpha)\sum_{i=1}^{\alpha}\frac{1}{\rho_i^L} - \left(\sum_{i=1}^{\alpha}\sqrt{\frac{1}{\rho_i^L}}\right)^2$$

and so

$$\left[\sum_{i=1}^{\alpha}\frac{1}{\rho_i^L} - \alpha\left(\frac{1}{\prod_{i=1}^{\alpha}\rho_i^L}\right)^{\frac{1}{\alpha}}\right] \leq \alpha\sum_{i=1}^{\alpha}\frac{1}{\rho_i^L} - \left(\sum_{i=1}^{\alpha}\sqrt{\frac{1}{\rho_i^L}}\right)^2.$$

This inequality transforms into

$$\left[\frac{Kf(G)}{n} - \alpha \frac{1}{(nt(G))^{\frac{1}{\alpha}}}\right] \le \frac{\alpha Kf(G)}{n} - \left(\sum_{i=1}^{\alpha} \frac{1}{\sqrt{\rho_i^L}}\right)^2$$

i.e.

$$\left[\frac{1}{LEL(G)^2} - \frac{\alpha}{(nt(G))^{\frac{1}{\alpha}}}\right] \le \frac{(n-2)}{n} Kf(G).$$

Hence,

$$Kf(G) \ge \frac{n}{n-2}\left(\frac{1}{LEL(G)^2} - \frac{n-1}{(nt(G))^{\frac{1}{n-1}}}\right).$$

Equality provides if and only if $G \cong K_n$ and $G \cong K_{1,n-1}$.■

Corollary 3.12. *Let T be a tree of order $n \ge 2$ then*

$$Kf(G) \ge \frac{n}{n-2}\left(\frac{1}{LEL(G)^2} - (n-1)\right).$$

Proof. For a tree T, $t = 1$. ■

Theorem 3.13. *Let G be a connected graph and $n \ge 2$ vertices. Then*

$$Kf'(G) \ge \frac{2mn(n-1)}{n+2R_{-1}(G)}, \tag{14}$$

if and only if $G \cong K_n$, $G \cong K_{n,m}$ and $G \cong K_{1,n-1}$.

Proof. From the Chebyshev inequality (see [7]) for $p_i = \frac{1}{\rho_i'}$, $a_i = (\rho_i')^2$, $b_i = \rho_i'$ $i = 0,1,\ldots,n-2$, it is obtained that

$$\sum_{i=0}^{n-2} \frac{1}{\rho_i'} \sum_{i=0}^{n-2} \frac{1}{\rho_i'} (\rho_i')^2 \rho_i' \ge \sum_{i=0}^{n-2} \frac{1}{\rho_i'} (\rho_i')^2 \sum_{i=0}^{n-2} \frac{1}{\rho_i'} \rho_i'.$$

This inequality gets,

$$\frac{Kf'(G)}{2m}(n + 2R_{-1}(G)) \ge (n^2 - n).$$

Hence, $Kf'(G) \ge \frac{2mn^2-2mn}{n+2R_{-1}(G)}$.

Equality provides if and only if $G \cong K_n$, $G \cong K_{n,m}$ and $G \cong K_{1,n-1}$. ■

Theorem 3.14. *Let G be a connected graph with $n \ge 3$ vertices. Then*

$$Kf'(G) \ge \frac{(2m)^{\frac{n-2}{n-1}}(n-1)}{\left(\frac{t(G)}{\Delta}\right)^{\frac{1}{n-1}}}. \tag{15}$$

Equality provides if and only if $G \cong K_n$, $G \cong K_{n,m}$ and $G \cong K_{1,n-1}$.

Proof. Applying the Arithmetic-Geometric Mean Inequality,

$$\sum_{i=0}^{n-2} \frac{1}{\rho_i'} \ge (n-1)\left(\prod_{i=0}^{n-2} \frac{1}{\rho_i'}\right)^{\frac{1}{n-1}},$$

then $\frac{Kf'(G)}{2m} \geq \frac{n-1}{(\frac{2mt(G)}{\Delta})^{\frac{1}{n-1}}}$. Hence, the claim holds. ■

Corollary 3.15. *Let T be a tree of order $n \geq 3$ then*

$$Kf'(G) \geq (2m)^{\frac{n-2}{n-1}}(n-1)\Delta^{\frac{1}{n-1}}. \quad (16)$$

Proof. For a tree T, $t = 1$.■

Corollary 3.16. *Let G be a complete graph with m edges and $n \geq 4$ vertices. Then*

$$Kf'(G) = \frac{\Delta(n-1)^{n+1}}{2mn^{n-3}}. \quad (17)$$

Proof. Let $n-1 = \alpha$. From the eigenvalues of normalized Laplacian matrix for complete graphs in [8]. Then $Kf'(G) = 2m(\alpha)\frac{1}{(\frac{n}{\alpha})^{\alpha}} = n(\alpha)^2(\frac{\alpha}{n})^{\alpha}$ and so

$$Kf'(G) = n(\alpha)^2(\frac{\alpha}{n})^{\alpha} = \frac{\Delta n^2(\alpha)^2(\frac{\alpha}{n})^{\alpha}}{2m}.$$

Hence, $Kf'(G) = \frac{\Delta(n-1)^{n+1}}{2mn^{n-3}}$. ■

4. CONCLUSION

In this paper, we first express some known inequalities and descriptors and then continue our study by observing the Kirchhoff matrix and its eigenvalues. In the sequel, some connections for Kirchhoff energy in terms of the distances and eigenvalues are obtained. Throughout the remainder of this paper, some information for the Kirchhoff and degree Kirchhoff indices of graphs are found.

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