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The Schultz Index for Product Graphs

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ABSTRACT

Among the binary operations made with graphs, the cartesian, corona, and lexicographic are three well-known products, as well as the cartesian sum. Topological indices are graph invariants used to describe graphs associated with molecules, one of these is the Schultz index, which can be obtained as $\sum_{u \neq v} (\deg u + \deg v) d(u, v)$, where the sum runs over all pairs of distinct vertices of the graph. In this paper, we give explicit expressions for the Schultz index of cartesian and corona, with alternative proofs to those given in the literature, as well as for lexicographic product and the cartesian sum, all of these formulas involve order and size of factors, additionally, the first three involve both Wiener and Schultz indices of factors, corona and lexicographic also involve Zagreb index and the last one just Zagreb.

1. INTRODUCTION

A number that can be used to characterize the graph associated with a molecule is called a *topological index,* this number is also known as a *graph invariant* by graph theorists [20]. It is said that the first of these graph invariants was the number of carbon atoms in hydrocarbon molecules, which is precisely the number of vertices in a graph of a molecule

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with hydrogens suppressed, this number was used around 1842 [6]. However, the term "topological index" was used for the first time in 1971 by Hosoya [13], in his paper, he defined the invariant Z in three steps and called it *topological index*.

Among the best known and studied indices are the Wiener, Zagreb, Randić, Hosoya, Balaban, and Schultz. These indices are given by formulas that involve properties of the graph, for example, degrees of vertices, number of edges, cyclomatic number, distances between vertices or matchings.

Generally, topological indices are correlated with some physical or chemical properties of a molecule, the first one for being used this way was the Wiener index [21], proposed in 1947, back then called "the path number", defined as the sum of distances between two carbon atoms of a molecule and used to compute the boiling points of alkanes. Clearly, this index was not defined in terms of graph theory. In the same way, Hosoya [13] pointed out that this number could be obtained as half of the sum of the entries of the distance matrix of the graph associated with the molecule. This index has been studied for a long time since its first appearance from different perspectives, for example, in [15] it is compared with the Szeged index, in [2] it is computed for the semi-complete product and [14] shows an explicit formula for it of Dutch windmill graphs.

Another index widely studied, and that has a close relationship with that of Wiener, is the Schultz index. This one was proposed in 1989 by Harry P. Schultz [19], and its original purpose was to give a technique for determining a molecular topological index to describe the structure of alkanes. Later, Gutman [10] studied this number, its relation with the Wiener index, called it *Schultz index* (also called *degree distance* in the literature) and defined a modification of it. As the Wiener index, Schultz index has been broadly studied and compared with other indices, for example, in [16] an explicit relation between Wiener and Schultz indices is found for acyclic graphs, in [3] is analyzed this index under the join and the strong product of graphs and in [4] an extension of the cut method is applied to this index.

It is known that there are some binary operations (products) between graphs: cartesian, strong, lexicographic, corona, and tensor, to name a few; and these have been studied from several perspectives, for example, [11] is a book which is a standard reference on graph products since it deals with algebraic aspects, some algorithms, and invariants; in [9] the Wiener index is computed for the cartesian product; [15] gives a formula for the Szeged index of the cartesian product; in [5] complete information about the spectrum and the Laplacian spectrum of the corona product is given; [22] shows the Szeged, vertex PI, first and second Zagreb indices for the corona product, in [8] a characterization for the hyperbolicity of lexicographic product of two graphs is given in terms of the factors and in [17] the chromatic number and the circular chromatic number for the cartesian sum is investigated. Even though the significance of most graph products in chemistry is not apparent until now, some of them at times are used in problems in chemical reactivity [6], moreover, there are examples of chemical structures which can be seen as products: the alkane C_3H_6 is the corona of P_3 and E_2, the cyclohexane C_6H_{12} is the corona of C_6 and E_2, the nanotube $TUC_4(m,n)$ is also the cartesian product of P_m and P_2, and C_m and P_2 and a fence and closed fence are the lexicographic product of P_m and P_2, and C_m and P_2, respectively [7].

In this paper, we give explicit formulas for the Schultz index of the cartesian, corona, and lexicographic product graphs as well as of cartesian sum, besides we compute explicitly the Wiener and Schultz indices for some graph families.

2. PRELIMINARIES

In this section we set some notation and concepts used throughout the paper, these are taken from [12] and [20]. By *graph* we mean a simple graph with no loops and it is denoted by $\Gamma = (V, E)$, where V and E are the *vertices* and *edges* sets, respectively, $|V|$ is called the *order* of Γ and $|E|$ its *size*. Let $\Gamma = (V, E)$ be a graph:

- for $v \in V$, $\deg v$ denotes its *degree*, that is, $\deg v = |\{x \in V : xv \in E\}|$. If $\deg v = k$, for all $v \in V$, then we say that Γ is k-*regular*;
- for $u, v \in V$, a *walk* from u to v is a sequence of vertices $u = x_0, x_1, \dots, x_{r-1}, x_r = v$ such that $x_i x_{i+1} \in E$, for $i = 0,1, \dots, r-1$, this sequence is called a $u - v$ *walk*, a $u - v$ walk which does not repeat vertices is called a *path*;
  - if for any $u, v \in V$ there is a $u - v$ walk, Γ is called *connected*;
  - for Γ connected and $u, v \in V$, the *distance* between u and v is
  $$d(u, v) = \min\{\text{length of u} - \text{v walks}\},$$
  where the length of a $u - v$ walk is the number of edges in such a walk;
  - for Γ connected, the *diameter* of Γ is
  $$diam\Gamma = \max\{d(u, v) : u, v \in V\};$$
  - the *Zagreb* index of Γ is defined as
  $$M_1(\Gamma) = \sum_{v \in V} \deg^2 v.$$

Next we recall the definition of some known families of graphs. Let n be a positive integer, then

- the *path* graph is defined as the graph $P_n = (V, E)$ with
$$V = \{v_1, \dots, v_n\} \quad \text{and} \quad E = \{v_i v_{i+1} : i = 1, \dots, n-1\};$$
- for $n \geq 3$, the *cycle* graph is defined as $C_n = (V, E)$, where
$$V = \{v_1, \dots, v_n\} \quad \text{and} \quad E = \{v_i v_{i+1} : i = 1, \dots, n\},$$
where $n + 1$ is taken as 1;
- the *star* graph is $S_n = (V, E)$ with
$$V = \{v_0, v_1, \dots, v_n\} \quad \text{and} \quad E = \{v_0 v_i : i = 1, \dots, n\};$$
- the *wheel* graph is the graph $W_n = (V, E)$ with
$$V = \{v_0, v_1, \dots, v_n\} \quad \text{and} \quad E = \{v_0 v_i, v_i v_{i+1} : i = 1, \dots, n\},$$

where $n+1$ is taken as 1;

• the *complete* graph is the graph $\mathrm{K}_n = (V, E)$ with

$$V = \{v_1, \dots, v_n\} \quad \text{and} \quad E = \{v_i v_j : i, j = 1, \dots, n \text{ and } i \neq j\}.$$

3. THE SCHULTZ INDEX

In this section, the Wiener and Schultz indices are defined and formulas for these applied to some known families are stated.

Definition 1. [20] *Let $\Gamma = (V, E)$ be a graph, the Wiener index of Γ is defined by the formula $W(\Gamma) = \sum_{u \neq v} d(u, v)$, where the sum runs over all pairs of distinct vertices of* Γ.

Note that if $V = \{v_1, \dots, v_n\}$, the Wiener index of Γ can be written as

$$W(\Gamma) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(v_i, v_j) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(v_i, v_j).$$

The following proposition states the precise value of the Wiener index for some families of graphs.

Proposition 3.2. *Let n be a positive integer, then*

1. $W(\mathrm{P}_n) = n(n-1)(n+1)/6$;
2. $W(\mathrm{C}_n) = \begin{cases} n^3/8, & if \quad n = 2k; \\ (n^3 - n)/8, & if \quad n = 2k-1; \end{cases}$
3. $W(\mathrm{S}_n) = n^2$;
4. $W(\mathrm{W}_n) = n(n-1)$;
5. $W(\mathrm{K}_n) = n(n-1)/2$.

Proof. All these formulas follow from the very definition, we just show the proof for paths and wheels. For paths we may observe that $d(v_i, v_j) = j - i$, for $i \leq j$, thus,

$$\begin{aligned} W(\mathrm{P}_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(v_i, v_j) \\ &= (1 + 2 + \cdots + (n-1)) + (1 + 2 + \cdots + (n-2)) + \cdots + (1+2) + 1 \\ &= \frac{(n-1)n}{2} + \frac{(n-2)(n-1)}{2} + \cdots + \frac{2(3)}{2} + \frac{1(2)}{2} \\ &= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right) \\ &= \frac{1}{6} n(n-1)(n+1). \end{aligned}$$

And for wheels we have $d(v_{\cdot}, v_i) = 1$, for $i = 1, \dots, n$, and

$$d(v_i, v_j) = \begin{cases} 1, & if \quad j = i+1; \\ 2, & otherwise\,, \end{cases}$$

thus,

$$
\begin{aligned}
W(\mathrm{W}_n) &= \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} d(v_i, v_j) \\
&= (1+\cdots+1) + (1+2+\cdots+2+1) + (1+2+\cdots+2) + \cdots + (1+2) + 1 \\
&= n + (2 + 2(n-3)) + (1 + 2(n-3)) + \cdots + (1 + 2(1)) + 1 \\
&= (2n-2) + 2\left(\frac{(n-2)(n-1)}{2}\right) \\
&= n^2 - n.
\end{aligned}
$$

Next, the definition of the Schultz index is given. It can be seen that this index is related to that of Wiener, indeed Lemma 3.1 shows one explicit relation between them.

Definition 2. [10] *Let $\Gamma = (V, E)$ be a graph, the Schultz index of Γ is defined by the formula* $S(\Gamma) = \sum_{u \neq v} (\deg u + \deg v) d(u, v)$, *where the sum runs over all pairs of distinct vertices of* Γ.

Note that if $V = \{v_1, \ldots, v_n\}$, then the Schultz index of Γ can be computed as follows

$$
\begin{aligned}
S(\Gamma) &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\deg v_i + \deg v_j) d(v_i, v_j) \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\deg v_i + \deg v_j) d(v_i, v_j).
\end{aligned}
$$

The next lemma shows an explicit relation between Wiener and Schultz indices, and its proof follows from the definition.

Lemma 3.1. *Let Γ be a k-regular graph, then $S(\Gamma) = 2kW(\Gamma)$.*

Now, we state the value of the Schultz index for some families of graphs.

Proposition 3.2. *Let n be a positive integer, then*

1. $S(\mathrm{P}_n) = n(n-1)(2n-1)/3$;
2. $S(\mathrm{C}_n) = \begin{cases} n^3/2, & if \;\; n = 2k; \\ (n^3 - n)/2, & if \;\; n = 2k-1; \end{cases}$
3. $S(\mathrm{S}_n) = n(3n-1)$;
4. $S(\mathrm{W}_n) = n(7n-9)$;
5. $S(\mathrm{K}_n) = n(n-1)^2$.

Proof. These formulas are not difficult to prove, we just show the proof for those of paths and cycles. For paths, since every vertex of P_n has degree 2, but v_1 and v_n, we get

$$
\begin{aligned}
S(\mathrm{P}_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\deg v_i + \deg v_j) d(v_i, v_j) \\
&= 3(n-2)(n-1) + 2(n-1) \\
&\quad + 4\left(\frac{(n-3)(n-2)}{2} + \frac{(n-4)(n-3)}{2} + \cdots + \frac{1(2)}{2}\right)
\end{aligned}
$$

$$\begin{aligned}
&= 3(n-2)(n-1) + 2(n-1) \\
&\quad +2((n-2)^2 - (n-2) + (n-3)^2 - (n-3) + \cdots + 2^2 - 2) \\
&= 3(n-2)(n-1) + 2(n-1) \\
&\quad + 2\left(\frac{(n-2)(n-1)(2n-3)}{6} - \frac{(n-2)(n-1)}{2}\right) \\
&= \frac{n(n-1)(2n-1)}{3}.
\end{aligned}$$

And for cycles note that C_n is a 2-regular graph, thus, by Lemma 3.1. we have $S(C_n) = 4W(C_n)$, hence,

$$S(C_n) = \begin{cases} \frac{n^3}{2}, & if \quad n = 2k; \\ \frac{n^3 - n}{2}, & if \quad n = 2k-1. \end{cases}$$

There is another relation between Wiener and Schultz indices for trees, which is given in [10], it says that if Γ is a tree with n vertices, then

$$S(\Gamma) = 4W(\Gamma) - n(n-1).$$

We obtain immediately the following corollary which relates these indices for the families we have considered.

Corollary 3.1.

1. $S(P_n) = 4W(P_n) - n(n-1)$;
2. $S(C_n) = 4W(C_n)$;
3. $S(S_n) = 4W(S_n) - n(n+1)$;
4. $S(W_n) = 7W(W_n) - 2n$;
5. $S(K_n) = 2(n-1)W(K_n)$.

4. THE SCHULTZ INDEX FOR GRAPH PRODUCTS

In this section, explicit formulas are given for the Schultz index of the cartesian, corona, and lexicographic products as well as of the cartesian sum, the first three involve both Wiener and Schultz indices of factors, corona and lexicographic also involve Zagreb index and just Zagreb for the last one. It is worth mentioning that the Schultz index for the cartesian and corona product has been already computed in [18] and [1], respectively, nevertheless, we present the formulas and give alternative proofs for them. From now on the graphs we mention are connected, unless we say otherwise.

4.1 CARTESIAN PRODUCT

Definition 3. [11] *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. The cartesian product of Γ_1 and Γ_2 is defined as the graph $\Gamma = (V, E)$ given by $V = V_1 \times V_2$ and $E = \{(u,y)(u,y'): yy' \in E_2\} \cup \{(x,v)(x',v): xx' \in E_1\}$.* We denote this graph by $\Gamma_1 \times \Gamma_2$.

From the definition, we may observe immediately that this operation commutes, that is, $\Gamma_1 \times \Gamma_2 \cong \Gamma_2 \times \Gamma_1$, Figure 1 shows a representation of the cartesian product of P_3 and C_4. Moreover, it is worthy to note that for any vertices (u, v) and (x, y) in $\Gamma_1 \times \Gamma_2$ the following relation holds

$$\deg(u, v) = \deg u + \deg v.$$

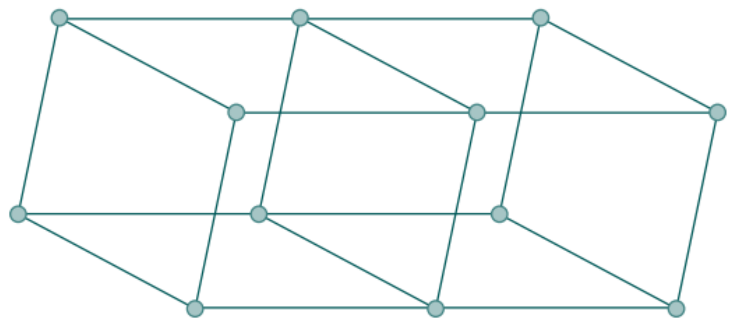

Figure 1: A representation of $P_3 \times C_4$.

The following lemma is proved in [11] and relates the distance in the Cartesian product with that of the factors.

Lemma 4.1. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ and let (u, v) and (x, y) be two vertices of Γ, then*

$$d((u, v), (x, y)) = d(u, x) + d(v, y).$$

Theorem 4.1. *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs, with $V_1 = \{u_1, \ldots, u_{n_1}\}$, $V_2 = \{v_1, \ldots, v_{n_2}\}$, $|E_1| = m_1$ and $|E_2| = m_2$, then $S(\Gamma_1 \times \Gamma_2) = {n_2}^2 S(\Gamma_1) + {n_1}^2 S(\Gamma_2) + 4n_2 m_2 W(\Gamma_1) + 4n_1 m_1 W(\Gamma_2)$.*

Proof. First note that

$$S(\Gamma_1 \times \Gamma_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s>j}^{n_2} (\deg(u_i, v_j) + \deg(u_i, v_s)) d((u_i, v_j), (u_i, v_s))$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r>i}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg(u_i, v_j) + \deg(u_r, v_s)) d((u_i, v_j), (u_r, v_s)),$$

that is, we may compute the Schultz index for this product by dividing the sum into two parts: for $i = r$ and for $i \neq r$. For $i = r$ we have

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (\deg(u_i, v_j) + \deg(u_i, v_s)) d((u_i, v_j), (u_i, v_s))$$
$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (2\deg u_i + \deg v_j + \deg v_s) d(v_j, v_s)$$
$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (2\deg u_i d(v_j, v_s) + (\deg v_j + \deg v_s) d(v_j, v_s))$$
$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} 2\deg u_i d(v_j, v_s) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (\deg v_j + \deg v_s) d(v_j, v_s)$$

$$= \sum_{i=1}^{n_1} 2\text{deg}u_i \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} d(v_j, v_s) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (\text{deg}v_j + \text{deg}v_s) d(v_j, v_s)$$
$$= \sum_{i=1}^{n_1} 2\text{deg}u_i W(\Gamma_2) + \sum_{i=1}^{n_1} S(\Gamma_2)$$
$$= 4m_1 W(\Gamma_2) + n_1 S(\Gamma_2).$$

While for $i \neq r$ we get

$$\sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}(u_i, v_j) + \text{deg}(u_r, v_s)) d((u_i, v_j), (u_r, v_s))$$
$$= \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}u_i + \text{deg}v_j + \text{deg}u_r + \text{deg}v_s)(d(u_i, u_r) + d(v_j, v_s))$$
$$= \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} ((\text{deg}u_i + \text{deg}u_r) d(u_i, u_r) + (\text{deg}u_i + \text{deg}u_r) d(v_j, v_s)$$
$$+ (\text{deg}v_j + \text{deg}v_s) d(u_i, u_r) + (\text{deg}v_j + \text{deg}v_s) d(v_j, v_s))$$
$$= \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}u_i + \text{deg}u_r) d(u_i, u_r)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}u_i + \text{deg}u_r) d(v_j, v_s)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}v_j + \text{deg}v_s) d(u_i, u_r)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}v_j + \text{deg}v_s) d(v_j, v_s)$$
$$= \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} (\text{deg}u_i + \text{deg}u_r) d(u_i, u_r)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} (\text{deg}u_i + \text{deg}u_r) \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} d(v_j, v_s)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} d(u_i, u_r) \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}v_j + \text{deg}v_s)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}v_j + \text{deg}v_s) d(v_j, v_s)$$
$$= \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} S(\Gamma_1) + \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} (\text{deg}u_i + \text{deg}u_r) 2W(\Gamma_2)$$
$$+ W(\Gamma_1) \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\text{deg}v_j + \text{deg}v_s) + \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} 2S(\Gamma_2)$$
$$= {n_2}^2 S(\Gamma_1) + 4(n_1 - 1) m_1 W(\Gamma_2) + 4n_2 m_2 W(\Gamma_1) + n_1(n_1 - 1) S(\Gamma_2).$$

Taking the summation of these computations we obtain the result.

4.2 CORONA PRODUCT

Definition 4. [12] *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. The corona product of Γ_1 and Γ_2 is the graph $\Gamma = (V, E)$ given by taking one copy of Γ_1 and $|V_1|$ copies of Γ_2, joining the r-th vertex of Γ_1 to every vertex in the r-th copy of Γ_2. In symbols,*

$$V = (V_1 \times \{v_0\}) \cup (V_1 \times V_2) \quad \text{and}$$
$$E = E_1 \cup \{(u_r, v_i)(u_r, v_j) : v_i v_j \in E_2\} \cup \{(u_r, v_0)(u_r, v_i) : v_i \in V_2\},$$

where $V_1 \times \{v_0\}$ are the vertices of the copy of Γ_1 and $V_1 \times V_2$ are those of the $|V_1|$ copies of Γ_2. We denote the corona product of these graphs by $\Gamma_1 \odot \Gamma_2$.

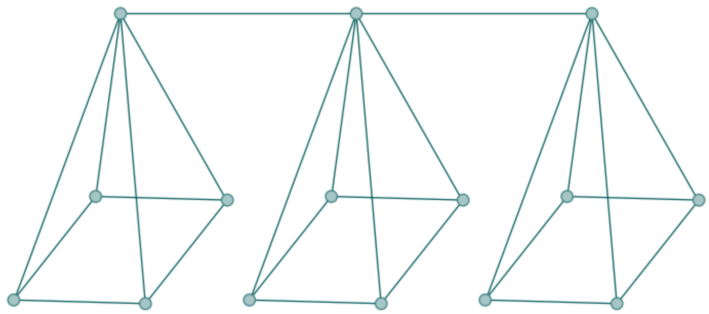

Figure 2: A representation of $P_3 \odot C_4$.

We may note that, in general, this operation is not commutative, that is, $\Gamma_1 \odot \Gamma_2 \ncong \Gamma_2 \odot \Gamma_1$, Figure 2 shows a representation of the corona product of P_3 and C_4. Moreover, it is straightforward to verify that for a vertex (u, v) of $\Gamma_1 \odot \Gamma_2$, we have

$$\deg(u,v) = \begin{cases} \deg u + |V_2|, & if \quad v = v_0; \\ \deg v + 1, & otherwise. \end{cases}$$

Observe that if a and b are two vertices of $\Gamma_1 \odot \Gamma_2$, then exactly one of the following cases holds.

- a and b are in the copy of Γ_1;
- a is in the copy of Γ_1 and b in the j-th copy of Γ_2 (the one which makes a cone with the j-th vertex of Γ_1);
- a is in the i-th copy of Γ_2 and b in the j-th copy of Γ_2;
- a and b are in the i-th copy of Γ_2.

The following result follows easily considering these cases.

Lemma 4.2. *Let $\Gamma = \Gamma_1 \odot \Gamma_2$ and consider $a = (u_i, v_r)$ and $b = (u_j, v_s)$ two vertices of Γ, then*

$$d(a,b) = \begin{cases} d(u_i, u_j), & for\ case\ i\,; \\ d(u_i, u_j) + 1, & for\ case\ ii\,; \\ d(u_i, u_j) + 2, & for\ case\ iii\,; \\ \min\{d(v_r, v_s), 2\}, & for\ case\ iv\,. \end{cases}$$

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs, with $V_1 = \{u_1, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, \ldots, v_{n_2}\}$ and consider $\Gamma = (V, E)$ as the corona product of Γ_1 and Γ_2. Note that we may compute $S(\Gamma)$ by calculating some sums separately, considering the cases for where are taken the pairs of vertices, as follows.

$$\begin{aligned} S(\Gamma) &= \textstyle\sum_{a \neq b} (\deg a + \deg b) d(a,b) \\ &= \textstyle\sum_{i=1}^{n_1 - 1} \sum_{r=i+1}^{n_1} (\deg(u_i, v_0) + \deg(u_r, v_0)) d((u_i, v_0), (u_r, v_0)) \end{aligned}$$

$$+\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg(u_i,v_0)+\deg(u_r,v_s))d((u_i,v_0),(u_r,v_s))$$
$$+\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}(\deg(u_i,v_j)+\deg(u_r,v_0))d((u_i,v_j),(u_r,v_0))$$
$$+\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg(u_i,v_j)+\deg(u_r,v_s))d((u_i,v_j),(u_r,v_s))$$
$$+\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg(u_i,v_j)+\deg(u_i,v_s))d((u_i,v_j),(u_i,v_s)).$$

Thus, the first part is determined for the case (i), the second and third for case (ii), the fourth for (iii) and the fifth for (iv).

For the first part of the sum we have

$$\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}(\deg(u_i,v_0)+\deg(u_r,v_0))d((u_i,v_0),(u_r,v_0))$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}(\deg u_i+\deg u_r+2n_2)d(u_i,u_r)$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}(\deg u_i+\deg u_r)d(u_i,u_r)+2n_2d(u_i,u_r)$$
$$=S(\Gamma_1)+2n_2W(\Gamma_1).$$

For the second part

$$\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg(u_i,v_0)+\deg(u_r,v_s))d((u_i,v_0),(u_r,v_s))$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg u_i+\deg v_s+n_2+1)(d(u_i,u_r)+1)$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}\deg u_i(d(u_i,u_r)+1)$$
$$+\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}\deg v_s(d(u_i,u_r)+1)$$
$$+(n_2+1)\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(d(u_i,u_r)+1)$$
$$=n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_i d(u_i,u_r)$$
$$+n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_i\quad\sum_{s=1}^{n_2}\deg v_s\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}(d(u_i,u_r)+1)$$
$$+(n_2+1)\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(d(u_i,u_r)+1)$$
$$=n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_i d(u_i,u_r)+n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_i+\sum_{s=1}^{n_2}\deg v_s W(\Gamma_1)$$
$$+\frac{n_1(n_1-1)}{2}\sum_{s=1}^{n_2}\deg v_s+n_2(n_2+1)W(\Gamma_1)+\frac{n_2(n_2+1)n_1(n_1-1)}{2}.$$

For the third one

$$\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}(\deg(u_i,v_j)+\deg(u_r,v_0))d((u_i,v_j),(u_r,v_0))$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}(\deg v_j+1+\deg u_r+n_2)(d(u_i,u_r)+1)$$
$$=\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}\deg u_r(d(u_i,u_r)+1)+\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}\deg v_j(d(u_i,u_r)+1)$$
$$+(n_2+1)\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\sum_{j=1}^{n_2}(d(u_i,u_r)+1)$$
$$=n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_r d(u_i,u_r)+n_2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1}\deg u_r+\sum_{j=1}^{n_2}\deg v_j W(\Gamma_1)$$
$$+\sum_{j=1}^{n_2}\frac{n_1(n_1-1)\deg v_j}{2}+n_2(n_2+1)W(\Gamma_1)+\frac{n_2(n_2+1)n_1(n_1-1)}{2}.$$

Now, for the fourth part

$$\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg(u_i,v_j)+\deg(u_r,v_s))d((u_i,v_j),(u_r,v_s))$$
$$=\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg v_j+\deg v_s+2)(d(u_i,u_r)+2)$$
$$=\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}(\deg v_j+\deg v_s)d(u_i,u_r)$$

$+2\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2}\left(\deg v_j+\deg v_s\right)$
$+2\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2} d(u_i,u_r) + \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2}\sum_{r=i+1}^{n_1}\sum_{s=1}^{n_2} 4$
$= 2n_2\sum_{j=1}^{n_2}\deg v_j\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1} d(u_i,u_r)+n_1(n_1-1)(2n_2)\sum_{j=1}^{n_2}\deg v_j$
$+2{n_2}^2\sum_{i=1}^{n_1-1}\sum_{r=i+1}^{n_1} d(u_i,u_r)+{n_2}^2 n_1(n_1-1)2$
$= 2n_2 W(\Gamma_1)\sum_{j=1}^{n_2}\deg v_j+2n_1n_2(n_1-1)\sum_{j=1}^{n_2}\deg v_j+2{n_2}^2W(\Gamma_1)+2{n_2}^2n_1(n_1-1).$

Finally, for the last part

$\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg(u_i,v_j)+\deg(u_i,v_s))d((u_i,v_j),(u_i,v_s))$
$=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s+2)(\min\{d(v_j,v_s),2\})$
$+\sum_{i=1}^{n_1}\sum_{s=1}^{n_2}(\deg u_i+n_2+\deg v_s+1)$
$=n_2\sum_{i=1}^{n_1}\deg u_i+n_1\sum_{s=1}^{n_2}\deg v_s+n_1n_2(n_2+1)$
$+n_1\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s+2)(\min\{d(v_j,v_s),2\})$
$=n_2\sum_{i=1}^{n_1}\deg u_i+n_1\sum_{s=1}^{n_2}\deg v_s+n_1n_2(n_2+1)$
$+n_1\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s)(\min\{d(v_j,v_s),2\})$
$+2n_1\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}\min\{d(v_j,v_s),2\}.$

Observe that

$$\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s)\min\{d(v_j,v_s),2\}=\sum_{v_jv_s\in E_2}(\deg v_j+\deg v_s)$$
$$+\sum_{v_jv_s\notin E_2}2(\deg v_j+\deg v_s),$$

since, $d((u_i,v_j),(u_i,v_s))=1$, for $v_jv_s\in E_2$, and $d((u_i,v_j),(u_i,v_s))=2$, when v_j and v_s are not adjacent, then v_j is $\deg v_j$ times in the first sum, for $j=1,\dots,n_2$, thus,

$$\sum_{v_jv_s\in E_2}(\deg v_j+\deg v_s)=\sum_{j=1}^{n_2}\deg^2 v_j$$

and in the second sum v_j is $n_2-(\deg v_j+1)$ times, which implies

$$\sum_{v_jv_s\notin E_2}(\deg v_j+\deg v_s)2=\sum_{j=1}^{n_2}2\deg v_j(n_2-(\deg v_j+1)),$$

obtaining

$\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s)\min\{d(v_j,v_s),2\}$
$=\sum_{j=1}^{n_2}\deg^2 v_j+\sum_{j=1}^{n_2}2\deg v_j(n_2-(\deg v_j+1))$
$=\sum_{j=1}^{n_2}\deg^2 v_j+2n_2\sum_{j=1}^{n_2}\deg v_j-2\sum_{j=1}^{n_2}\deg^2 v_j-2\sum_{j=1}^{n_2}\deg v_j$
$=4n_2m_2-\sum_{j=1}^{n_2}\deg^2 v_j-4m_2=4m_2(n_2-1)-M_1(\Gamma_2).$

We may note that

$$\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}2\min\{d(v_j,v_s),2\}=2(n_2(n_2-1)-m_2),$$

since each of the $n_2(n_2-1)/2$ pairs of the addends are at distance 2, but those m_2 which are adjacent and, obviously, are at distance 1. Hence, the last part can be written as

$2n_2m_1+2n_1m_2+n_1n_2(n_2+1)+4n_1m_2(n_2-1)-n_1M_1(\Gamma_2)+2n_1(n_2(n_2-1)-m_2).$

Taking the summation of the five parts we get

$S(\Gamma)=(n_2+1)S(\Gamma_1)+(2n_2+2n_2(n_2+1)+2n_2^2)W(\Gamma_1)$

$$+(4+4n_2)m_2W(\Gamma_1)+2m_2(n_1(n_1-1)+2n_2n_1(n_1-1)+n_1)$$
$$+2m_1(n_2(n_1-1)+n_2)+n_2n_1(n_2+1)(n_1-1)+2n_2{}^2n_1(n_1-1)$$
$$+n_2n_1(n_2+1)+4n_1m_2(n_2-1)-n_1M_1(\Gamma_2)+2n_1(n_2(n_2-1)-m_2)$$
$$=(n_2+1)S(\Gamma_1)+4n_2(n_2+1)W(\Gamma_1)+4m_2(n_2+1)W(\Gamma_1)$$
$$+m_2(4n_2n_1{}^2+2n_1{}^2-4n_2n_1)+2n_2n_1m_1+3n_2{}^2n_1{}^2+n_2n_1^2-2n_2{}^2n_1$$
$$+4n_1m_2(n_2-1)-n_1M_1(\Gamma_2)+2n_1(n_2(n_2-1)-m_2).$$

Summarizing, we have proved the following result.

Theorem 4.2. *Let $\Gamma_1=(V_1,E_1)$ and $\Gamma_2=(V_2,E_2)$ be two graphs, with $V_1=\{u_1,\dots,u_{n_1}\}$, $V_2=\{v_1,\dots,v_{n_2}\}$, $|E_1|=m_1$ and $|E_2|=m_2$, then*

$$S(\Gamma_1\odot\Gamma_2)=(n_2+1)S(\Gamma_1)+4(n_2+m_2)(n_2+1)W(\Gamma_1)+2n_1m_2(2n_2n_1+n_1-3)$$
$$+n_1n_2(2m_1+3n_1n_2+n_1-2)-n_1M_1(\Gamma_2).$$

The next corollary follows at once from this theorem.

Corollary 4.1. *Under the hypothesis of the last theorem, if $diam\Gamma_2\le 2$, then*

$$S(\Gamma_1\odot\Gamma_2)=(n_2+1)S(\Gamma_1)+4(n_2+m_2)(n_2+1)W(\Gamma_1)$$
$$+n_1n_2(2m_1+3n_1n_2+n_1-2n_2)+2n_1m_2(2n_1n_2+n_1-2n_2)$$
$$+n_1(S(\Gamma_2)+2W(\Gamma_2)).$$

4.3 LEXICOGRAPHIC PRODUCT

Definition 5. [11] *Let $\Gamma_1=(V_1,E_1)$ and $\Gamma_2=(V_2,E_2)$ be two graphs. The lexicographic product of Γ_1 and Γ_2 is defined as the graph $\Gamma=(V,E)$ given by $V=V_1\times V_2$ and*

$$E=\{(u,v)(x,y): ux\in E_1\}\cup\{(x,v)(x,y): vy\in E_2\}.$$

We denote this graph by $\Gamma_1\circ\Gamma_2$.

Observe that $\Gamma_1\circ\Gamma_2$ can be obtained by taking $|V_1|$ copies of Γ_2 and joining all vertices of $\Gamma_{2,u}$ with all the vertices of $\Gamma_{2,x}$ (the copies corresponding to vertices u and x, respectively), whenever $ux\in E_1$. Figure 3 shows the lexicographic product of P_3 with C_4.

We may note that, in general, this operation does not commute. Moreover, it is not difficult to verify that the degree of a vertex (u,v) of $\Gamma_1\circ\Gamma_2$ can be obtained as $\deg(u,v)=\deg v+\deg u\cdot|V_2|$.

Now, observe that given two vertices a and b of $\Gamma_1\circ\Gamma_2$, we have

- a and b are in the same copy $\Gamma_{2,u}$ or
- a is in a copy $\Gamma_{2,u}$ and b in another copy $\Gamma_{2,x}$.

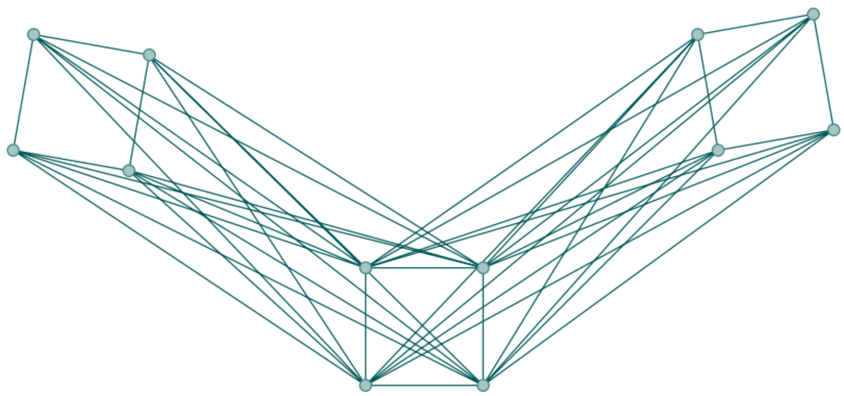

Figure 3: A representation of $P_3 \circ C_4$.

The following lemma follows quickly considering these cases.

Lemma 4.3. *Let $\Gamma = \Gamma_1 \circ \Gamma_2$ and consider $a = (u, v)$ and $b = (x, y)$ two vertices of Γ, then*

$$d(a,b) = \begin{cases} \min\{d(v,y), 2\}, & if \quad u = x; \\ d(u,x), & otherwise. \end{cases}$$

Theorem 4.3. *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs, with $V_1 = \{u_1, \dots, u_{n_1}\}$, $V_2 = \{v_1, \dots, v_{n_2}\}$, $|E_1| = m_1$ and $|E_2| = m_2$, then*

$$S(\Gamma_1 \circ \Gamma_2) = n_2^3 S(\Gamma_1) + 4n_2 m_2 W(\Gamma_1) + 4n_2 m_1 (n_2(n_2 - 1) - m_2) + 4n_1 m_2 (n_2 - 1) - n_1 M_1(\Gamma_2).$$

Proof. First note that

$$S(\Gamma_1 \circ \Gamma_2) = \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg(u_i, v_j) + \deg(u_r, v_s)) d((u_i, v_j), (u_r, v_s)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2-1} \sum_{s=j+1}^{n_2} (\deg(u_i, v_j) + \deg(u_i, v_s)) d((u_i, v_j), (u_i, v_s)).$$

thus, we may compute the Schultz index by dividing the sum into two parts: one for $i \neq r$ and the other for $i = r$. For $i \neq r$ we have

$$\sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg(u_i, v_j) + \deg(u_r, v_s)) d((u_i, v_j), (u_r, v_s))$$
$$= \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg v_j + n_2 \deg u_i + \deg v_s + n_2 \deg u_r) d(u_i, u_r)$$
$$= \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} n_2 (\deg u_i + \deg u_r) d(u_i, u_r)$$
$$+ \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg v_j + \deg v_s) d(u_i, u_r)$$
$$= n_2 \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} (\deg u_i + \deg u_r) d(u_i, u_r)$$
$$+ \sum_{j=1}^{n_2} \sum_{s=1}^{n_2} (\deg v_j + \deg v_s) \sum_{i=1}^{n_1-1} \sum_{r=i+1}^{n_1} d(u_i, u_r)$$
$$= {n_2}^3 S(\Gamma_1) + 2n_2 \sum_{j=1}^{n_2} \deg v_j W(\Gamma_1)$$
$$= {n_2}^3 S(\Gamma_1) + 4n_2 m_2 W(\Gamma_1).$$

And for $i = r$ we obtain

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg(u_i,v_j)+\deg(u_i,v_s))d((u_i,v_j),(u_i,v_s))$$
$$=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+n_2\deg u_i+\deg v_s+n_2\deg u_i)\min\{d(v_j,v_s),2\}$$
$$=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(2n_2\deg u_i+\deg v_j+\deg v_s)\min\{d(v_j,v_s),2\}$$
$$=2n_2\sum_{i=1}^{n_1}\deg u_i\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}\min\{d(v_j,v_s),2\}$$
$$+\sum_{i=1}^{n_1}\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s)\min\{d(v_j,v_s),2\}$$
$$=4n_2m_1\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}\min\{d(v_j,v_s),2\}$$
$$+n_1\sum_{j=1}^{n_2-1}\sum_{s=j+1}^{n_2}(\deg v_j+\deg v_s)\min\{d(v_j,v_s),2\}.$$

By the computations made for corona product, the last expression can be written as

$$4n_2m_1(n_2(n_2-1)-m_2)+4n_1m_2(n_2-1)-n_1M_1(\Gamma_2).$$

Taking the summation of these we get the result.

Corollary 4.2. *Under the hypothesis of the last theorem, if $diam\Gamma_2 \leq 2$, then*

$$S(\Gamma_1\circ\Gamma_2)={n_2}^3S(\Gamma_1)+4n_2m_2W(\Gamma_1)+4n_2m_1W(\Gamma_2)+n_1S(\Gamma_2).$$

4.4 CARTESIAN SUM

Definition 6. [17] *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. The cartesian sum of Γ_1 and Γ_2 is defined as the graph $\Gamma = (V, E)$ given by $V = V_1 \times V_2$ and*

$$E=\{(u,v)(x,y): ux\in E_1 \text{ or } vy\in E_2\}.$$

We denote this graph by $\Gamma_1 \oplus \Gamma_2$.

It is clear that cartesian sum of graphs commutes. Note that $\Gamma_1 \oplus \Gamma_2$ contains $\Gamma_1 \circ \Gamma_2$ as subgraph, in fact, the edges that are not considered in lexicographic product are those of the form $(u,v)(x,y)$, with $u \neq x$ and $vy \in E_2$. Figure 4 shows the cartesian sum of P_3 with C_4.

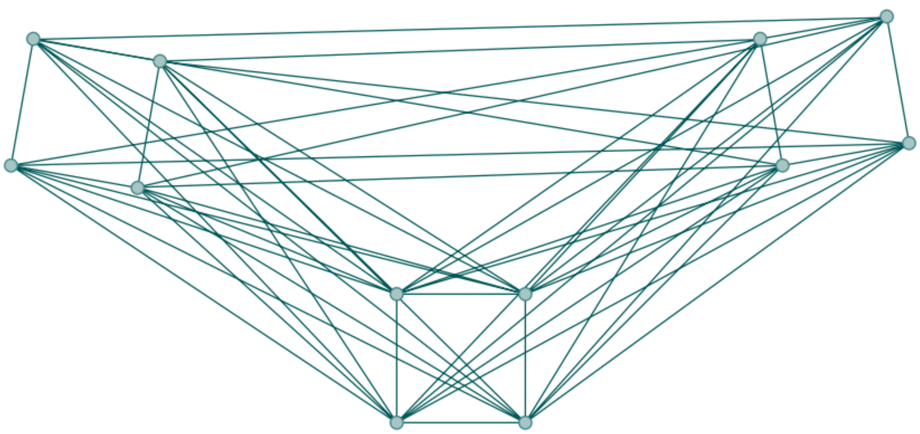

Figure 4: A representation of $P_3 \circ C_4$.

Consider (u,v) a vertex of $\Gamma_1 \oplus \Gamma_2$ and suppose that $u_1,\dots,u_k$ are the neighbors of u, then (u_i,y) are neighbors of (u,v), for $i=1,\dots,k$ and for all $y \in V_2$, analogously, if $v_1,\dots,v_l$ are the neighbors of v, then(x,v_j) are neighbors of (u,v), for $j=1,\dots,l$ and for all $x \in V_1$. Thus, $\deg(u,v) = \deg u \cdot |V_2| + \deg v \cdot |V_1| - \deg u \cdot \deg v$.

Lemma 4.4. *Let (u,v) and (x,y) be two vertices of $\Gamma_1 \oplus \Gamma_2$, then*

$$d\big((u,v),(x,y)\big) = \begin{cases} 1, & if \quad (u,v)(x,y) \in E; \\ 2, & otherwise\ . \end{cases}$$

Proof. If $(u,v)(x,y) \in E$, obviously they are at distance 1. If they are not adjacent, consider $a \in V_1$ and $b \in V_2$ such that $ua \in E_1$ and $vb \in E_2$, thus, $(u,v)(a,b)(x,y)$ is a path. Hence, $d((u,v),(x,y)) = 2$.

Theorem 4.4. *Let $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ be two graphs, with $V_1 = \{u_1,\dots,u_{n_1}\}$, $V_2 = \{v_1,\dots,v_{n_2}\}$, $|E_1| = m_1$ and $|E_2| = m_2$. If $\Gamma = (V,E)$ is the cartesian sum of Γ_1 and Γ_2, then*

$$S(\Gamma) = (4n_2m_2 - n_2^3)M_1(\Gamma_1) + (4n_1m_1 - n_1^3)M_1(\Gamma_2) - M_1(\Gamma_1)M_1(\Gamma_2)$$
$$+4(n_1^3n_2m_2 + n_2^3n_1m_1 + 2m_1m_2 - 4n_1n_2m_1m_2 - n_1^2m_2 - n_2^2m_1).$$

Proof. First note that the formula for the Schultz index of Γ can be split by taking a sum over all pairs of vertices which form edges and another sum over the rest of pairs of vertices, that is,

$$S(\Gamma) = \sum_{ab \in E} (\deg a + \deg b) d(a,b) + \sum_{ab \notin E} (\deg a + \deg b) d(a,b).$$

Analogous computations to those used for the corona product show that

$$S(\Gamma) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg^2(u_i,v_j) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} 2\deg(u_i,v_j)(n_1n_2 - (\deg(u_i,v_j) + 1)).$$

For the first sum

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg^2(u_i,v_j) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (n_2\deg u_i + n_1\deg v_j - \deg u_i \deg v_j)^2$$
$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (n_2^2\deg^2 u_i + n_1^2\deg^2 v_j + \deg^2 u_i \deg^2 v_j$$
$$+ 2n_1n_2\deg u_i \deg v_j - 2n_2\deg v_j \deg^2 u_i - 2n_1 \deg u_i \deg^2 v_j)$$
$$= n_2^2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg^2 u_i + n_1^2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg^2 v_j$$
$$+ \sum_{i=1}^{n_1} \deg^2 u_i \sum_{j=1}^{n_2} \deg^2 v_j$$
$$+ 2n_1n_2 \sum_{i=1}^{n_1} \deg u_i \sum_{j=1}^{n_2} \deg v_j - 2n_2 \sum_{i=1}^{n_1} \deg^2 u_i \sum_{j=1}^{n_2} \deg v_j$$
$$- 2n_1 \sum_{i=1}^{n_1} \deg u_i \sum_{j=1}^{n_2} \deg^2 v_j$$
$$= n_2^3 \sum_{i=1}^{n_1} \deg^2 u_i + n_1^3 \sum_{j=1}^{n_2} \deg^2 v_j + \sum_{i=1}^{n_1} \deg^2 u_i \sum_{j=1}^{n_2} \deg^2 v_j$$
$$+ 8n_1n_2m_1m_2 - 4n_2m_2 \sum_{i=1}^{n_1} \deg^2 u_i - 4n_1m_1 \sum_{j=1}^{n_2} \deg^2 v_j$$
$$= (n_2^3 - 4n_2m_2)M_1(\Gamma_1) + (n_1^3 - 4n_1m_1)M_1(\Gamma_2) + M_1(\Gamma_1)M_1(\Gamma_2)$$
$$+ 8n_1n_2m_1m_2.$$

And for the second

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} 2\deg(u_i,v_j)(n_1n_2-(\deg(u_i,v_j)+1))$$
$$=2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(n_2\deg u_i+n_1\deg v_j-\deg u_i\deg v_j)(n_1n_2-n_2\deg u_i$$
$$-n_1\deg v_j+\deg u_i\deg v_j-1)$$
$$=2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_1^2n_2\deg v_j-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_1^2\deg^2 v_j+2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_1n_2^2\deg u_i$$
$$-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} 3n_1n_2\deg u_i\deg v_j+2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} 2n_1\deg u_i\deg^2 v_j$$
$$-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_1\deg v_j-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_2^2\deg^2 u_i+2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} 2n_2\deg^2 u_i\deg v_j$$
$$-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_2\deg u_i-2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2 u_i\deg^2 v_j+2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg u_i\deg v_j$$
$$=4n_1^3n_2m_2-2n_1^3\sum_{j=1}^{n_2}\deg^2 v_j+4n_1n_2^3m_1-24n_1n_2m_1m_2$$
$$+8n_1m_1\sum_{j=1}^{n_2}\deg^2 v_j-4n_1^2m_2-2n_1^3\sum_{i=1}^{n_1}\deg^2 u_i+8n_2m_2\sum_{i=1}^{n_1}\deg^2 u_i$$
$$-4n_2^2m_1-2\sum_{i=1}^{n_1}\deg^2 u_i\sum_{j=1}^{n_2}\deg^2 v_j+8m_1m_2$$
$$=(8n_1m_1-2n_1^3)M_1(\Gamma_2)+(8n_2m_2-2n_2^3)M_1(\Gamma_1)+4n_1^3n_2m_2+4n_2^3n_1m_1$$
$$-24n_1n_2m_1m_2-4n_1^2m_2-4n_2^2m_1+8m_1m_2-2M_1(\Gamma_1)M_1(\Gamma_2).$$

Taking the sum of these parts we obtain the formula.

5. CONCLUDING REMARKS

In this work we have obtained explicit formulas for the Schultz index of the cartesian, corona and, lexicographic products, as well as of the cartesian sum, we observed that the Schultz index of the first three products can be written in terms of the Schultz and Wiener indices and the order and size of the factors, but for the cartesian sum is different, it involves order, size and, Zagreb index of factors, this it also appears in the formula for the lexicographic product. We noted once again how the close relationship between Schultz and Wiener indices is reaffirmed by their emergence in those expressions.

There are still some questions to be answered; for example, can we find explicit formulas for the Schultz index of some other products of graphs or for other indices?

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