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On the Minimal Unicyclic and Bicyclic Graphs with respect to the Neighborhood First Zagreb Index

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ABSTRACT

The neighborhood first Zagreb index has recently been introduced for characterizing the topological structure of molecular graphs. In the present study, we characterize the graphs having minimum neighborhood first Zagreb index in the class of unicyclic/bicyclic graphs on n vertices for every fixed integer $n \geq 5$.

1. INTRODUCTION

All the graphs discussed here are simple, connected, finite and undirected. For further basic notions of graph theory, we refer the reader to some relevant books [12, 14, 29].

The first Zagreb index M_1 (appeared within a formula derived in [20]) and the second Zagreb index M_2 (introduced in [18]) for a graph H can be defined as: $M_1(H) = \sum_{v_1 \in V(H)} d(v_1)^2 = \sum_{v_1 v_2 \in E(H)} \big(d(v_1) + d(v_2)\big)$ and $M_2(H) = \sum_{v_1 v_2 \in E(H)} d(v_1) d(v_2)$.

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The theory of Zagreb indices is deep rooted; for example, see the papers [1, 2, 8, 15, 16, 19, 20, 23, 24, 26, 28], recent surveys [3, 4, 6, 17] and related references listed therein.

For a vertex $w \in V(H)$, different researchers use different notations for representing the sum of degrees of the adjacent to w in literature, however, we use the notations $S_H(w)$ or Simply $S(w)$ or S_w, due to the simple reason, as S used for sum. The average-degree [32](also known as dual degree [10]) of a vertex $w \in V(H)$ is the number $\frac{s(w)}{d(w)}$ and we denote it by $a(w)$. Consider the following general graph invariants

$$\Gamma_1(H) = \sum_{w \in V(H)} g_1\big(S(w)\big) \text{ and } \Gamma_2(H) = \sum_{vw \in E(H)} g_2\big(S(v), S(w)\big).$$

Most of the cases of the above invariants Γ_1 *and* Γ_2 have already been presented in mathematical chemistry. For example, if we take $g_1(S(u)) = S(u)$ *or* $1/\sqrt{S(u)}$ then Γ_1 gives the first Zagreb index M_1 [7] or first extended zeroth−order connectivity index [5, 30, 31,33], respectively and if we take $g_2\big(S(v), S(w)\big) = S(v) + S(w)$ or $1/\sqrt{S(v)\,S(w)}$ then Γ_2 gives M_2 (see Lemma 2.6 in [7]), the first extended first-order connectivity index [5], fourth atom-bond connectivity index [11] or fifth geometric-arithmetic index [13], respectively. On the same lines, it is natural to consider [27] the following revised version of the first and second Zagreb indices:

$$\mathrm{NM}_1(H) = \sum_{v \in V(H)} (s(v))^2 \text{ and } \mathrm{NM}_2(H) = \sum_{v \in V(H)} s(u)s(v).$$

The invariant NM_1 and NM_2 was referred [27] to as the neighborhood first Zagreb index and neighborhood second Zagreb index. In this current paper, we are concerned with the neighborhood first Zagreb index NM_1, which was initially presented in Refs. [9, 25] and referred to as the neighborhood first Zagreb index [25].Clearly, the invariant NM_1 can rewritten [9] as $\mathrm{NM}_1(H) = \sum_{v \in V(H)} (d(v)a(v))^2$.

The main objective of the present study is to establish extremal results regarding the unicyclic graphs and bicyclic graph of order n with respect to NM_1. In Section 2, we define some transformations which will decrease then neighborhood first Zagreb index. Throughout this paper, graph under discussion is either a unicycle graph or a bicyclic graph on n vertices for every fixed integer $n \geq 5$.

2. MINIMUM NEIGHBORHOOD FIRST ZAGREB INDEX OF UNICYCLIC AND BI-CYCLIC GRAPHS

We provide two transformations which will reduce the neighborhood first Zagreb index as follows:

Transformation 2.1. Let G be a simple, connected graph and select $u \in V(G)$. G^* is created from G by identifying u along with the vertex v'_j of a simple path $v'_1, v'_2, \dots, v'_n, 1 < j < n$. G^{**} is created from G^* by removing $v'_{j-1}v'_j$ and adding $v'_{j-1}v'_n$.

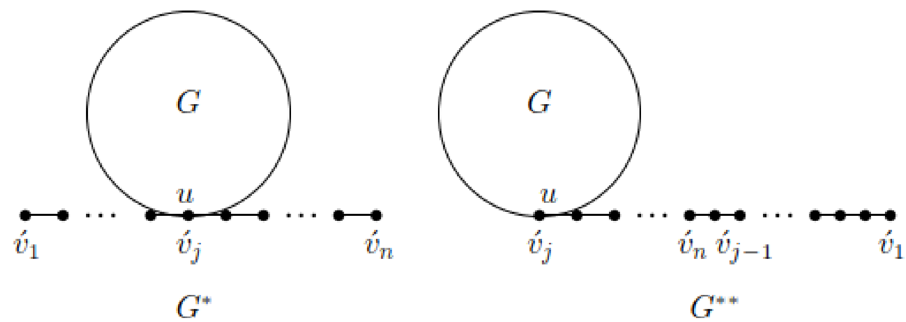


Figure 1: Graphs $G^*$and G^{**} (used within the Transformation 2.1).

Lemma 2.1. Suppose G^{**} and G^* be the graphs as in Transformation 2.1. Then $\mathrm{NM}_1(G^*) > \mathrm{NM}_1(G^{**})$.

Proof. Choose $u(= v\acute{}_j) \in V\,(G),\ d(u) \geq 4$ and $N_G(u) = \{u_1, u_2\}$ and $N_{G^*}(u) \backslash N_G(u) = \{v\acute{}_{j-1}, v\acute{}_{j+1}\}$. There will be four cases regarding the length of the path.

Case I: If $j = 2$ and $n = 3$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I} = \left[\left(\sum_{\Re \epsilon N_G(u_i)} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_G(u_i)} d(\Re) - 1\right)^2\right] + d(u)^2 - 4 > 0.$$

Since $d(u)^2 - 4 \geq 12$ for $d_G(u) \geq 4$.

Case II: If $j = 2$ and $n > 3$.

Sub-Case II(a): If $j = 2$ and $n = 4$,

$$\mathrm{NM}_1(\mathrm{G}^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_G(u_i)} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_G(u_i)} d(\Re) - 1\right)^2\right]$$

$$+ d(u)^2 - 9 > 0.$$

Since $d_G(u) \geq 4$.

Sub-Case II(b): If $j = 2$ and $n \geq 5$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_G(u_i)} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_G(u_i)} d(\Re) - 1\right)^2\right]$$

$$+ (\mathrm{d(u)} + 2)^2 - (\mathrm{d(u)} + 1)^2 + \mathrm{d(u)}^2 - 16 > 0.$$

Since $d_G(u) \geq 4$.

Case III: If $j > 2$ and $n = j + 1$.

Sub-Case III(a): If $j = 3$ and $n = j + 1$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(u_i)} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_G(u_i)} d(\Re) - 1\right)^2\right]$$
$$+ d(u)^2 - 9 > 0.$$

Since $d_G(u) \geq 4$.

Sub-Case III(b): If $j \geq 4$ and $n = j + 1$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(u_i)} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_G(u_i)} d(\Re) - 1\right)^2\right]$$
$$+ (d(u) + 2)^2 - (d(u) + 1)^2 + d(u)^2 - 16 > 0.$$

Since $d_G(u) \geq 4$.

Case IV: If $j > 2$ and $n > j + 1$.

Sub-case IV(a): If $j = 3$ and $n = j + 2$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(ui)} d(\Re)\right)^2 - \left(\sum_{(\Re \epsilon N_G(ui))} d(\Re) - 1\right)^2\right]$$
$$+ (d(u) + 1)^2 - 21 > 0.$$

Since $d_G(u) \geq 4$.

Sub-case IV(b): If $j = 3$ and $n = j + 3$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(ui)} d(\Re)\right)^2 - \left(\sum_{(\Re \epsilon N_G(ui))} d(\Re) - 1\right)^2\right]$$
$$+ (d(u) + 2)^2 - 28 > 0.$$

Since $d_G(u) \geq 4$.

Sub-case IV(c): If $j = 4$ and $n = j + 2$,

$$\mathrm{M}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(ui)} d(\Re)\right)^2 - \left(\sum_{(\Re \epsilon N_G(ui))} d(\Re) - 1\right)^2\right]$$
$$+ (d(u)+2)^2 - 28 > 0.$$

Since $d_G(u) \geq 4$.

Sub-case IV(d): If $j \geq 4$ and $n \geq j+3$,
$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_G(ui)} d(\Re)\right)^2 - \left(\sum_{(\Re \epsilon N_G(ui))} d(\Re) - 1\right)^2\right]$$
$$+ (d(u)+2)^2 - (d(u)+1)^2 + (d(u)+2)^2 - 35 > 0.$$

Since $d_G(u) \geq 4$.

■

Transformation 2.2. Let $u, v \in V(G)$. G^* is constructed from G by attaching paths $u_o' u_1' u_2' \dots u_i'$ and $v_o' v_1' v_2' \dots v_j'$ with the vertex $u (= u_o')$ and the vertex $v (= v_o')$, respectively. Construct $G^{**} = G^* - uu_1' + v_j' u_1'$.

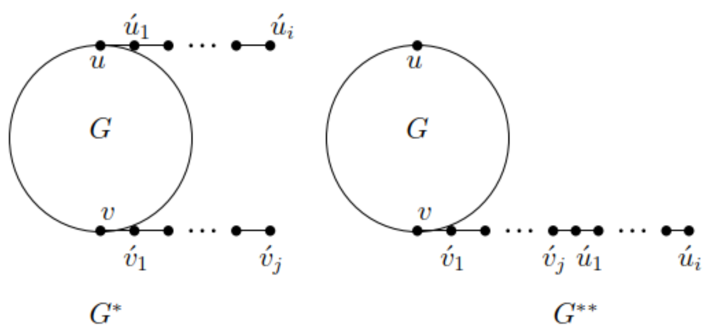


Figure 2: Graphs $G^*$and G^{**} (used within the Transformation 2.2).

Lemma 2.2. G^{**} and G^* be the graphs as appear in Transformation 2.2. If $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$, $i \geq 1$ and $j \geq 1$, then $\mathrm{NM}_1(G^*) > \mathrm{NM}_1(G^{**})$.

Proof. Bearing in mind the assumption that $j > 0$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$, there will be five cases regarding the position (location) of u and v.

Case I: If $uv \in E(G^*)$ and $N_{G^*}(u) \cap N_{G^*}(v) = \varphi$.

Sub-Case I(a): If $i = 1$ and $j = 1$,
$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ d(u)^2 + d(v)^2 - (d(v)+1)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $\mathrm{d(u)}^2 + \mathrm{d(v)}^2 \geq 2\mathrm{d(v)}^2 \geq (d(v)+1)^2$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case I(b): If $i = 1$ and $j > 1$. When $i = 1$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 9\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ d(u)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $d(u)^2 + (d(v)+1)^2 \geq \mathrm{d(v)}^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 1$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 8\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ d(u)^2 - 8 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ *and* $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case I(c): If $i > 1$ and $j = 1$. When $i = 2$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 9\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ d(v)^2 + (d(u)+1)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$, $\mathrm{d(v)}^2 + (d(u)+1)^2 \geq \mathrm{d(v)}^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ d(v)^2 + (d(u)+2)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ *and s* $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case I(d): If $i > 1$ and $j > 1$. When $i = 2$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 21\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+ (d(u)+1)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7, (d(u)+1)^2 + (d(v)+1)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.When $i \geq 3$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+ (d(u)+2)^2 - (d(v)+2)^2 + (d(v)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 2$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 12\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+ (d(u)+1)^2 - 16 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 19\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v, \acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+ (d(u)+2)^2 - 16 > 0$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Case II: If $uv \in E(G^*)$ and $|N_{G^*}(u) \cap N_{G^*}(v)| = 1$.

Sub-Case II(a): If $i = 1$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$$

$$+\left[\left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+d(u)^2 + d(v)^2 - (d(v)+1)^2 > 0.$$

Since $\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $d(u)^2 + d(v)^2 \geq 2d(v)^2 \geq (d(v)+1)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case II(b): If $i = 1$ and $j > 1$. When $i = 1$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 9\right]$$
$$+\left[\left(\sum_{\gamma'_\alpha\epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha\epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+\left[\left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+d(u)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $d(u)^2 + (d(v)+1)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.When $i = 1$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 8\right]$$
$$+\left[\left(\sum_{\gamma'_\alpha\epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha\epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+\left[\left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+d(u)^2 - 8 > 0.$$

Since $\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case II (c): If $i > 1$ and $j = 1$. When $i = 2$ and $j = 1$,

$NM_1(G^*) - NM_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 9\right]$$
$$+\left[\left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+1)^2 + d(v)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$, $(d(u)+1)^2 + d(v)^2 \geq d(v)^2(d(u)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+\left[\left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re\epsilon N_{G^*}(\gamma_\alpha\setminus\{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+2)^2 + d(v)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha\epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case II (d): If $i > 1$ and $j > 1$. When $i = 3$ and $j = 2$,

$NM_1(G^*) - NM_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 21\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+1)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$, $(d(u)+1)^2 + (d(v)+1)^2 \geq 2(d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+2)^2 + (d(v)+1)^2 - (d(v)+2)^2 - 12 > 0.$$

Since $\sum_{x \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 2$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.When $i \geq 3$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 19\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) - 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+2)^2 - 16 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 7$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Case III: If $uv \notin E(G^*)$ and $N_{G^*}(u) \cap N_{G^*}(v) = \varphi$.

Sub-Case III(a): If $i = 1$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$$

$$+\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)+1\right)^2\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right]$$
$$+d(u)^2 + d(v)^2 - (d(v)+1)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, $d(u)^2 + d(v)^2 \geq 2d(v)^2 \geq (d(v)+1)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. $\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)+1\right)^2\right] + \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right] > 0$.

Sub-Case III(b): If $i = 1$ and $j > 1$. When $i = 1$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-1\right)^2 - 9\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right]$$
$$+d(u)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, $d(u)^2 + (d(v)+1)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.When $i = 1$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-1\right)^2 - 8\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right]$$
$$+d(u)^2 - 8 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case III(c): If $i > 1$ and $j = 1$. When $i = 2$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-2\right)^2 - 9\right]$$
$$+\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)+1\right)^2\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right]$$
$$+(d(u)+1)^2 + d(v)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $(d(u)+1)^2 + d(v)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. $\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)+1\right)^2\right] + \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\acute{u}_1\})} d(\Re)-1\right)^2\right] > 0$. When $i \geq 3$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-2\right)^2 - 16\right]$$

$$+\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+ (d(u)+2)^2 + d(v)^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

$\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right] + \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{v,\acute{u}_1\})} d(\Re) - 1\right)^2\right] > 0.$

Sub-Case III(d): If $i > 1$ and $j > 1$. When $i = 2$ and $j = 2$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 20\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+(d(u)+1)^2 + (d(v)+1)^2 - (d(v)+2)^2 - 1 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) \geq 6, (d(u)+1)^2 + (d(v)+1)^2 \geq 2(d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j = 2$

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+(d(u)+2)^2 - (d(v)+2)^2 + (d(v)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $\mathrm{i} = 2$ and $j \geq 3$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+(d(u)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j \geq 3$,

$$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = I$$
$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 19\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re - 1\right)^2\right]$$
$$+(d(u)+2)^2 - 16 > 0.$$

Since $\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Case IV: If $uv \notin E(G^*)$ and $|N_{G^*}(u) \cap N_{G^*}(v)|$=1.

Sub-case IV(a): If $i = 1$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \left[\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$
$+ \left[\left(\sum_{\gamma'_\alpha \in N_{G^*}(u)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \in N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right] \left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re\right)^2 -\right.$
$\left.\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re - 1\right)^2\right] + d(u)^2 + d(v)^2 - (d(v+1))^2 > 0.$
Since $\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) \geq 5, d_{G^*}(u) \geq d_{G^*}(v) \geq 3$ and $d(u)^2 + d(v)^2 \geq 2d(v)^2 \geq (d(v)+1)^2$.
$\left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re\right)^2 - \left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re - 1\right)^2\right] + \left[\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha)\right)^2 -\right.$
$\left.\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) + 1\right)^2\right] > 0.$

Sub-Case IV(b): If $i = 1$ and $j > 1$. When $i = 1$ and $j = 2$,

$$\begin{aligned}\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) &= \mathrm{I} \\ &= \left[\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 9\right] \\ &+ \left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re\right)^2 - \left(\sum_{\Re \in N_{G^*}(\gamma_\alpha\{ú_1\})} d\Re - 1\right)^2\right] \\ &+ d(u)^2 + (d(v)+1)^2 - (d(v)+2)^2 > 0.\end{aligned}$$

Since $\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, $d(u)^2 + (d(v)+1)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 1$ and $j \geq 3$,

$$\begin{aligned}\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) &= \mathrm{I} \\ &= \left[\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 8\right] \\ &+ \left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re) - 1\right)^2\right] \\ &+ d(u)^2 - 8 > 0.\end{aligned}$$

Since $\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) \geq 5$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case IV(c): If $i > 1$ and $j = 1$. When $i = 2$ and $j = 1$,

$$\begin{aligned}\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) &= \mathrm{I} \\ &= \left[\left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 9\right] \\ &+ \left[\left(\sum_{\gamma'_\alpha \in N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \in N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right] \\ &+ \left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re) - 1\right)^2\right] \\ &+ (d(u)+1)^2 + \mathrm{d}(\mathrm{v})^2 - (d(v)+2)^2 > 0.\end{aligned}$$

Since $\sum_{\gamma_\alpha \in N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $(d(u)+1)^2 + d(v)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.
$\left[\left(\sum_{\gamma'_\alpha \in N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \in N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right] + \left[\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re)\right)^2 -\right.$
$\left.\left(\sum_{\Re \in N_{G^*}(\gamma_\alpha \setminus \{v, ú_1\})} d(\Re) - 1\right)^2\right] > 0.$

When $i \geq 3$ and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+2)^2 + \mathrm{d(v)}^2 - (d(v)+2)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

$\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right] + \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})} d(\Re) - 1\right)^2\right] > 0.$

Sub-case IV(d): if $i > 1$ and $j > 1$. When $i = 2$ and $j = 2$

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 20\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})} d(\Re)\right)^2 - \left(\sum_{\left(\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})\right)} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+1)^2 + (d(v)+1)^2 - (d(v)+2)^2 - 1 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6, d_{G^*}(u) \geq d_{G^*}(v) \geq 3$ and $(d(u)+1)^2 + (d(v)+1)^2 \geq 2(d(v)+1)^2 \geq (d(v)+2)^2$. When $i \geq 3$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})} d(\Re)\right)^2 - \left(\sum_{\left(\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})\right)} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+2)^2 - (d(v)+2)^2 + (d(v)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 2$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})} d(\Re)\right)^2 - \left(\sum_{\left(\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{u'1\})\right)} d(\Re) - 1\right)^2\right]$$
$$+(d(u)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ *and* $j \geq 3$,

$\mathrm{NM}_1(\mathrm{G}^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 19\right]$$
$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \{\acute{u}_1\})} d\Re - 1\right)^2\right]$$

$$+(d(u)+2)^2-16>0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Case V: If $uv \notin E(G^*)$ and $N_{G^*}(u) \cap N_{G^*}(v)| = 2$. $Let\ t_{1,} t_2 \epsilon N_{G^*}(u) \cap N_{G^*}(v)$.

Sub-Case V (a): If i =1 and $j = 1$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 4\right]$$

$$+ \left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)\right)^2 - \left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha) + 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}_{i=1}^2)} d\Re\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}_{i=1}^2)} d\Re - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)_{i=1}^2} d\Re''\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*}(t_i)_{i=1}^2} d\Re'' - 1\right)^2\right]$$

$$+d(u)^2 + d(v)^2 - (d(v)+1)^2 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, $\sum_{\acute{\Re} \epsilon N_{G^*}} (t_i)_{i=1}^{2}\ d\Re'' > \sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)$, $d_{G^*}(u) \geq d_{G^*(v)} \geq 3$ and $d(u)^2 + d(v)^2 \geq 2d(v)^2 \geq (d(u)+1)^2$.

Sub-Case V (b): If $i = 1$ and $j > 1$. When $i = 1$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 9\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}^2{}_{i=1})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}^2{}_{i=1})} d(\Re) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'') - 1\right)^2\right] + d(u)^2$$

$$+(d(v)+1)^2 - (d(v)+2)^2 > 0$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, $d(u)^2 + (d(v)+1)^2 \geq d(v)^2 + (d(v)+1)^2 \geq (d(v)+2)^2$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 1$ and $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 1\right)^2 - 8\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}^2{}_{i=1})} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1, t_i\}^2{}_{i=1})} d(\Re) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'') - 1\right)^2\right]$$

$$+d(u)^2-8>0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 5$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

Sub-Case V (c): If $i>1$ and $j=1$. When $i=2$ and $j=1$,

$\mathrm{NM}_1(G^*)-\mathrm{NM}_1(G^{**})=\mathrm{I}$

$$=\left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2-\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-2\right)^2-9\right]$$
$$+\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)\right)^2-\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)+1\right)^2\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\}^2{}_{i=1})} d(\Re)\right)^2-\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\}^2{}_{i=1})} d(\Re)-1\right)^2\right]$$
$$+\left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')\right)^2-\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')-1\right)^2\right]$$
$$+(d(u)+1)^2+d(v)^2-(d(v)+2)^2>0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'') > \sum_{\gamma'{}_\alpha \epsilon N_{G^*}(u)} d(\gamma'{}_\alpha)$, $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$ and $d(u)^2+(d(u)+1)^2+d(v)^2 \geq d(v)^2+(d(v)+2)^2$. When $i \geq 3$ and $j=1$,

$\mathrm{NM}_1(G^*)-\mathrm{NM}_1(G^{**})=\mathrm{I}$

$$=\left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2-\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-2\right)^2-16\right]$$
$$+\left[\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(u)} d(\gamma'_\alpha)\right)^2-\left(\sum_{\gamma'_\alpha \epsilon N_{G^*}(v)} d(\gamma'_\alpha)+1\right)^2\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\}^2{}_{i=1})} d(\Re)\right)^2-\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\}^2{}_{i=1})} d(\Re)-1\right)^2\right]$$
$$+\left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')\right)^2-\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')-1\right)^2\right]$$
$$+(d(u)+2)^2+d(v)^2-(d(v)+2)^2>0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6, \sum_{\Re'' \epsilon N_{G^*}(,t_i)_{i=1}^2} d(\Re'') > \sum_{\gamma_\alpha' \epsilon N_{G^*}(v)} d(\gamma_\alpha')$, $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$ and $(d(u)+1)^2+d(v)^2 \geq d(v)^2+(d(v)+1)^2 \geq (d(v)+2)^2$.

Sub-Case V (d): If $i>1$ and $j>1$, When $i=2$ and $j=2$,

$\mathrm{NM}_1(G^*)-\mathrm{NM}_1(G^{**})=\mathrm{I}$

$$=\left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2-\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)-2\right)^2-20\right]$$
$$+\left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\})_{i=1}^2} d(\Re)\right)^2-\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \backslash \{\acute{u}_1,t_i\})_{i=1}^2} d(\Re)-1\right)^2\right]$$
$$+\left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')\right)^2-\left(\sum_{\Re'' \epsilon N_{G^*}(t_i)^2{}_{i=1}} d(\Re'')-1\right)^2\right]$$
$$+(d(u)+1)^2+(d(v)+1)^2-(d(v)+2)^2-1>0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, $(d(u)+1)^2 + (d(v)+1)^2 \geq 2(d(v)+1)^2 \geq (d(v)+2)^2$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ and $j = 2$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})_{i=1}^2} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})_{i=1}^2} d(\Re) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(,t_i)_{i=1}^2} d(\Re'')\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*}(,t_i)_{i=1}^2} d(\Re'') - 1\right)^2\right]$$

$$+ (d(u)+2)^2 - (d(v)+2)^2 + (d(v)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i = 2$ *and* $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 16\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})_{i=1}^2} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})_{i=1}^2} d(\Re) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(,t_i)_{i=1}^2} d(\Re'')\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*}(,t_i)_{i=1}^2} d(\Re'') - 1\right)^2\right]$$

$$+ (d(u)+1)^2 - 12 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$, and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$. When $i \geq 3$ *and* $j \geq 3$,

$\mathrm{NM}_1(G^*) - \mathrm{NM}_1(G^{**}) = \mathrm{I}$

$$= \left[\left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha)\right)^2 - \left(\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) - 2\right)^2 - 19\right]$$

$$+ \left[\left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})^2 i=1} d(\Re)\right)^2 - \left(\sum_{\Re \epsilon N_{G^*}(\gamma_\alpha \setminus \{\acute{u}_1, t_i\})^2 i=1} d(\Re) - 1\right)^2\right]$$

$$+ \left[\left(\sum_{\Re'' \epsilon N_{G^*}(t_i) 2_{i=1}} d(\Re'')\right)^2 - \left(\sum_{\Re'' \epsilon N_{G^*(t_i) 2_{i=1}}} d(\Re'') - 1\right)^2\right]$$

$$+ (d(u)+2)^2 - 16 > 0.$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*}(u)} d(\gamma_\alpha) \geq 6$ and $d_{G^*}(u) \geq d_{G^*}(v) \geq 3$.

■

Lemma 2.3. Suppose there exists a path $u'_1, u'_2, \ldots, u'_i$, $i \geq 1$ attached to the vertex $u \in G^*$, where u is identified with the vertex u'_1 and $u'_1 v \in E(G^*)$. Construct $G^{**} = G^* - u'_1 v + u'_i v$, then $\mathrm{NM}_1(G^*) > \mathrm{NM}_1(G^{**})$.

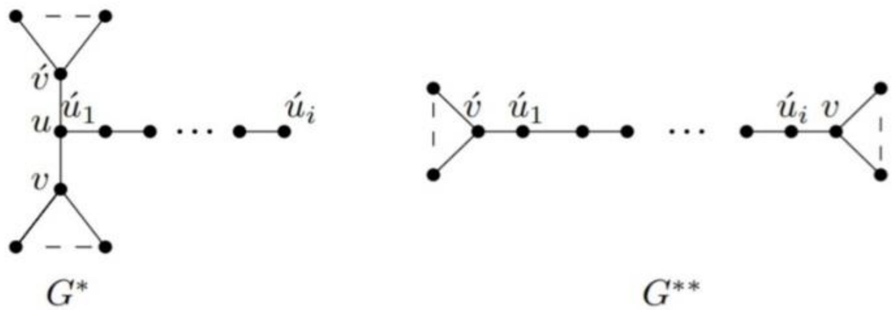


Figure 3: Graphs $G^*$and G^{**} (used within Lemma 2.3).

Proof. Since $d(u) \geq 3$, there will be three cases regarding the length of the path.

Case I: If $i = 1$,

$$\begin{aligned}\mathrm{NM_1(G^*)} - \mathrm{NM_1(G^{**})} = \mathrm{I} \\ &= \Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha)\Big)^2 - \Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) - 1\Big)^2 \\ &+\Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(v)}} d(\gamma'_\alpha)\Big)^2 - \Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(v)}} d(\gamma'_\alpha) - d(u) + 2\Big)^2 \\ &+\Big(\sum_{\Re \epsilon N_{G^*(\gamma_\alpha\{v,u'_2\}}} d(\Re)\Big)^2 - \Big(\sum_{\Re \epsilon N_{G^*(\gamma_\alpha\{v,u'_2\}}} d(\Re) - 1\Big)^2 \\ &+d(u)^2 - (d(u)+1)^2 > 0.\end{aligned}$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) \geq \mathrm{d}(u) + 1$, implies that

$$\left[\Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha)\Big)^2 - \Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) - 1\Big)^2 - 1\right] > (d(u)+1)^2 - d(u)^2.$$

Case II: If $i = 2$,

$$\begin{aligned}\mathrm{NM_1(G^*)} - \mathrm{NM_1(G^{**})} = \mathrm{I} \\ &= \left[\Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha)\Big)^2 - \Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) - 2\Big)^2 - 12\right] \\ &+ \left[\Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(u)}} d(\gamma'_\alpha)\Big)^2 - \Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(u)}} d(\gamma'_\alpha) - d(u) + 2\Big)^2\right] \\ &+ \left[\Big(\sum_{\Re \epsilon N_{G^*(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})}} d(\Re)\Big)^2 - \Big(\sum_{\Re \epsilon N_{G^*(\gamma_\alpha \backslash \{v,\, \acute{u}_1\})}} d(\Re) - 1\Big)^2\right] > 0.\end{aligned}$$

Since $\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) \geq 6$.

Case III: If $i > 2$.

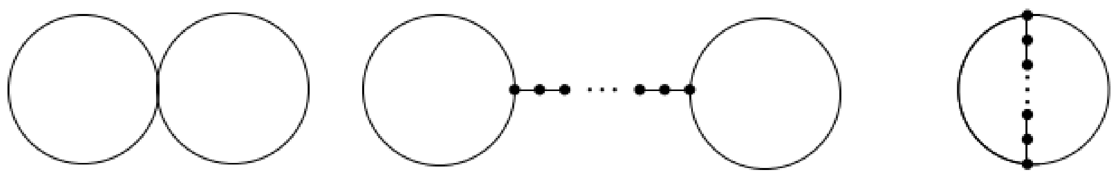
$$\begin{aligned}\mathrm{NM_1(G^*)} - \mathrm{NM_1(G^{**})} = \mathrm{I} \\ &= \left[\Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha)\Big)^2 - \Big(\sum_{\gamma_\alpha \epsilon N_{G^*(u)}} d(\gamma_\alpha) - 2\Big)^2\right] \\ &+ \left[\Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(u)}} d(\gamma'_\alpha)\Big)^2 - \Big(\sum_{\gamma'_\alpha \epsilon N_{G^*(u)}} d(\gamma'_\alpha) - d(u) + 2\Big)^2 - 19\right]\end{aligned}$$

$$+\left[\left(\sum_{\mathfrak{R}\epsilon N_{G^*}(\gamma_\alpha\backslash\{v,\acute{u}_1\})} d(\mathfrak{R})\right)^2-\left(\sum_{\mathfrak{R}\epsilon N_{G^*}(\gamma_\alpha\backslash\{v,\acute{u}_1\})} d(\mathfrak{R})-1\right)^2\right]$$
$$+(d(u)+2)^2-(d(u)+1)^2>0.$$

■

Let U_n^i be the unicyclic graph collection derived by joining a path of length $n-i$ to the cycle C_i of length i. From Lemmas 2.1 and 2.2, we have

Theorem 2.1. Let G^* be an unicyclic graph of order n and girth i. If $G^* \notin U_n^i$, then $\mathrm{NM}_1(\mathrm{G}^*) > \mathrm{NM}_1(U_n^i)$.



D_1 D_2 D_3

Figure 4: Graphs D_1 , D_2 and D_3 (used in Theorem 2.2).

Theorem 2.2. Let C_n be the optimal (minimal) unicyclic graph from the collection of U_n with minimum neighborhood first Zegrab index.

Let D_1 , D_2 and D_3 be the n-vertex bicyclic graphs showed in figure 4. From Lemmas 2.2 and 2.3, it is obvious that bicyclic graph with the minimum neighborhood first Zagreb index is one of the graphs D_1 , D_2 and D_3.

$\mathrm{NM}_1(D_1) = 16n + 128,$

$\mathrm{NM}_1(D_2) = \mathrm{NM}_1(D_3)$

$$= \begin{cases} 16n+102 \ \ if\ uv \in E(D_3) \\ 16n+96 \ \ if\ \ uv \notin E(D_2)\ or\ uv \notin E(D_3)\ and\ \ |N(u)\cap N(v)N(u)\cap N(v)| = 1; \\ 16n+96 \ \ if\ \ uv \notin E(D_2)\ or\ uv \notin E(D_3)\ and\ \ N(u)\cap N(v) = \phi, \end{cases}$$

So, the above findings brings closer to our extremal result that is stated below.

Theorem 2.3. The optimal (minimal) bicyclic graphs of order n with minimum neighborhood first Zagreb index are the graphs D_2 and D_3, in which non-adjacent vertices of degree three exists without any common neighbor.

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