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The Effect of Radial Basis Functions (RBFs) Method in Solving Coupled Lane-Emden Boundary Value Problems in Catalytic Diffusion Reactions

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ABSTRACT

In this paper, the radial basis functions (RBFs) method is applied to solve the coupled Lane-Emden boundary value problems arising in catalytic diffusion reactions. First, we multiply the equations by x to overcome the difficulties of the singularity at the origin. Then, the Kansa collocation method based on radial basis functions is used to approximate the unknown functions. By this technique, the problem with boundary conditions is reduced to a system of algebraic equations. We solve this system and compare the maximal residual error with the results previously, which show the presented method is efficient and produces very accurate and rapidly convergent numerical results in considerably low computational effort and easy implementation.

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1. INTRODUCTION

The Lane-Emden equation is a second-order nonlinear ordinary differential equation that describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules that is subject to the classical laws of thermodynamics [1, 2]. This equation was first studied by astrophysicists named Jonathan Homer Lane and Robert Emden [3, 4] and solved by Aris and Metha as a boundary value problem [5, 6].

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Lane-Emden equation is used in astrophysics, mathematical physics, biology, and biochemistry, such as solidification of a cylindrical, reaction-diffusion process, the theory of stellar structure, analysis of the diffusive transport, etc [7, 8]. Due to the widespread use of this equation in modeling fundamental problems and the lack of an exact solution of it in many cases, plenty of articles have been published that solve it in various methods. In [9] Adomian decomposition method (ADM) in the form of a power series was used, and a modified decomposition method was applied in [10]. Bender et al. applied a new technique called $\delta-$method [11] and J. H. He obtained an analytical solution of Lane-Emden equation by Ritz's method [12]. Series solutions of this equation have been presented as a Volterra integral equation in [13] and Homotopy analysis method (HAM) is used by Dehghan, Singh et al. [14, 15]. Parand et al. in [16, 17] applied Hermite functions collocation method and Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type. Also, Parand et al. solved nonlinear Lane-Emden type equations with unsupervised combined artificial neural networks in [1]. The variational iteration method and hybrid function approximations are used by Yildirim, Marzban et al. [18, 19]. Recently, a numerical algorithm based upon compact finite difference is used by Bisheh-Niasar [20].

Systems of Lane-Emden equations which are called the coupled of Lane-Emden equation, are applied in chemical reactions, population evolution, pattern formation, and modeling some of the physical and chemical phenomena [21, 22]. We consider the coupled Lane-Emden equation, which is boundary value problem in the following form with $i = 1, 2$:

$$\begin{cases} u_i''(x) + \dfrac{l_i}{x}u_i'(x) + f_i(u_1(x), u_2\,(x)) = 0, \\ u_i'(0) = 0, \qquad u_i(1) = b_i\,, \end{cases} \tag{1.1}$$

where for $i = 1, 2,\ l_i \geq\ 1,\ b_i$ are real constants and $x \in (0,1)$.

Several numerical methods have been proposed for solving such systems including, the Homotopy analysis method (HAM) by Singh and Wazwaz [23] and Adomian decomposition method (ADM) in [2]. A rapid convergence series solution is used by the author of [24]. Saadatmandi et al. and Zabihi applied the Chebyshev finite difference method (ChFD) in [25, 26]. Also, Sinc-collocation method is presented by Saadatmandi et al. [27]. We used meshless methods to solve the coupled of Lane-Emden equation that have never been used for solving such systems. In these methods, a set of scattered nodes is used in the domain and on its boundaries without the need to generating mesh and it makes nodes easily distributed, especially in irregular domains and hence, meshless methods are highly flexible with respect to the geometry of the computational domain. Among the meshless methods, radial basis functions (RBFs) which lie in the concept of positive definite functions are a powerful tool for interpolation and can be considered as mathematical parsley since they have been used in all mathematical problems requiring a powerful, i.e. efficient, and stable, approximation tool. RBFs were studied in 1968 by Roland Hardy and used for solving differential equations (DEs) by Kansa for the first time [28]. This method can get a better approximation than most methods because interpolation of scattered data and smooth with some RBFs has spectral accuracy and exponential convergence, the error in using them is reduced rapidly. In the last decades, these methods have frequently been used for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) that appear in mathematical modeling and engineering problems and ease of implementation in high dimensions. A lot of articles are provided by this method, for example, the meshfree collocation method based on radial basis functions (RBFs) [16, 29, 30] and the radial point interpolation method [31]. The interested reader is referred to the books by Buhmann [32] and Wendland [33].

The most common types of RBFs are listed in Table 1, where $r = \|x - x_i\|$ and the positive parameter ε is named shape parameter. By decreasing the shape parameter, the accuracy increases, but the method becomes less stable and vice versa. Optimization of this value, which is an open problem, leads to both the accuracy and stability of the method. Researchers such as Hardy [34], Franke [35], Rippa [36], Fornberg and Wright [37] and Rad et al. [38] have presented ideas to choose an appropriate value for this parameter. Although these efforts lead to finding values for the shape parameter, these values depend on the specific conditions of the problem. Hence, research continues to find the optimal shape parameter. In this paper, the shape parameter is achieved via a trial and error approach.

Table 1. Some radial basis functions.

Name of Functions	Formulate
Inverse quadric (IQ)	$(\varepsilon^2 + r^2)^{-1}$
Inverse multi quadric (IMQ)	$(\varepsilon^2 + r^2)^{-1/2}$
Multi quadric (MQ)	$(\varepsilon^2 + r^2)^{1/2}$
Gaussian (GA)	$e^{-\varepsilon r^2}$
Hyperbolic secant (SECH)	$sech(\varepsilon r)$

In the current paper, we approximate the problem by applying the Kansa collocation method based on radial basis functions and then solve the resulting algebraic equation system. Overall, our focus in this paper is more devoted to providing a high-order accurate, computationally fast, convergence, and simple technique to evaluate the coupled Lane-Emden boundary value problems.

The remainder of this paper is structured as follows: In the second section, a detailed description of our method is presented. In Section 3, the numerical results of two test examples are reported and compared with the numerical results of [2, 23] and the results obtained by applying the method in [25, 26] for this problem. Finally, some conclusions are mentioned.

2. METHODOLOGY

Consider the coupled Lane-Emden equation (1.1) with nonlinear functions as [21],

$$f_i\big(u_1(x), u_2(x)\big) = -c_{i1}u_1^2(x) - c_{i2}u_1(x)u_2(x), \quad i = 1,2.$$

As is clear from system (1.1), the coupled Lane-Emden equation is singular in $x = 0$, and this is one of the major difficulty of this type of equations. We multiply both these equations by x, to overcome this difficulty. Thus system (1.1) is rewritten as follows by substituting the above f_i and multiplying by x,

$$\begin{cases} xu_1''(x) + 2u_1'(x) - c_{11}xu_1^2(x) - c_{12}xu_1(x)u_2(x) = 0, \\ xu_2''(x) + 2u_2'(x) - c_{21}xu_1^2(x) - c_{22}xu_1(x)u_2(x) = 0, \end{cases} \tag{2.1}$$

subject to the same boundary conditions as mentioned in the previous section:

$$u_1'(0) = 0, \qquad u_1(1) = b_1, \qquad u_2'(0) = 0, \qquad u_2(1) = b_2, \tag{2.2}$$

where l_1, l_2 in (1.1) replaced to 2 and $0 < x < 1$.

System (2.1) now as a system of nonlinear ordinary differential equations has to be solved in each space step. In this section, the Kansa collocation method based on radial basis functions is used in both equations to reach an algebraic system of equations. Therefore, this section is allocated into two subsections, radial basis functions and implementation of the proposed method.

2.1. RADIAL BASIS FUNCTIONS

Definition 2.1. *A function* $\Phi: \mathbb{R}^d \to \mathbb{R}$ *is called radial if a real-valued function* $\phi: [0,\infty) \to \mathbb{R}$ *exists that* $\Phi(x) = \phi(\|x\|)$, *where* $\|.\|$ *is usually the Euclidean norm on* $\mathbb{R}^d$.

The radial basis functions (RBFs) depend on the radial distance between the data points x and the centers x_j and are symmetric around x_j for each j, hence it can be written in the form $\Phi_j(x)= \phi(\|x-x_j\|)$. These functions are divided into three categories [39]:

1. INFINITELY SMOOTH RBFS: The functions of this category which are defined in Table 1, require tuning a shape parameter. By using these RBFs in numerical computational of differential equations, full matrices are created [38-41].

2. PIECEWISE SMOOTH RBFS: Piecewise smooth RBFs that are less accurate than the basic functions displayed in the category 1, are not infinitely differentiable e.g. conical splines $\phi(r) = r^{2k+1}$, thin plate splines (TPS) $\phi(r) = (-1)^{k+1}r^{2k}log(r)$, etc [39].

3. COMPACTLY SUPPORTED RBFS: These functions are only non-zero within a radius of $\frac{1}{\varepsilon}$, and so have sparse derivative matrices. In general, it can be said that these radial basis functions are more stable and less accurate than the smooth RBFs [32, 33]. For example, Wendland's, Wu's and Buhmann's.

2.1.1. RBF INTERPOLATION AND COLLOCATION METHOD

Interpolation of the scattered data is referred as the main objective of the meshless methods and is one of the essential problems in the approximation theory and data modeling. Suppose $\{(\mathbf{x}_i, y_i)\}_{i=1}^{n}$ is a given set of distinct points $\mathbf{x}_i \in \mathbb{R}^d$ and their corresponding values $y_i \in \mathbb{R}$, we find the continuous function $S(\mathbf{x})$ as an approximation solution of y, where

$$S(\mathbf{x}_i) = y_i, \ i = 1,2,\ldots,n, \tag{2.3}$$

and consider it as a linear combination of radial basis functions ϕ in the following form,

$$y(\mathbf{x}) \approx S(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j \phi\left(\left\|\mathbf{x} - \mathbf{x}_j\right\|_2\right) = \sum_{j=1}^{n} \lambda_j \Phi_j(\mathbf{x}) \tag{2.4}$$

Then, by collocating the recent combination in $\mathbf{x}_i$, that is substituting these points instead of $\mathbf{x}$ in (2.4) and using (2.3), we get $y_i = \sum_{j=1}^{n} \lambda_j \Phi_j(\mathbf{x}_i),\ i = 1,2,\ldots,n$. By solving this system, which is an algebraic system of equations, the unknown coefficients λ_j can be determined and hence the approximation of the function y is obtained.

2.2. IMPLEMENTATION OF PROPOSED METHOD

First, the functions $u_1(x),\ l = 1,2$ are approximated by radial basis function Φ as (2.4),

$$u_l(x) \approx \tilde{u}_1(x) = \sum_{j=1}^{n} \lambda_{lj} \Phi_j(x),\ l = 1,2. \tag{2.5}$$

Next, the scattered points distributed $\{x_1, x_2, \ldots, x_n\}$ in [0,1] are supposed as following,

$$x_i = \frac{1}{2}\left(1 - cos\left(\frac{i-1}{n-1}\pi\right)\right),\ i = 1,2,\ldots,n,$$

where $x_1 = 0,\ x_n = 1$.

Now, substituting $\tilde{u}_1(x),\ l = 1,2$ in system (2.1) and collocating the obtained system in the points $\{x\}_{i=2}^{n-1}$, we have the following absolute residual errors (Res):

$$\begin{cases} Res_1(x) = x_i \tilde{u}_1''(x_i) + 2\tilde{u}_1'(x_i) - c_{11} x_i \tilde{u}_1^2(x_i) - c_{12} x_i \tilde{u}_1(x_i)\tilde{u}_2(x_i), \\ Res_2(x) = x_i \tilde{u}_2''(x_i) + 2\tilde{u}_2'(x_i) - c_{21} x_i \tilde{u}_1^2(x_i) - c_{22} x_i \tilde{u}_1(x_i)\tilde{u}_2(x_i), \end{cases} \tag{2.6}$$

where $i = 2, 3, \ldots, n-1$.

Above system consists of $2n-4$ equations and $2n$ unknowns which using the following boundary conditions as added equations to (2.6), is obtained a system of $2n$ equations which by solving it, unknowns can be calculated.

$$\tilde{u}_1'(x_1) = 0, \quad \tilde{u}_1(x_n) = b_1, \quad \tilde{u}_2'(x_1) = 0, \quad \tilde{u}_2(x_n) = b_2.$$

By finding unknown coefficients $\{\lambda_{lj}\}_{j=1}^{n}$, $l = 1,2$, approximate solutions $\tilde{u}_1$ and $\tilde{u}_2$ are computed.

3. NUMERICAL RESULTS

In this section, two particular case of the coupled Lane-Emden problems (2.1)–(2.2) are considered [2, 23]. The accuracy of our method by using the Gaussian (GA) of RBFs is shown in several Tables, and graphs. The results are compared with the numerical solutions of [2, 23]. We also obtained the results with the Chebyshev finite difference method (ChFD) that is introduced in [25, 26], which has never been used to solve the current problem, and put them in some Tables for comparison. The approximated and residual error of functions by applying some radial basis functions and with different numbers of collocation points are obtained. The examples are as below:

Example 1. Consider that the following values are substituted in the problem (2.1) and conditions (2.2), $c_{11} = c_{22} = 1$, $c_{12} = \frac{2}{5}$, $c_{21} = \frac{1}{2}$, $b_1 = 1$, $b_2 = 2$.

In Table 2 and Figure 1, the approximated solutions of functions u_1, u_2 obtained from ADM, HAM, ChFD methods of [2, 23, 25, 26], and the proposed technique with $n = 7$, $\varepsilon = 0.05$ in the interval [0,1] have been reported. It is clear that the RBF method is in good adaptation to the HAM and ADM methods. Also, the absolute residual errors of these functions are displayed in Figure 2. and Table 3 The maximum absolute residual error is of $O(10^{-3})$ in the RBF method while it is of $O(10^{-1}), O(10^{-2})$ and $O(10^{-3})$ in the ADM, HAM and ChFD methods respectively, so the proposed method has good accuracy.

Table 2. Obtained values of approximated functions by ADM, HAM, ChFD methods and the present method (with the GA of RBFs and $\varepsilon = 0.05$, $n = 7$) in Example 1.

$\tilde{u}_1$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	0.763624868	0.780767047	0.781368743	0.781372911
0.1	0.765831758	0.782684342	0.783267006	0.783273949
0.2	0.772463013	0.788465717	0.789001310	0.789006647
0.3	0.783553024	0.798200840	0.798666754	0.798664977
0.4	0.799167888	0.812043169	0.812416845	0.812409269
0.5	0.819418576	0.830215964	0.830476787	0.830470338
0.6	0.844479370	0.853020705	0.853156774	0.853158195
0.7	0.874611567	0.880847918	0.880865280	0.880875221
0.8	0.910192446	0.914190405	0.914122353	0.914133488
0.9	0.951749513	0.953658877	0.953572904	0.953575814
1.0	1.000000000	1.000000000	1.000000000	1.000000000

Table 2. (Continued)

$\tilde{u}_2$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	1.668215608	1.689598095	1.690662449	1.690667671
0.1	1.671315683	1.692335084	1.693373144	1.693381877
0.2	1.680631694	1.700585517	1.701557399	1.701564111
0.3	1.696214314	1.714469358	1.715342059	1.715339784
0.4	1.718159052	1.734191780	1.734932268	1.734922655
0.5	1.746622830	1.760051017	1.760628360	1.760620144
0.6	1.781847192	1.792449348	1.792842745	1.792844445
0.7	1.824188148	1.831907230	1.832116803	1.832129258
0.8	1.874152645	1.879080563	1.879137769	1.879151761
0.9	1.932441674	1.934781102	1.934755628	1.934759289
1.0	2.000000000	2.000000000	2.000000000	2.000000000

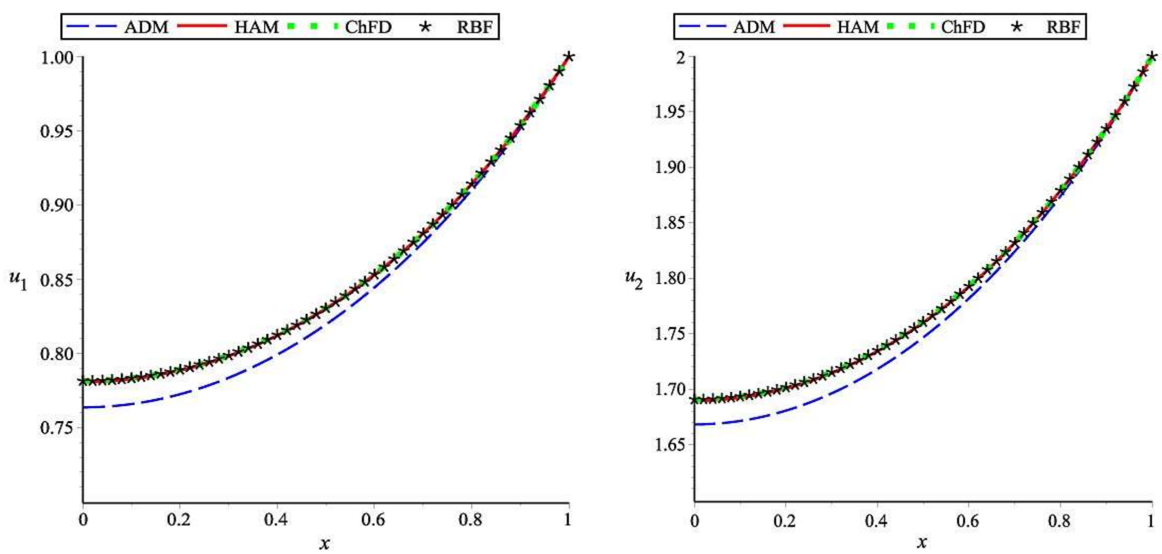


Figure 1. Graphs of the functions $u_1(x)$ and $u_2(x)$ with ADM, HAM, ChFD methods and the GA of RBFs with $n = 7,\ \varepsilon = 0.05$ in Example 1.

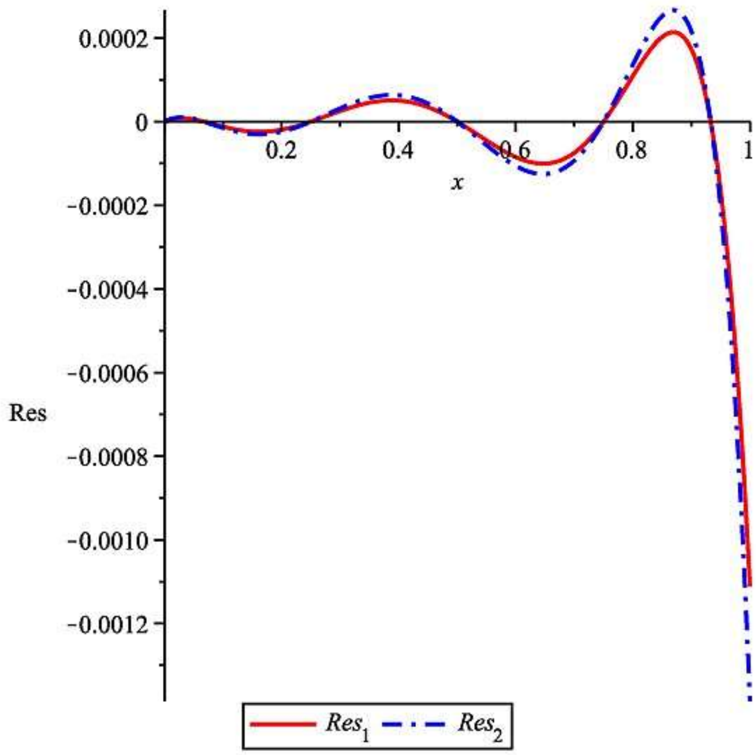


Figure 2. Graphs of the absolute residual errors $u_1(x)$ and $u_2(x)$ with the GA of RBFs and $n = 7,\ \varepsilon = 0.05$ in Example 1.

Table 3. Obtained absolute residual errors by ADM, HAM, ChFD methods and the present method (with the GA of RBFs and $\varepsilon = 0.05, n = 7$) in Example 1.

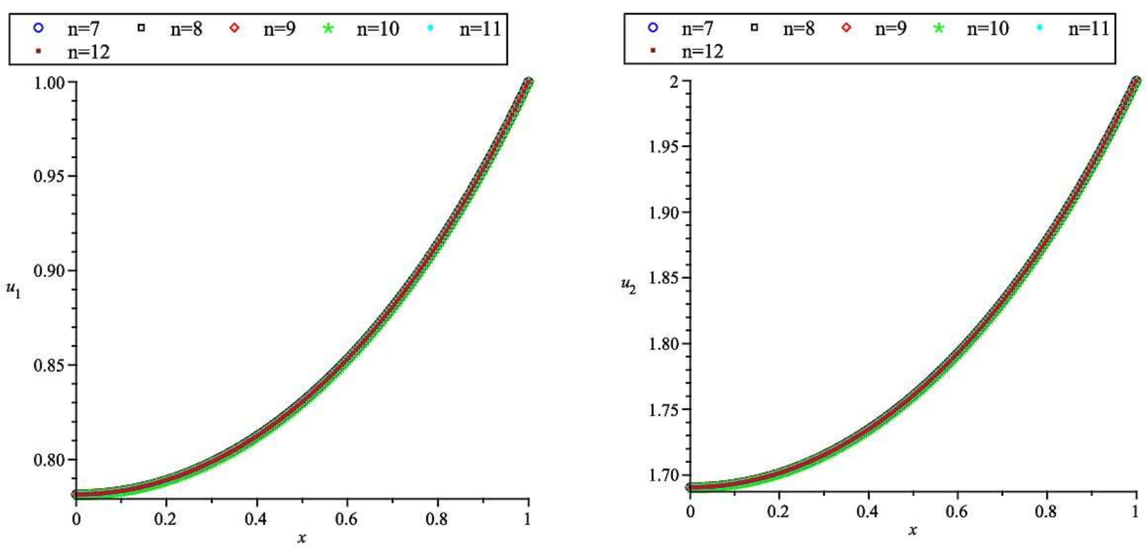
$Res_1(x)$				
x	**ADM method**	**HAM method**	**CHFD method**	**RBF method**
0.0	0.230942914	0.011640586	0.000000000	0.000000000
0.1	0.226867128	0.011385744	0.000010285	0.000012019
0.2	0.214888858	0.010641294	0.000210365	0.000019215
0.3	0.195746198	0.009458900	0.000134203	0.000025298
0.4	0.170637696	0.007895555	0.000198966	0.000050290
0.5	0.141171286	0.005967766	0.000474940	0.000000000
0.6	0.109285297	0.003582957	0.000324541	0.000084744
0.7	0.077133580	0.000443210	0.000370354	0.000074538
0.8	0.046923639	0.004085554	0.001082808	0.000109205
0.9	0.020692579	0.011148438	0.000129675	0.000173217
1.0	8.4134 E -17	0.022633272	0.005995341	0.001111697

Table 3. (Continued)

$Res_2(x)$				
x	**ADM method**	**HAM method**	**CHFD method**	**RBF method**
0.0	0.766598470	0.016249657	0.000000000	0.000000000
0.1	0.763378418	0.015921395	0.000013021	0.000015028
0.2	0.754062831	0.014966847	0.000266186	0.000024026
0.3	0.739676243	0.013466018	0.000169714	0.000031631
0.4	0.721894603	0.011515827	0.000251441	0.000062867
0.5	0.702994203	0.009172894	0.000599730	0.000000000
0.6	0.685772693	0.006368090	0.000409452	0.000105872
0.7	0.673434203	0.002786970	0.000466795	0.000093078
0.8	0.669427479	0.002292183	0.001363308	0.000136288
0.9	0.677221825	0.010214085	0.000163077	0.000216029
1.0	0.700000000	0.023231283	0.007530095	0.001385391

Now, we examine the accuracy of the presented method for approximations u_1, u_2 by solving the problem for different values of collocation points. Graphs of these functions in Figure 3 and results of their maximum residual errors in Table 4 and Figure 4 with Gaussian base are displayed. Also, the following criterion with the number of different collocation points has been used to show the numerical convergence of the RBF technique. Clearly, the accuracy of the solutions increases as n increases and the order of maximum absolute residual errors reaches $O(10^{-8})$ in $n = 12$.

$$res = \sqrt{\int_0^1 (Res_1(x))^2 dx} + \sqrt{\int_0^1 (Res_2(x))^2 dx}.$$



(a) u_1 (b) u_2

Figure 3. Graphs of the functions $u_1(x)$ and $u_2(x)$ with the GA of RBFs and different values of collocation points of Example 1.

Table 4. Obtained maximum absolute residual errors and res with GA of RBFs in Example 1.

n	$MAx\ Res_1(x)$	$MAx\ Res_2(x)$	res
7	0.001111697	0.001385391	0.000361389
8	0.000137317	0.000168885	0.000039137
9	0.000018606	0.000022681	0.000004701
10	0.000002533	0.000003072	0.000000579
11	0.000000323	0.000000389	-
12	0.000000043	0.000000051	-

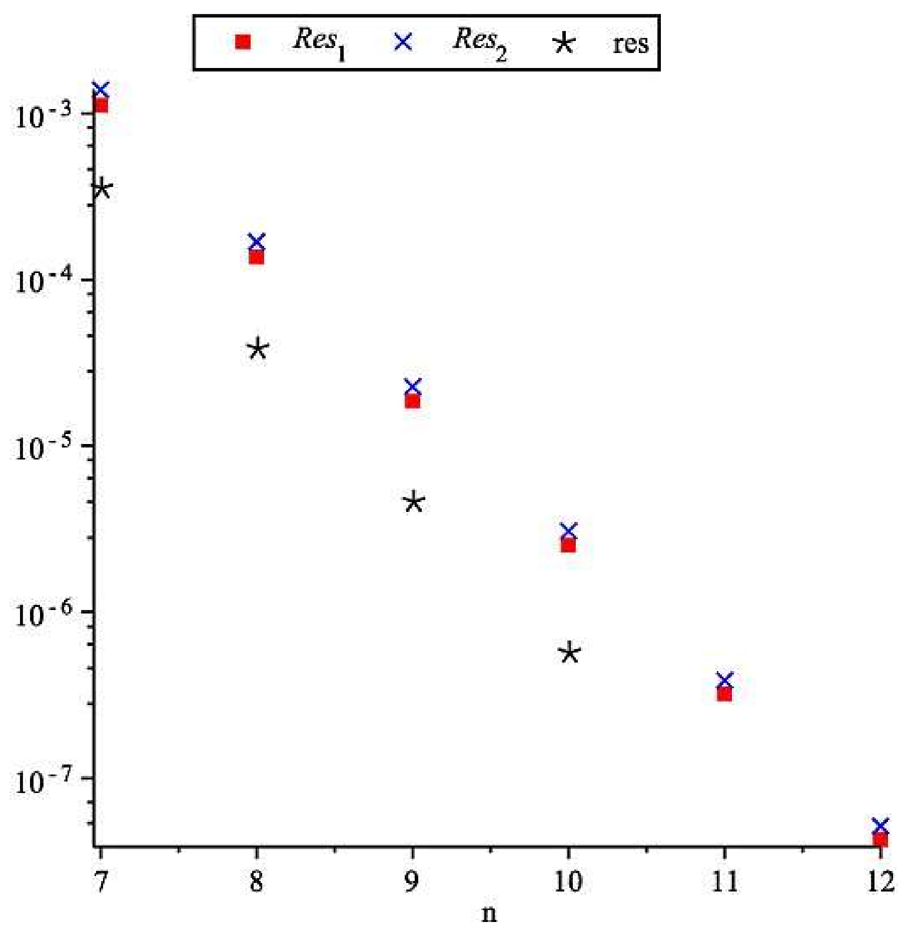
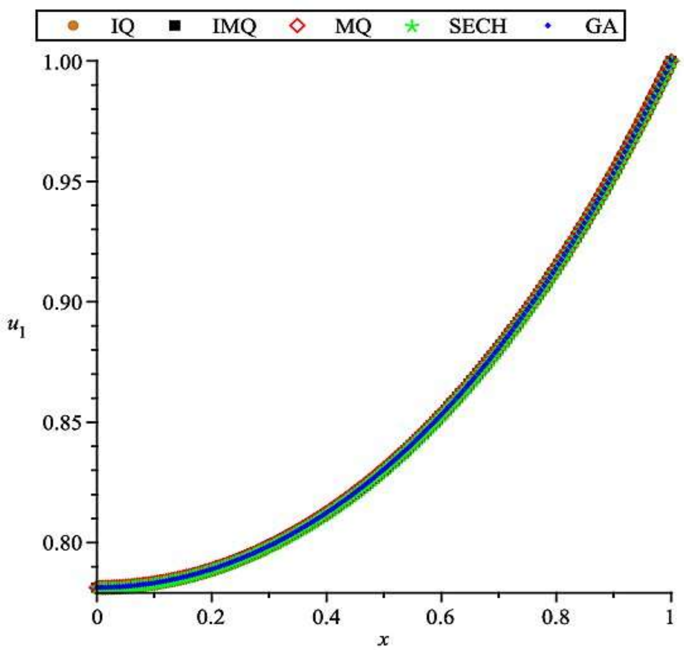


Figure 4. Graph of the res with the GA of RBFs in Example 1.

As the final experiment, we have used all the functions listed in Table 1 to show the impact of the base's functions on the RBF method with $n = 8$. Figure 5 is plotted to show an approximation of functions u_1, u_2 and Figure 6, is graphs of absolute residual errors. Figure 6 indicates that the error order in IQ and IMQ is one, in MQ, two and both SECH and GA is of $O(10^{-4})$, so the most accuracy with GA and SECH functions and the least accuracy of IQ and IMQ bases achieved.



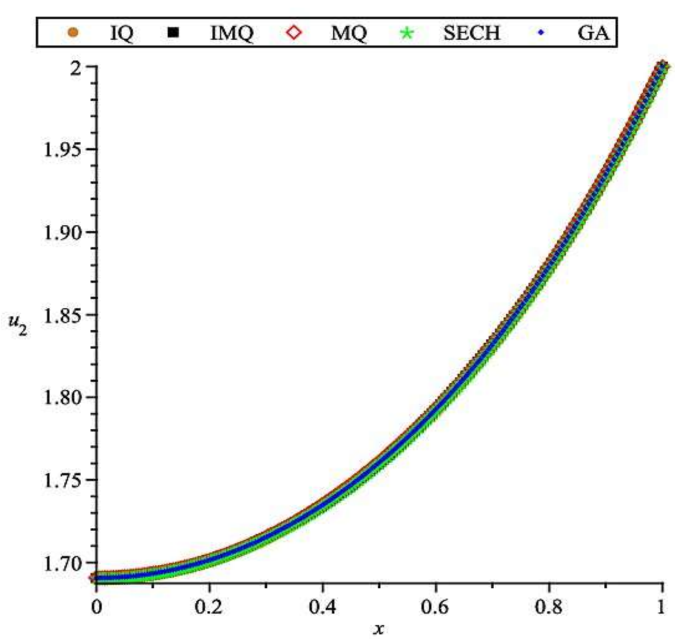
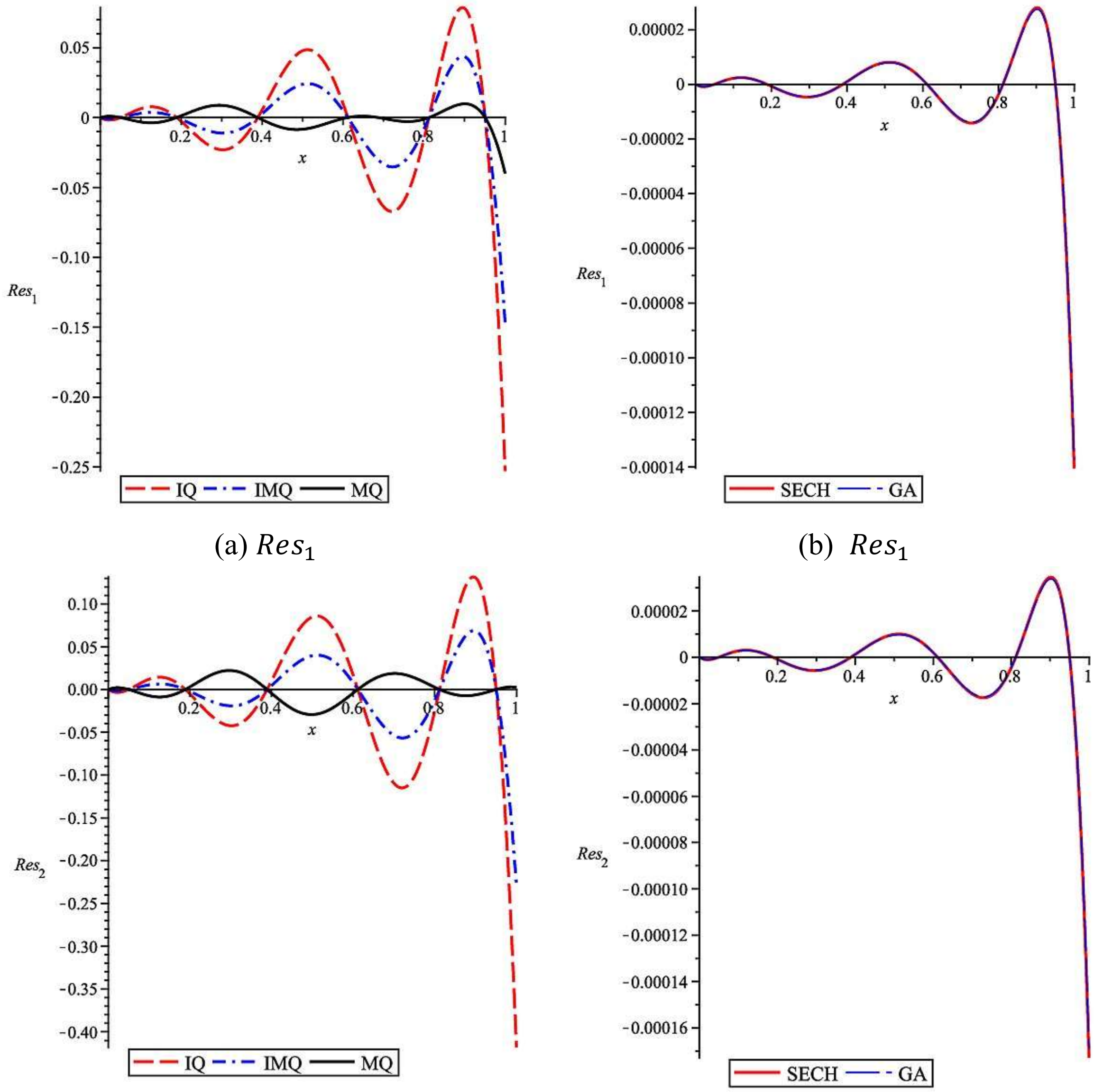


Figure 5. Graphs of the functions $u_1(x)$ and $u_2(x)$ with some of RBFs with $n = 8$ in Example 1.



(a) Res_1 (b) Res_1

(c) Res_2 (d) Res_2

Figure 6. Graphs of the functions $Res_1(x)$ and $Res_2(x)$ with some of RBFs and $n = 8$ in Example 1.

Example 2. The unknown data in the problem mentioned in (2.1) and (2.2) are assumed as $c_{11} = c_{12} = c_{21} = c_{22} = 1,\ b_1 = 1,\ b_2 = 2$. The approximate solutions of functions u_1, u_2 are shown in Table 5 and Figure 7. The results are reported using the ADM, HAM, ChFD techniques of [2, 23, 25, 26], and the RBF method with $n = 7,\ \varepsilon = 0.01$ in the interval [0,1], indicating that the RBF method is acted better than the ADM, HAM and ChFD methods. Figure 8 and Table 6, are displayed to observe $Res_1,\ Res_2$, and it can be seen that in the RBF method, maximum error is in $x = 1$ and the errors are of $O(10^{-3})$, hence it has a well accuracy compared to ADM, HAM and ChFD methods.

Table 5. Obtained values of approximated functions by ADM, HAM, ChFD methods and the present method (with the GA of RBFs and $\varepsilon = 0.01,\ n = 7$) in Example 2.

$\tilde{u}_1$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	0.592658730	0.674423143	0.676513976	0.676526834
0.1	0.596753063	0.677139742	0.679164176	0.679185633
0.2	0.609002667	0.685344640	0.687201228	0.687217728
0.3	0.629317007	0.699206350	0.700811927	0.700806240
0.4	0.657577333	0.719016401	0.720290027	0.720266142
0.5	0.693684896	0.745205361	0.746074728	0.746054479
0.6	0.737628444	0.778365260	0.778789161	0.778794205
0.7	0.789571015	0.819278422	0.819278884	0.819311572
0.8	0.849956000	0.868952703	0.868650367	0.868686920
0.9	0.919632507	0.928663139	0.928309484	0.928318638
1.0	1.000000000	1.000000000	1.000000000	1.000000000

$\tilde{u}_2$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	1.592658730	1.673518521	1.676513976	1.676526834
0.1	1.596753063	1.676240889	1.679164176	1.679185633
0.2	1.609002667	1.684463213	1.687201228	1.687217728
0.3	1.629317007	1.698354439	1.700811927	1.700806240
0.4	1.657577333	1.718207120	1.720290027	1.720266142
0.5	1.693684896	1.744453865	1.746074728	1.746054479
0.6	1.737628444	1.777690385	1.778789161	1.778794205
0.7	1.789571015	1.818705108	1.819278884	1.819311572
0.8	1.849956000	1.868515392	1.868650367	1.868686920
0.9	1.919632507	1.928410306	1.928309484	1.928318638
1.0	2.000000000	2.000000000	2.000000000	2.000000000

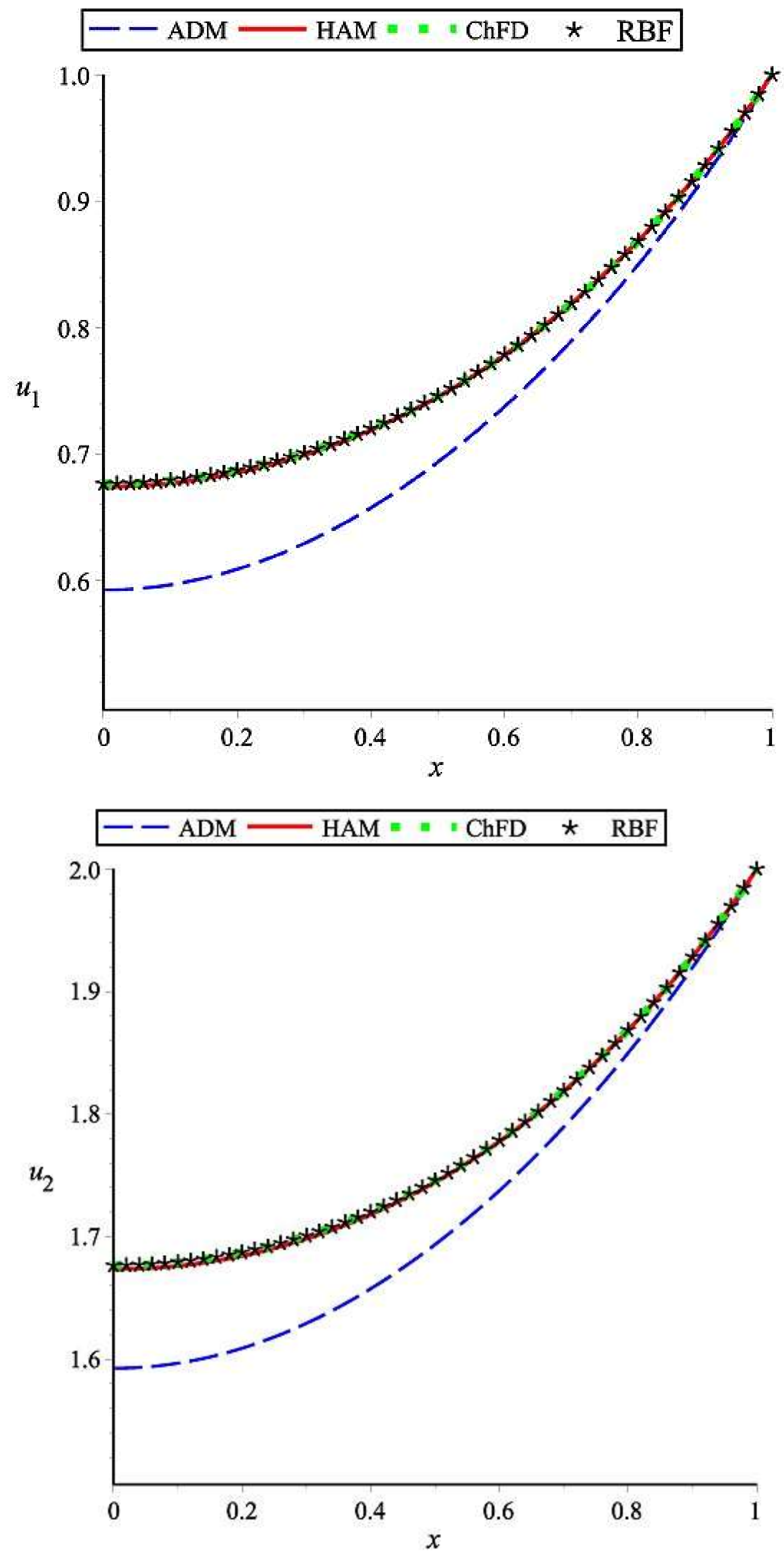


Figure 7. Graphs of the functions $u_1(x)$ and $u_2(x)$ with ADM, HAM, ChFD methods and the GA of RBFs with $n = 7,\ \varepsilon = 0.01$ in Example 2.

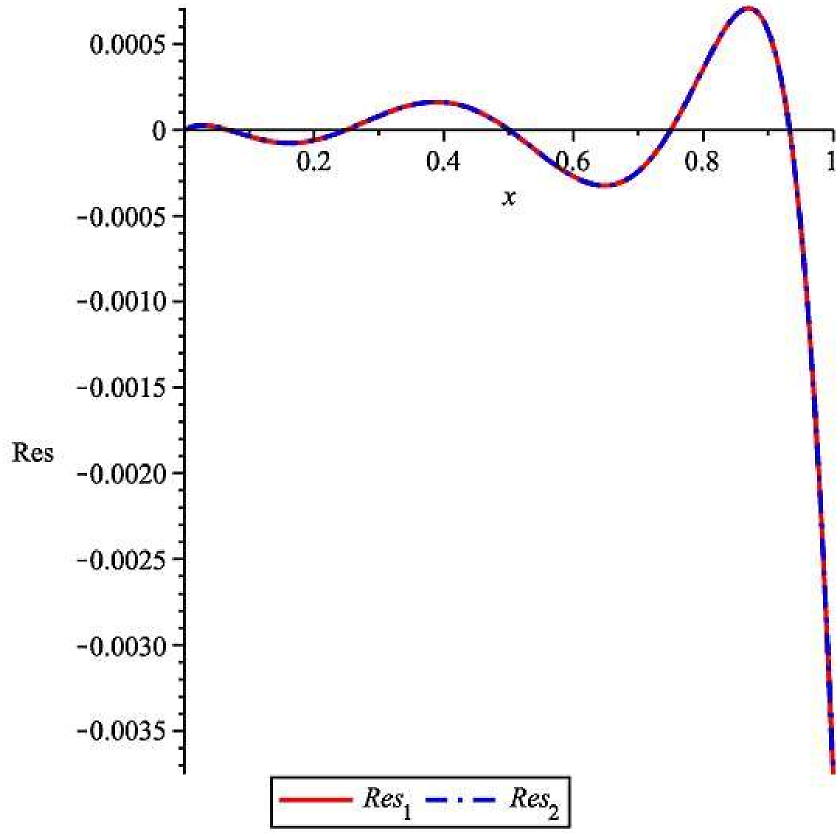


Figure 8. Graph of the absolute residual errors $u_1(x)$ and $u_2(x)$ with the GA of RBFs and $n = 7$, $\varepsilon = 0.01$ of Example 2.

Table 6. Obtained absolute residual errors by ADM, HAM, ChFD methods and the present method (with the GA of RBFs and $\varepsilon = 0.01, n = 7$) of Example 2.

$Res_1(x)$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	1.163185862	0.043719198	0.000000000	0.000000000
0.1	1.143631002	0.042789989	0.000031868	0.000038266
0.2	1.086028837	0.040067370	0.000658492	0.000061051
0.3	0.993549036	0.035715976	0.000424505	0.000080377
0.4	0.871406768	0.029906016	0.000636200	0.000160106
0.5	0.726730135	0.022646550	0.001535815	0.000000000
0.6	0.568313445	0.013530025	0.001061902	0.000272634
0.7	0.406196710	0.001363158	0.001226923	0.000241846
0.8	0.250993596	0.016352278	0.003634507	0.000358156
0.9	0.112865431	0.044140746	0.000441362	0.000575572
1.0	1.1796 E -16	0.089549318	0.020709797	0.003751641

Table 6. (Continued)

$Res_2(x)$				
x	ADM method	HAM method	CHFD method	RBF method
0.0	0.163185862	0.047175755	0.000000000	0.000000000
0.1	1.143631002	0.046264512	0.000031868	0.000038266
0.2	1.086028837	0.043608077	0.000658492	0.000061051
0.3	0.993549036	0.039407953	0.000424505	0.000080377
0.4	0.871406768	0.033895791	0.000636200	0.000160106
0.5	0.726730135	0.027166674	0.001535815	0.000000000
0.6	0.568313445	0.018923646	0.001061902	0.000272634
0.7	0.406196710	0.008108600	0.001226923	0.000241846
0.8	0.250993596	0.007616937	0.003634507	0.000358156
0.9	0.112865431	0.032593096	0.000441362	0.000575572
1.0	1.1796 E -16	0.074158043	0.020709797	0.003751641

Graphs of u_1, u_2 by the number different collocation points and GA of the RBFs are displayed in Figure 9. Also, the maximum residual errors of these functions and the criterion res mentioned in the previous example are shown in Table 7 and Figure 10. With more collocation points, we get more accuracy, which indicates the rapid convergence of this method, so that for $n = 12$, the order of the maximum absolute residual errors is $O(10^{-7})$.

Table 7. Obtained maximum absolute residual errors and res with GA of RBFs in Example 2.

n	$Max\ Res_1(x)$	$Max\ Res_2(x)$	Res
7	0.003751641	0.003751641	0.001077158
8	0.000682858	0.000682858	0.000173185
9	0.000109421	0.000109421	0.000024760
10	0.000018290	0.000018290	0.000003755
11	0.000002769	0.000002769	-
12	0.000000439	0.000000439	-

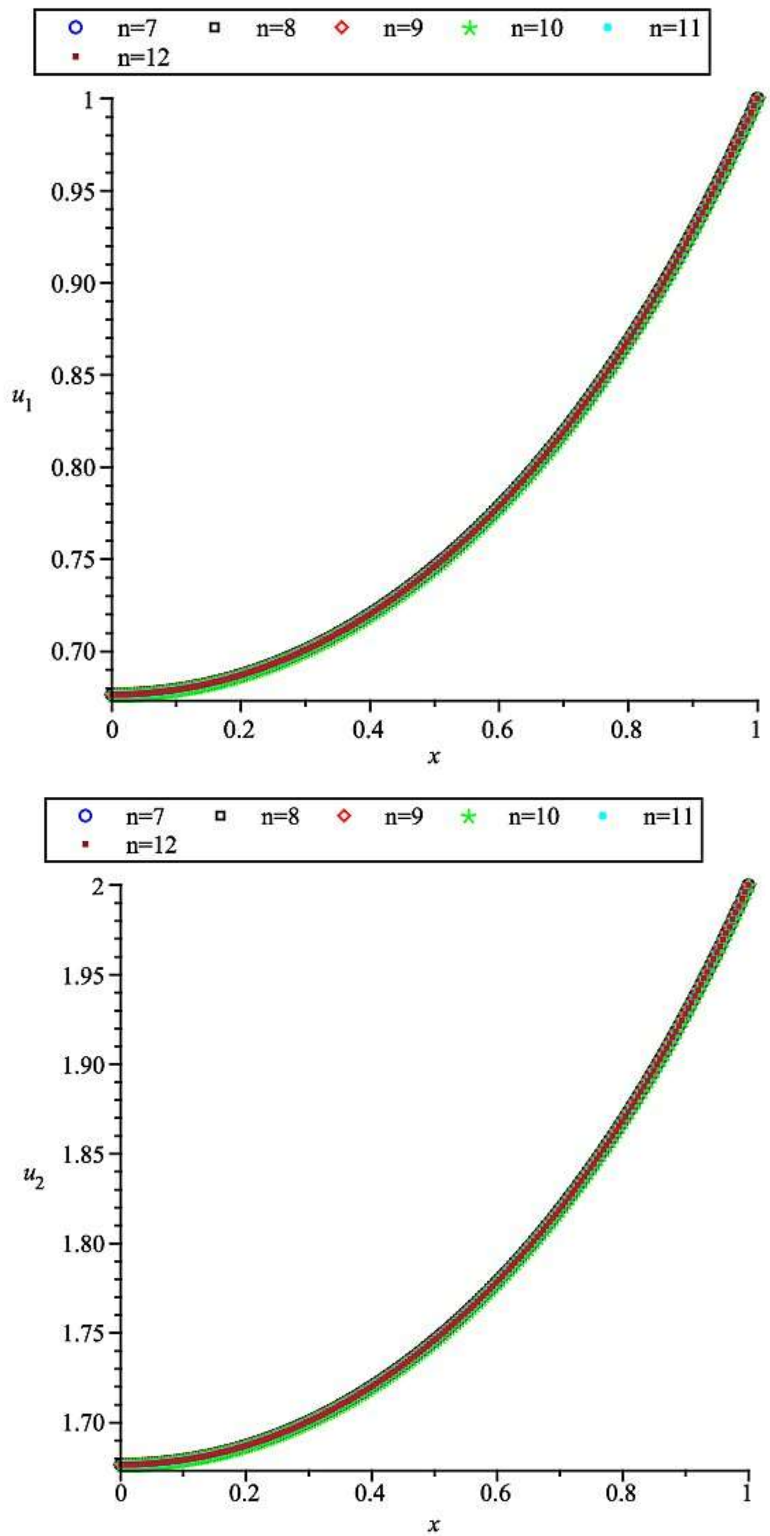


Figure 9. Graphs of the functions $u_1(x)$ and $u_2(x)$ with the GA of RBFs and different values of collocation points in Example 2.

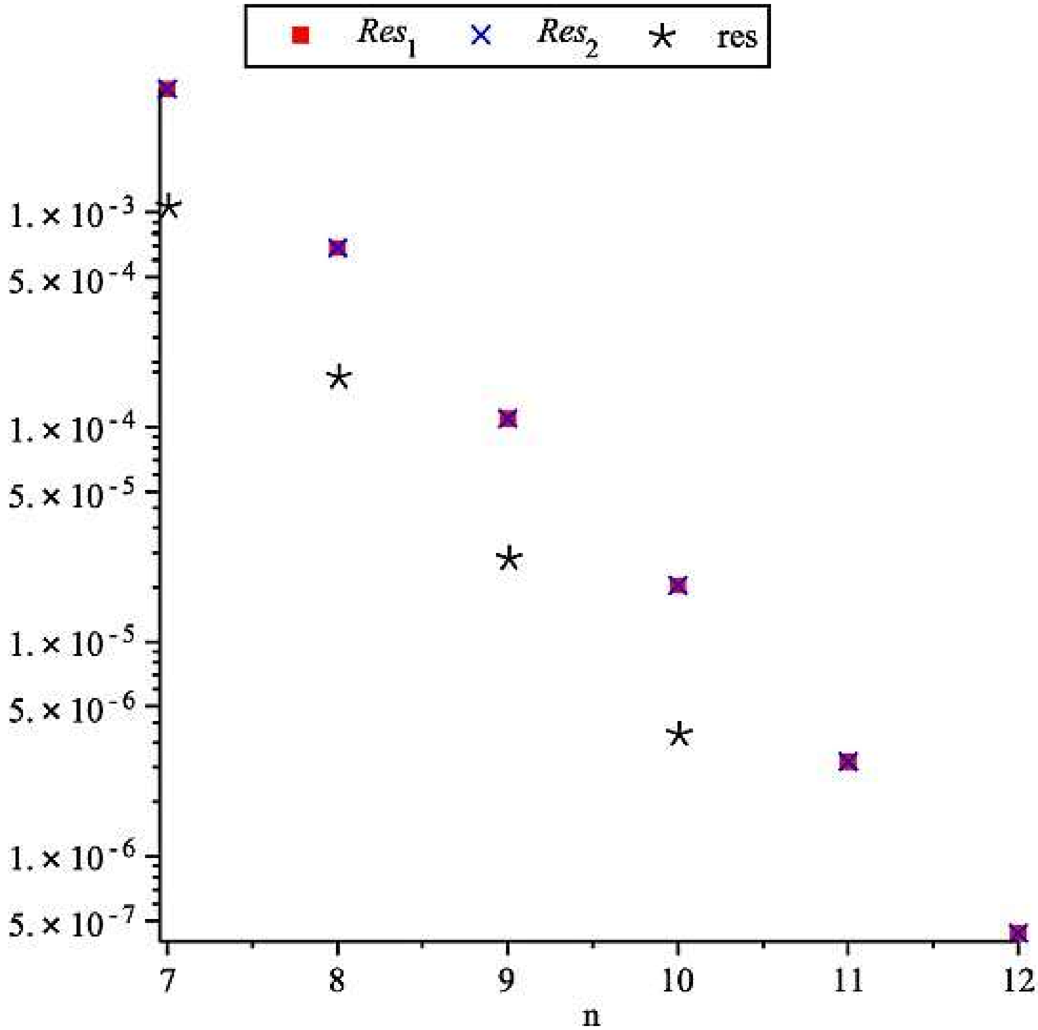


Figure 10. Graph of the res with the GA of RBFs in Example 2.

To illustrate the effect of other basic functions on the RBF method, all the functions listed in Table 1 with $n = 7$ have been applied. The approximate functions of u_1, u_2 derived from this examination are plotted in Figure 11, and the graphs of their absolute residual errors are displayed in Figure 12. It is clear that the most accuracy is related to the GA and SECH functions and the least is the IQ , IMQ, and MQ, respectively.

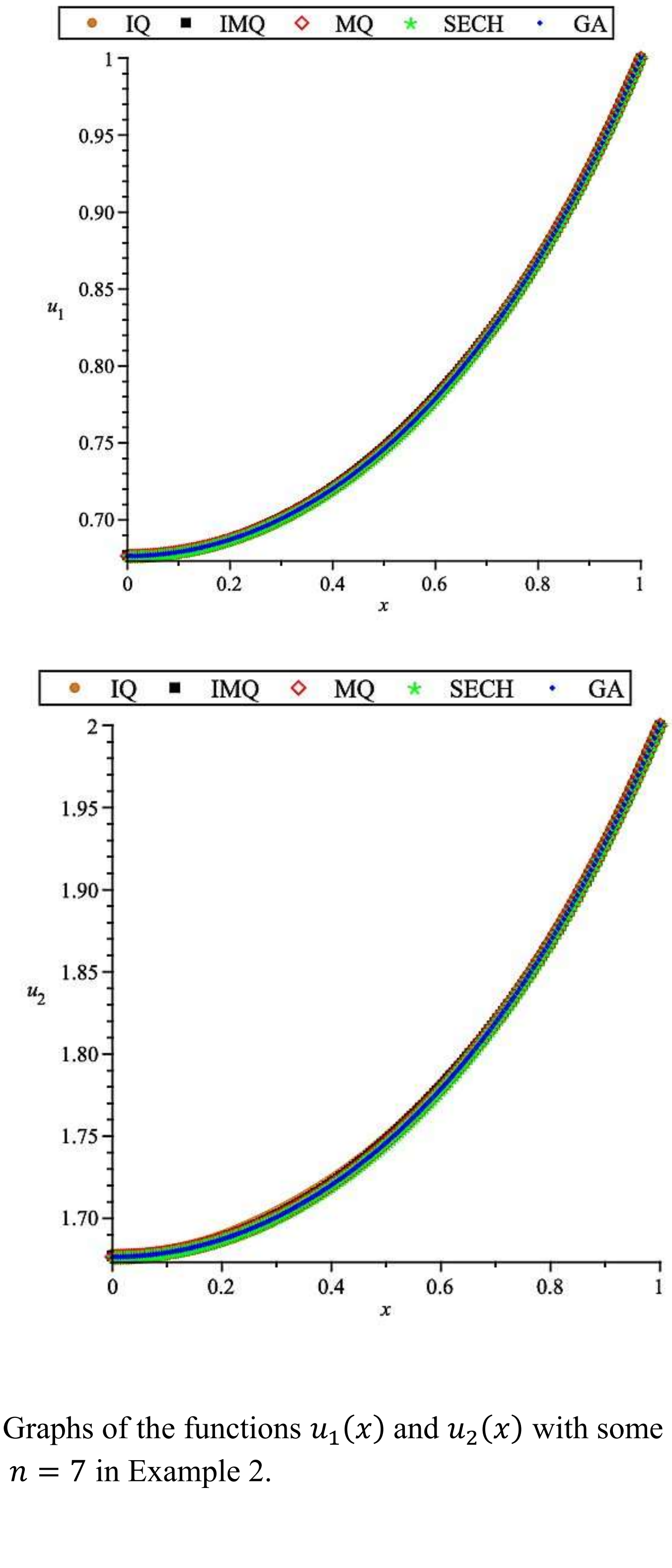
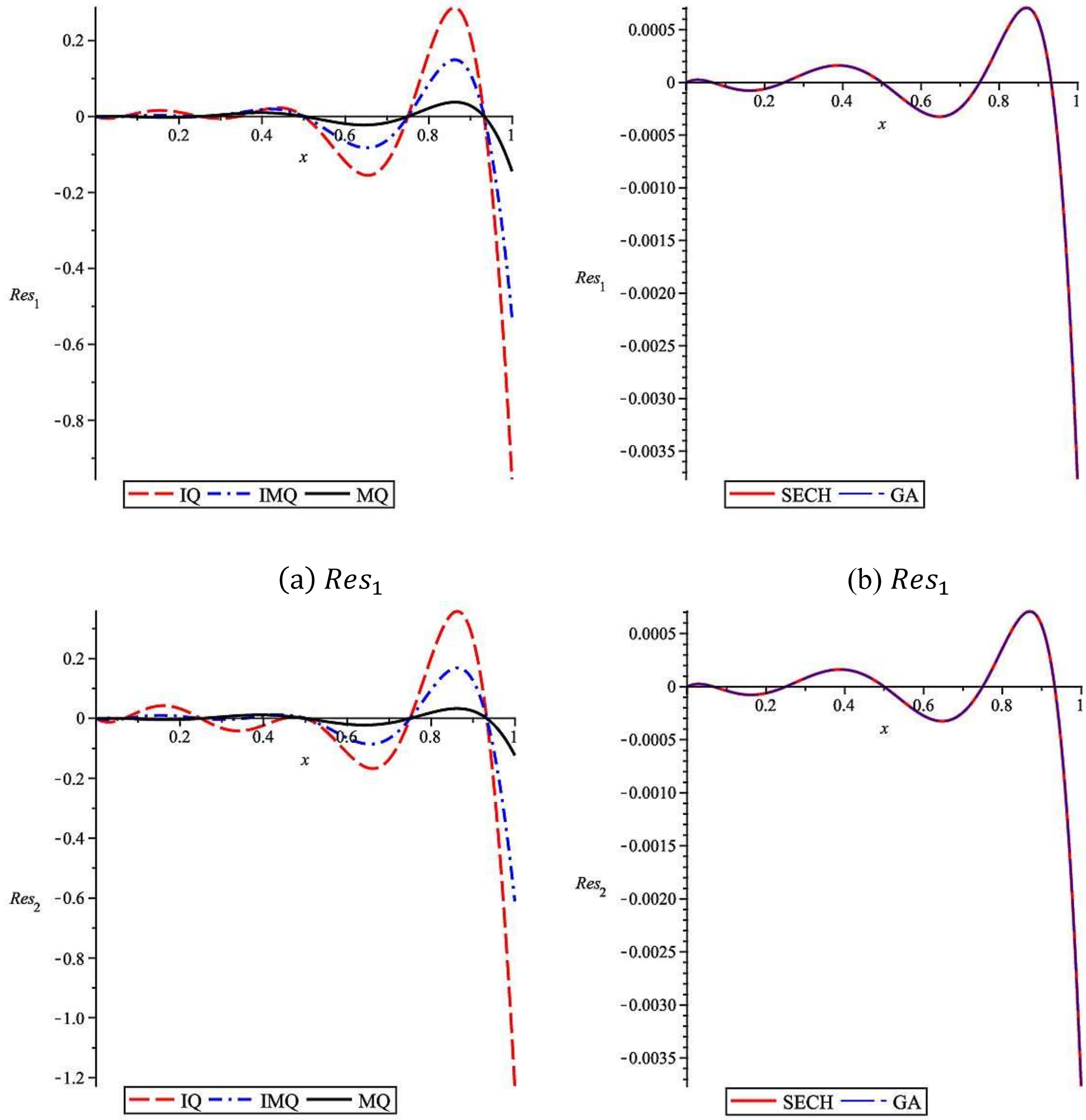


Figure 11. Graphs of the functions $u_1(x)$ and $u_2(x)$ with some of RBFs and $n = 7$ in Example 2.



(a) Res_1 (b) Res_1

(c) Res_2 (d) Res_2

Figure 12. Graphs of the functions $Res_1(x)$ and $Res_2(x)$ with some of RBFs and $n = 7$ in Example 2.

4. CONCLUSION

In the current paper, an efficient algorithm has been presented to solve the coupled Lane-Emden boundary value problems. In this problem, there is the singularity at the origin that in first, the equations are multiplied in x. Next, space dimension is discretized using Kansa collocation based on RBF methods. Finally, the obtained nonlinear system of ordinary equations is solved. The technique is easy to implement, and the numerical results in compared to the results obtained in [2, 23, 25, 26] demonstrate the good accuracy of the method. Also, rapid convergence and not require any linearization confirmed that our technique is capable.

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