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Resolvent Energy of Digraphs

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ABSTRACT

The resolvent energy of a graph G is defined as $ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of G. We extend this concept to directed graphs with two approaches. The first approach, consider $ER(G) = \sum_{i=1}^{n} \frac{1}{n-\sigma_i}$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values of G. The second approach, define the resolvent energy of a digraph G by $ER(G) = \sum_{i=1}^{n} \frac{1}{n-R\ (z_i)}$, where $z_1, \dots, z_n$ are the eigenvalues of G and $Re(z_i)$ denotes the real part of z_i. We prove some properties of resolvent energy for some special digraphs and determine the resolvent energy of unicyclic and bicyclic digraphs and present lower bound for resolvent energy of directed cycles.

1. INTRODUCTION

The energy of a graph is a quantity based on the graph spectrum. Nowadays, it is so interesting topic between researchers who study mathematical chemistry or spectral graph theory. The energy of a graph, defined as the sum of the absolute values of its eigenvalues, was first defined by Ivan Gutman in 1978 at a conference in Austria [11]. For finding more information on graph energy we suggest to see [7, 10, 26]. There are many kinds of graph energies, such as Laplacian energy [6] and Randić energy [20].

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In 2008, Pena and Rada generalized the concept of energy to a digraph G as the sum of the absolute values of the real part of eigenvalues of graph, [24]. Also, the digraph energy was extensively studied [2, 17, 18, 22, 27, 28]. Nikiforov defined the energy of a digraph as the sum of its singular values [23].

Very recently, Gutman et al. introduced the resolvent energy [14], and it is defined by $ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of G. They made conjectures about resolvent energy of unicyclic, bicyclic and tricyclic graphs. Afterward a lot of research has been done on it. In [1, 5], all of those conjectures have been proven. In [14], established a number of bounds on the resolvent energy and characterize trees, unicyclic and bicyclic graphs with smallest and largest resolvent energies. For more results about resolvent energy see [8, 9, 12, 13]. The Laplacian resolvent energy, signless Laplacian resolvent energy and normalized Laplacian resolvent energy were recently put forward in [3, 25].

In this paper, we extend the concept of resolvent energy to directed graphs. We prove some properties of resolvent energy for the directed unicyclic and bicyclic digraphs and make two conjecture about it. Moreover, we present lower bound for resolvent energy of directed cycles.

2. PRELIMINARIES

Let $G = (V, E)$ be a finite and directed graph. We denote by $V = \{v_1, \ldots, v_n\}$ the vertex set of G and by $E = \{e_1, \ldots, e_m\}$ the directed edge (arc) set of G. Since G is directed, elements of E are ordered pairs of elements of V. For $v_i, v_j \in V$, denote by $v_i v_j$ the directed edge (arc) from v_i to v_j. Moreover, we call v_j the head and v_i the tail of $v_i v_j$, respectively.

Let $v_i v_j$ be an arc. We call it simple if $v_j v_i$ is not an arc in G. Also, if every arc in G is simple, then G is called simple. Further, if both $v_i v_j$ and $v_j v_i$ are arcs in G, then they called symmetric arcs.

Let $deg^+(v_i) = |\{v_j \in V, v_i v_j \in E\}|$ denote the outdegree of the vertex v_i, and $deg^-(v_i) = |\{v_j \in V, v_j v_i \in E\}|$ denote the indegree of the vertex v_i in the digraph G.

A directed path of length $n-1$ $(n \geq 2)$, denoted by $\vec{P}_n$, is a graph with vertex set $\{v_i | i = 1, \ldots, n\}$ and arc set $\{v_i v_{i+1} | i = 1, \ldots, n-1\}$. We call v_1 the initial vertex and v_n the terminal vertex of the directed path $\vec{P}_n$, respectively. A cycle of length n, denoted by $\vec{C}_n$, is the digraph with the vertex set $\{v_i | i = 1, \ldots, n\}$ and arc set $\{v_i v_{i+1} | i = 1, \ldots, n-1\} \cup \{v_n v_1\}$. A complete digraph on n vertices denoted by $\overleftrightarrow{K}_n$, with vertex set $\{v_i | i = 1, \ldots, n\}$ and arc set $\{v_i v_j, v_j v_i | 1 \leq i, j \leq n,\ i \neq j\}$.

A directed graph is connected if there is an undirected path between any pair of vertices, and strongly connected if there is a directed path between every pair of vertices. A directed tree is a simple, directed, connected and acyclic graph.

Let $G = (V, E)$ be a digraph with n vertices. The adjacency matrix $A = A(G)$ of the graph G is defined such that its (i, j)-entry is equal to 1 if $v_i v_j \in E$ and 0 otherwise. The characteristic polynomial $\Phi_G(\lambda) = |\lambda I - A|$ of the adjacency matrix A of G is called the characteristic polynomial of G and the eigenvalues of A are called the eigenvalues of G.

Let A be any n by n matrix with real entries and A^T is the transpose of A. The (i, j)–entry of $AA^T(A^T A)$ of G is equal to the number of common out-neighbours (in-neighbours) of v_i and v_j. Diagonal entries of the matrix $AA^T(A^T A)$ represent outdegrees (indegrees) of the vertices of G (see [19]).

The singular values of the matrix A are the square roots of the eigenvalues of AA^T. The singular values of graph G is the singular values of its adjacency matrix. The set of all singular values of a graph G is denoted by $Singular(G)$.

Throughout this paper, graphs are assumed to be finite and without loop, multiple and symmetric arcs.

3. First Approach: Resolvent Energy via Singular Values

Let G be a simple directed graph of order n and $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of its adjacency matrix. If A is adjacency matrix of G, then by [15], $\sigma_1 \leq [||A||_1 ||A||_\infty]^{1/2}$. Since G is simple, $||A||_1, ||A||_\infty \leq n - 1$. Consequently, $\sigma_1 < n$.

We now define the resolvent energy for digraph as follow:

Definition 3.1. *Let G be a digraph on n vertices with singular values $\sigma_1, \dots, \sigma_n$. Its resolvent energy is $ER(G) = \sum_{i=1}^{n} \frac{1}{n - \sigma_i}$.*

Lemma 3.1. *Let G be a digraph. $ER(G) = 1$ if and only if G has no arcs.*

Proof. $ER(G) = 1$ if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$, and this equivalent to $A = 0$. ■

Theorem 3.1. *Let G be a connected digraph of order n and there is at least one vertex v_j in G such that $deg^-(v_j) = 1$. Let G' be a digraph obtained from G by deleting the arc $v_i v_j$. Then $ER(G) \geq ER(G')$.*

Proof. Let $A(G)$ be the adjacency matrices of G with singular values $\sigma_1 \geq \sigma_2 \dots \geq \sigma_n \geq 0$ and $A(G')$ be the adjacency matrix of G' with singular values $\sigma'_1 \geq \sigma'_2 \dots \geq \sigma'_n \geq 0$. By [16, Theorem 4.2], $\sigma_1 \geq \sigma'_1 \geq \sigma_2 \geq \sigma'_2 \geq \cdots \sigma_n \geq \sigma'_n \geq 0$. Therefore, $ER(G) \geq ER(G')$. ■

Remark 3.1. *Let G be a connected digraph of order n in which there is at least one vertex v_j such that $deg^{+}(v_j) = 1$. Let G' be a digraph obtained from G by deleting $v_j v_i$, for some vertex v_i. Then $ER(G) \geq ER(G')$.*

Theorem 3.2. *Let G be a connected digraph of order n and $v_i \in V(G)$ such that $deg^{-}(v_i) = 0$. Define G' by adding a pendant edge $v_{n+1} v_i$ at the vertex v_i. Then*

$$ER(G') = ER(G) + \frac{1}{n-1}.$$

Proof. By [16, Proposition 4.5], we have $Singular(G') = Singular(G) \cup \{1\}$, so the theorem holds. ■

In the following examples, we will show that by deleting an arc, the resolvent energy may increase or decrease.

Example 3.1. *Let G be a graph with the vertex set $V = \{v_1, \ldots, v_9\}$ and arc set*

$$E = \{v_1v_3, v_1v_9, v_2v_1, v_2v_3, v_2v_5, v_2v_6, v_2v_7, v_3v_5, v_3v_6, v_3v_8, v_3v_9, v_4v_1, v_4v_2, v_4v_5, v_4v_6, v_4v_7, v_5v_2, v_5v_3, v_5v_6, v_5v_7, v_5v_8, v_5v_9, v_6v_4, v_7v_5, v_7v_6, v_7v_8, v_7v_9, v_8v_4, v_8v_9, v_9v_4\}.$$

The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $ER(G) = 1.2140$. If $G_1 = G \backslash \{v_7v_6\}$, then $ER(G_1) = 1.2148$.

Example 3.2. *Let G be a graph with the vertex set $V = \{v_1, \ldots, v_5\}$ and arc set*

$$E = \{v_1v_2, v_1v_4, v_2v_1, v_2v_3, v_2v_4, v_2v_5, v_3v_2, v_3v_5, v_4v_2, v_4v_5, v_5v_4\}.$$

The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $ER(G) = 1.3904$. If $G_2 = G \backslash \{v_2v_5\}$, then $ER(G_2) = 1.3596$.

3.1 Resolvent Energy of Some Special Digraphs

In the following, we compute the resolvent energy of some kind of directed graph, for this purpose we need the next theorem.

Theorem 3.3. *Suppose that G is a digraph, $V(G) = \{v_1, \dots, v_n\}$ and $deg^-(v_i) \leq 1$, for $1 \leq i \leq n$. Then $Singular(G) = \{\sqrt{deg^+(v_i)}\colon\ 1 \leq i \leq n\}$.*

Proof. Let A be the adjacency matrix of G and $C = AA^T$. We have $c_{ij} = \sum_{r=1}^{n} a_{ir}a_{jr}$. Since $deg^-(v_i) \leq 1$, so

$$c_{ij} = \begin{cases} deg^+(v_i) & i = j \\ 0 & i \neq j \end{cases},$$

which completes the proof. ■

Similar to the above theorem we can prove the following theorem:

Theorem 3.4. *Suppose that G is a digraph, $V(G) = \{v_1, \dots, v_n\}$ and $deg^+(v_i) \leq 1$, for $1 \leq i \leq n$. Then $Singular(G) = \{\sqrt{deg^-(v_i)}\colon\ 1 \leq i \leq n\}$.*

Remark 3.2. *By Theorem* 3.3, *we have $Singular(\vec{P}_n) = \{(1)^{n-1}, 0\}$.* This implies that $ER(\vec{P}_n) = 1 + \frac{1}{n}$. Therefore, $ER(\vec{P}_3) > \cdots > ER(\vec{P}_n)$.

An out star (in star) is a directed star graph such that for every vertex v, $deg^-(v) \leq 1$ $(deg^+(v) \leq 1)$.

Remark 3.3. *Let $\vec{S}_n$ be the directed out star of order n. By Theorem 3.3, we have*

$$Singular(\vec{S}_n) = \{\sqrt{n-1}, (0)^{n-1}\},$$

and so

$$ER(\vec{S}_n) = \frac{n-1}{n} + \frac{1}{n-\sqrt{n-1}}.$$

Consider the function

$$f(x) = \frac{x-1}{x} + \frac{1}{x-\sqrt{x-1}}.$$

Then we have

$$f'(x) = \frac{1}{x^2} - \frac{1-\frac{1}{2}(x-1)^{\frac{-1}{2}}}{(x-\sqrt{x-1})^2},$$

and $f(x)$ is an increasing function on $[2, \infty)$, which implies that $ER(\vec{S}_3) > \cdots > ER(\vec{S}_n)$. It is possible to obtain the same result for in star.

Remark 3.4. *By Theorem* 3.3*, we get that*

$$Singular(\vec{C}_n) = \{(1)^n\}.$$

So

$$ER(\vec{C}_n) = \frac{n}{n-1} = 1 + \frac{1}{n-1}.$$

Then

$$ER(\vec{C}_3) > \cdots > ER(\vec{C}_n).$$

Remark 3.5. *Let $K_{a,b}$ be the complete bipartite graph with vertex partition $V = \{v_1, \dots, v_a\}$ and $U = \{u_1, \dots, u_b\}$. Assume that asymmetric digraph $\vec{K}_{a,b}$ obtained from $K_{a,b}$ such that each edge $v_i u_j$ of $K_{a,b}$ is changed by an arc $v_i u_j$. Then*

$$Singular(\vec{K}_{a,b}) = \{\sqrt{ab}, (0)^{a+b-1}\}. \tag{1}$$

Hence,

$$ER(\vec{K}_{a,b}) = \frac{1}{a+b-\sqrt{ab}} + \frac{a+b-1}{a+b}. \tag{2}$$

An out tree (in tree) of order n, denoted by $\vec{\mathbb{T}}_n$, is a directed tree such that for every vertex v of vertex set, we have $deg^-(v) \le 1$ $(deg^+(v) \le 1)$.

Theorem 3.5. *Let $\vec{P}_n$, $\vec{\mathbb{T}}_n$ and $\vec{S}_n$ be path, out tree and out star of order* n*, respectively. Then* $ER(\vec{S}_n) \le ER(\vec{\mathbb{T}}_n) \le ER(\vec{P}_n)$.

Proof. Let $\vec{P}_n = v_1 v_2 \dots v_n$ be the path, hence $deg^-(v_1) = 0 = deg^+(v_n)$. Moreover, let $\overleftarrow{\mathbb{T}}_n$ be an out tree of order n. We construct $\vec{\mathbb{T}}_n$ from $\vec{P}_n$. For this purpose, we cut edge $v_{n-1}v_n$ from $\vec{P}_n$ and attach it to v_i (we choose v_i such that the resulting tree is more similar to $\vec{\mathbb{T}}_n$). We call the new tree $\vec{\vec{T}}_1$. By doing this, the vertex v_{n-1} of outdegree equal to 1, change to a vertex with outdegree equal to 0, in $\vec{\vec{T}}_1$. Also the vertex v_i of outdegree equal to 1 changes to vertex of outdegree equal to 2. Therefore,

$$\begin{aligned} ER(\vec{\vec{T}}_1) &= ER(\vec{P}_n) - \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-\sqrt{2}} \\ &= ER(\vec{P}_n) - \frac{2}{n-1} + \frac{1}{n} + \frac{1}{n-\sqrt{2}}. \end{aligned}$$

Since $\frac{2}{n-1} \ge \frac{1}{n} + \frac{1}{n-\sqrt{2}}$, $ER(\vec{P}_n) \ge ER(\vec{\vec{T}}_1)$. Now, we cut edge $v_{n-2}v_{n-1}$ of $\vec{\vec{T}}_1$ and attach it to another vertex in order to change $\vec{\vec{T}}_1$ to a tree more similar to $\vec{\mathbb{T}}_n$. We will continue this process and every time we change a vertex of outdegree equal to 1 to a vertex of outdegree equal to 0 (cut the pendant edge of $\vec{\vec{T}}_i$) and also change a vertex of outdegree equal to d to a vertex of outdegree equal to $d+1$ (add the cut pendent edge to a vertex). Consequently,

$$ER(\vec{\vec{T}}_{i+1}) = ER(\vec{\vec{T}}_i) - \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n-\sqrt{d}} + \frac{1}{n-\sqrt{d+1}}.$$

Since $\frac{1}{n-1}+\frac{1}{n-\sqrt{d}}>\frac{1}{n}+\frac{1}{n-\sqrt{d+1}}$, $ER(\vec{\vec{T}}_i) \geq ER(\vec{\vec{T}}_{i+1})$. Therefore, $ER(\vec{P}_n) \geq ER(\vec{\mathbb{T}}_n)$. By the same argument, it can be shown that $ER(\vec{S}_n) \leq ER(\vec{\mathbb{T}}_n)$, it follows that $ER(\vec{S}_n) \leq ER(\vec{\mathbb{T}}_n) \leq ER(\vec{P}_n)$. ■

3.2 Resolvent Energy of the Unicyclic Digraph

In the sequel of this section, we denote by $\vec{U}_{m,n}$ the connected unicyclic digraph with $n \geq 3$ vertices and unique cycle of order m, where $3 \leq m \leq n$. We consider that the vertices in the cycle are $v_1, v_2, \ldots, v_m$. It is clear that $\vec{U}_{m,n} = \vec{C}_m \cup \vec{T}_1 \cup \ldots \cup \vec{T}_m$, where $\vec{T}_i$ is a directed pendant tree of order t_i hangs of v_i, $1 \leq i \leq m$. Hence $t_i \geq 1$ and so $\sum_{i=1}^{m} t_i = n$. Let $V(T_i) = \{v_i = v_1^i, v_2^i \ldots, v_{t_i}^i\}$, for $1 \leq i \leq m$. Moreover, consider $A_i = A(T_i) = [a_{i_{x,y}}]$ to be the adjacency matrix of T_i.

Theorem 3.6. *Let $\vec{U}_{m,n}$ be an unicyclic digraph such that all directed trees hang of vertices in the cycle are out tree (in tree). The set of singular value of $\vec{U}_{m,n}$ is the square roots of outdegree (indegree) of all vertices.*

Proof. Let T_i be an out tree, so $A_iA_i^T = diag(deg^+(v_1^i), \ldots, deg^+(v_{t_i}^i))$, for $1 \leq i \leq m$, by Theorem 3.3. Let A be the adjacency matrix of $\vec{U}_{m,n}$. Suppose that the order of the vertices on rows and columns of A are $\{v_1, \ldots, v_m, v_2^1, \ldots, v_{t_1}^1, \ldots, v_2^m, \ldots, v_{t_m}^m,\}$. Let

$$n_i = \sum_{j=1}^{i} (t_j - 1), \quad 1 \leq i \leq m-1,$$

and

$$k_i = m + n_{i-1} + 1, \quad 1 \leq i \leq m,$$

where $n_0 = 0$. Then, the adjacency matrix A is a block matrix as follows:

$$A = \begin{bmatrix} A(C_m) & B_{1,2} & & \ldots & B_{1,m+1} \\ 0 & B_{2,2} & 0 & \ldots & 0 \\ 0 & 0 & \ddots & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & & & B_{m+1,m+1} \end{bmatrix},$$

where $B_{1,1} = A(C_m), B_{1,j}$ is an m by $(t_{j-1} - 1)$ matrix with zero elements except elements on $(j-1)$-th row, which are $(a_{j_{1,2}}, \ldots, a_{j_{1,t_j}})$, for $2 \leq j \leq m+1$, and

$$B_{i,i} = (a_{t,s}), \quad k_i \leq t, s < k_{i+1}.$$

Let $C = AA^T$. Then, C is a block diagonal matrix as $C = (C_{i,j})$, for $1 \leq i, j \leq m+1$, such that

$$\begin{aligned}
C_{1,1} &= B_{1,1}B_{1,1}^T + \cdots + B_{1,m+1}B_{1,m+1}^T \\
&= A(C_m)A^T(C_m) + \begin{bmatrix} \sum_{j=k_1}^{k_2-1} a_{1,j} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \end{bmatrix} + \cdots \\
&+ \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & & \\ 0 & \dots & 0 & \sum_{j=k_m}^{n} a_{m,j} \end{bmatrix} \\
&= I + \begin{bmatrix} \sum_{j=2}^{t_1} a_{1,j} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sum_{j=2}^{t_m} a_{m_{1,j}} \end{bmatrix} \\
&= diag(deg^+(v_1), \dots, deg^+(v_m))
\end{aligned}$$

and

$$\begin{aligned}
C_{i,i} &= \sum_{r=1}^{m+1} B_{i,r}B_{i,r} = B_{i,i}^2 \\
&= diag(deg^+(v_2^{i-1}), \dots, deg^+(v_{t_{i-1}}^{i-1})),
\end{aligned}$$

for $2 \leq i \leq m+1$. ■

Similarly, if T_i is an in tree, for $1 \leq i \leq m$, then we get the same result.

Remark 3.6. *Let $\vec{G} = \vec{U}_{m,n}$ be an unicyclic digraph such that all directed trees hang of vertices in the cycle are out tree. Therefore, $ER(\vec{G}) = \sum_{i=1}^{n} \frac{1}{n-deg^+(v_i)}$.* Similarly, if all directed trees hang of vertices in the cycle are in tree, then $ER(\vec{G}) = \sum_{i=1}^{n} \frac{1}{n-deg^-(v_i)}$.

Lemma 3.2. *Let $\vec{G}_1$ and $\vec{G}_2$ be unicyclic digraphs of order n with cycle of order k, which is shown in the figure 1. Then $ER(\vec{G}_1) \leq ER(\vec{G}_2)$*

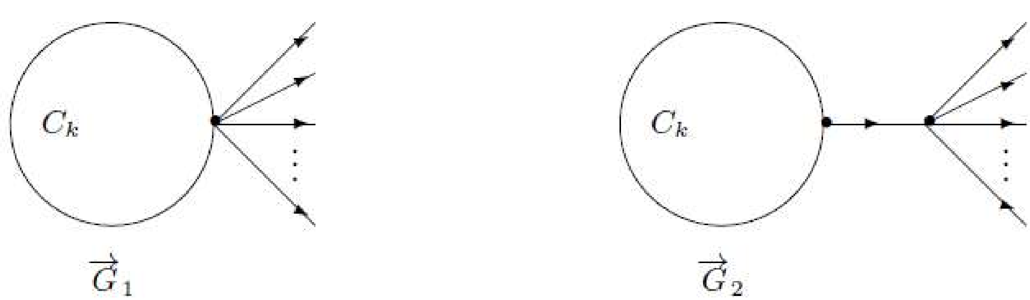

Figure 1: Digraphs $\vec{G}_1$ and $\vec{G}_2$.

Proof. By Theorem 3.6, the singular values of $\vec{G}_1$ and $\vec{G}_2$ are

$$Singular(\vec{G}_1) = \{(1)^{k-1}, \sqrt{n-k+1}, (0)^{n-k}\}$$

and

$$Singular(\vec{G}_2) = \{(1)^{k-1}, \sqrt{n-k-1}, \sqrt{2}, (0)^{n-k-1}\}.$$

Then,

$$ER(\vec{G}_1) - ER(\vec{G}_2) = \frac{1}{n} + \frac{1}{n-\sqrt{n-k+1}} - \frac{1}{n-\sqrt{2}} - \frac{1}{n-\sqrt{n-k-1}} \leq 0,$$

which implies that $ER(\vec{G}_1) \leq ER(\overleftarrow{G}_2)$. ■

Theorem 3.7. *Let $\vec{G} = \vec{U}_{m,n}$ be an unicyclic digraph such that all directed trees hang of the vertices in the cycle are out tree. Then $ER(\vec{U}_{m,n}) \leq ER(\vec{C}_n)$.*

Proof. Let $\vec{C}_n = v_1v_2 \ldots v_nv_1$ be a cycle and $\vec{U}_{m,n}$ be an unicyclic digraph. Similar to the Theorem 3.5, we construct $\vec{U}_{m,n}$ from $\vec{C}_n$. Hence, we cut an arbitrary edge v_iv_{i+1} from $\vec{C}_n$ and adjacent v_i to v_{i+2}. Moreover, attach the edge v_iv_{i+1} to v_k (we choose v_k such that the resulting graph is more similar to $\vec{U}_{m,n}$). We call the new graph $\vec{\tilde{C}}_1$. It implies that the outdegree of v_i changes from 1 to 0 and the outdegree of v_k changes from 1 to 2. Therefore,

$$ER(\vec{\tilde{C}}_1) = ER(\vec{C}_n) - \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-\sqrt{2}}.$$

Since $\frac{2}{n-1} \geq \frac{1}{n} + \frac{1}{n-\sqrt{2}}$, $ER(\vec{C}_n) \geq ER(\vec{\tilde{C}}_1)$. It is clear that $\vec{\tilde{C}}_1$ is an unicyclic digraph. We will continue this process and every time we cut an edge of cyclic of graph such that the head vertex of it has outdegree equal to one. Then attach that edge to a vertex in order to change $\vec{\tilde{C}}_i$ to an unicyclic digraph more similar to $\vec{U}_{m,n}$. In addition, similar to above, we have $ER(\vec{\tilde{C}}_{i+1}) = ER(\vec{\tilde{C}}_i) - \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n-\sqrt{d}} + \frac{1}{n-\sqrt{d+1}}$. Therefore, $ER(\vec{\tilde{C}}_i) \geq ER(\vec{\tilde{C}}_{i+1})$, which implies that $ER(\vec{C}_n) \geq ER(\vec{U}_{m,n})$. ■

We did a lot of tests with MATLAB 2020 about resolvent energy of unicyclic digraphs, finally we state the following conjecture:

Conjecture 3.1. *Let $\vec{X}_n$ be an unicyclic digraphs of order n such that the cycle is from order three with $\mathrm{n}-3$ vertices which have outdegree equal to one. Then $ER(\vec{X}_n) \leq ER(\vec{U}_{m,n})$.*

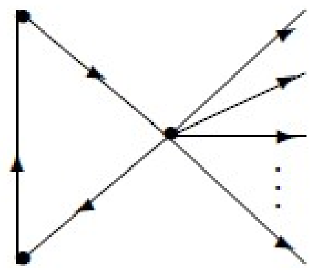

Figure 2. Unicyclic digraph $\vec{X}_n$.

3.3. RESOLVENT ENERGY OF THE BICYCLE DIGRAPH

Suppose $\vec{B}_n$ to be a bicyclic digraph of order n and $\vec{K}$ is a subgraph of it such that $\vec{K}$ is the unique bicyclic subdigraph of $\vec{B}_n$ containing no pendant vertices. In the other word, $\vec{B}_n$ obtain from $\vec{K}$ by attaching directed trees to some vertices of $\vec{K}$. It is well known that there are three types of bicyclic digraphs containing no pendant vertex.

Type 1. Let $\vec{B}(p,q)$, where $p,q \geq 3$ be the bicyclic digraph obtained from two directed cycles $\overleftarrow{C}_p$ and $\overleftarrow{C}_q$, which have just one common vertex.

Type 2. Let $\vec{B}(p,l,q)$, where $p,q \geq 3$ be the bicyclic digraph obtained from two directed cycles $\vec{C}_p$ and $\vec{C}_q$ with one unique directed path $\vec{P}_l$ connecting $\vec{C}_p$ and $\vec{C}_q$.

Type 3. Let $\vec{B}(P_s,P_l,P_m)$, where $s,l,m \geq 3$ be the bicyclic digraph obtained from three pairwise disjoint directed paths from a vertex x to a vertex y, these three directed paths are with length s,l and m and are not in the same direction, for instance, P_s and P_m are from left to right and P_l is from right to left.

We consider that the vertices on $\vec{K}$ are $\{v_1,\dots,v_t\}$. It is clear that

$$\vec{B}_n = \vec{K} \cup \vec{T}_1 \cup \dots \cup \vec{T}_t,$$

where $\vec{T}_i$ is a directed pendant tree of order a_i hangs of v_i, $1 \leq i \leq t$. Hence $a_i \geq 1$ and so $\sum_{i=1}^{t} = n$.

In the following, we compute the singular values of bicyclic digraphs. We consider the following cases:

Case 1: Let $\vec{K} = \vec{B}(p,q)$, $|\vec{K}| = k = p + q - 1$ and A be its adjacency matrix. In addition let $C = AA^T$, hence $c_{ij} = \sum_{r=1}^{n} a_{ir}a_{jr}$ and we have:

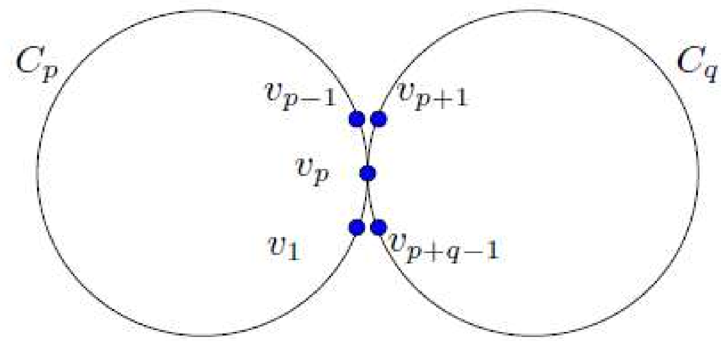

Figure 3. Bicyclic digraph $B(p,q)$.

We consider the direction of both cycles are clockwise. Hence,

$$c_{ij}=\begin{cases}1 & i=j\neq p,(i,j)=(p-1,k),(k,p-1)\\ 2 & i=j=p\\ 0 & o.w\end{cases}.$$

Therefore,

$$\begin{aligned}det(C-\lambda I)&=(1-\lambda)^{p-2}((1-\lambda)^{q}(2-\lambda)-(2-\lambda)(1-\lambda)^{q-2})\\ &=(1-\lambda)^{p+q-4}(2-\lambda)^{2}\lambda.\end{aligned}$$

Then $Singular(\vec{K})=\{(\sqrt{2})^2,(1)^{k-3},0\}$. For other cases of the direction on the cycles, one can see its matrix AA^T is similar to the above matrix, so their singular values are the same.

Case 2: Let $\vec{K}=\vec{B}(p,l,q)$ and $|\vec{K}|=k=p+q+l-2$

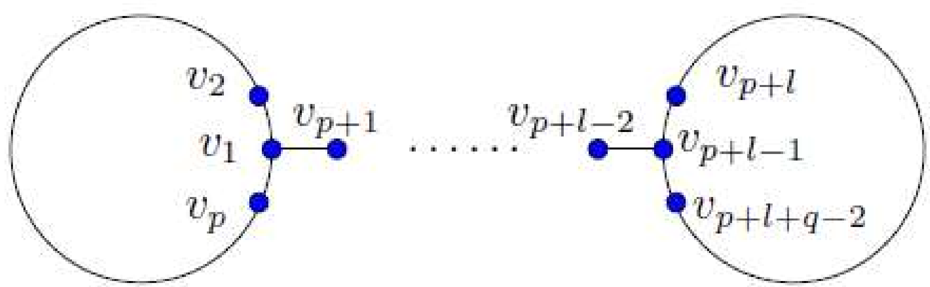

Figure 4. Bicyclic digraph $\vec{B}(p,l,q)$.

Let the direction of both cycles are clockwise and the direction of the path be from left to right. In this case, we have

$$c_{ij}=\begin{cases}1 & i=j\neq 1,(i,j)=(k-q,k),(k,k-q)\\ 2 & i=j=1\\ 0 & o.w\end{cases}$$

similar to the above case, we have $Singular(\vec{K}) = \{(\sqrt{2})^2, (1)^{k-3}, 0\}$. For other cases of the direction on the cycles and path, one can see its matrix AA^T is similar to the above matrix, so their singular values are the same.

Case 3: Let $\vec{K} = \vec{B}(P_s, P_l, P_m)$, $|\vec{K}| = k = s + l + m - 4$ and A be its adjacency matrix. In addition let $C = AA^T$, hence $c_{ij} = \sum_{r=1}^{n} a_{ir}a_{jr}$ and we have:

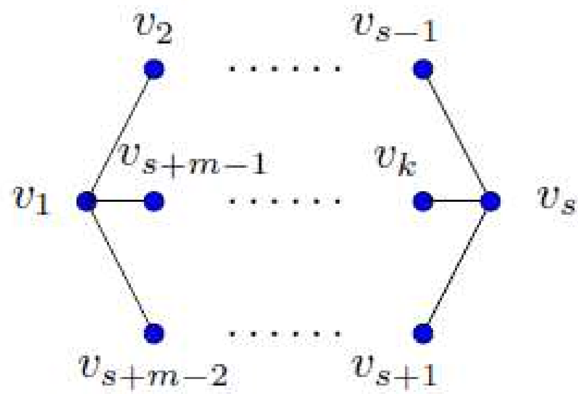

Figure 5. Bicyclic digraph $B(P_k, P_l, P_m)$.

Let top and middle paths be from left to right and the down path be from right to left.

$$c_{ij} = \begin{cases} 1 & i = j \neq 1, (i,j) = (s-1,k), (k,s-1), \\ 2 & i = j = 1, \\ 0 & o.w. \end{cases}$$

Therefore, $Singular(\vec{K}) = \{(\sqrt{2})^2, (1)^{k-3}, 0\}$. For other cases of the direction on the paths, one can see its matrix AA^T is similar to the above matrix, so their singular values are the same.

Remark 3.7. *Let $\vec{G}$ be a bicyclic digraph of order n of any type, without any directed pendant trees, so $ER(\vec{G}) = \frac{2}{n-\sqrt{2}} + \frac{n-3}{n-1} + \frac{1}{n}$.*

Similar to Theorem 3.5 and 3.7, the next theorem follows.

Theorem 3.8. *Let $\vec{\vec{B}}_n$ be a bicyclic digraphs of order n such that all pendent directed trees hang of it are out tree. Also let $\vec{H} \in \{\vec{B}(p,q), \vec{B}(p,l,q), \vec{B}(P_s, P_l, P_m)\}$ be a bicyclic digraphs of order n. Then $ER(\vec{\vec{B}}_n) \leq ER(\vec{H})$.*

4. SECOND APPROACH: RESOLVENT ENERGY VIA EIGENVALUES

Let G be a simple directed graph of order n and $z_1, \ldots, z_n$ be the eigenvalues of its adjacency matrix. The spectral radius of G is defined by $\rho = \rho(G)$ and defined as

$$\rho(G) = \max_{1 \le i \le n} |z_i|.$$

Here, we introduce the resolvent energy of a digraph by the real part of its eigenvalues. First we recall some results.

Lemma 4.1. ([21]) *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. Then*

$$\min_{1 \le i \le n} \sum_{j=1}^{n} a_{ij} \le \rho(A) \le \max_{1 \le i \le n} \sum_{j=1}^{n} a_{ij}.$$

Corollary 4.1. *Let G be a simple digraph with n vertices and $A = (a_{ij})$ be its adjacency matrix with eigenvalues $z_1, \ldots, z_n$. Then $|\sum_{j=1}^{n} a_{ij}| < n$, for $i = 1, \ldots, n$, and we have*

$$|z_i| \le \rho(A) < n.$$

Now we define resolvant energy for digraph as follow:

Definition 4.1. *Let G be a graph on n vertices with eigenvalues $z_1, \ldots, z_n$. Its resolvent energy is*

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - Re(z_i)}, \tag{3}$$

where $Re(z_i)$ denotes the real part of z_i.

Since G is a simple digraph, so by Corollary 4.1, $|Re(z_i)| < n$ and hence equality (3) is well-defined.

Definition 4.2. *Let G be a digraph with n vertices and eigenvalues $z_1, \ldots, z_n$. The k-th spectral moment of the digraph G is defined as*

$$M_k = M_k(G) = \sum_{i=1}^{n} Re(z_i^k).$$

Lemma 4.2. *Let G be a digraph. Then $M_k(G)$ is equal to the number of self-returning walks of length k.*

Proof. By [4], the sum of k-th power of all eigenvalues of a graph is equal to the number of self-returning walks of length k. Let $z_1, \ldots, z_n$ be eigenvalues of the digraph G. Assume that $z_1, \ldots, z_s$ are complex numbers and $z_{s+1}, \ldots, z_n$ are real numbers. If G has complex eigenvalues then they will always occur in complex conjugate pairs. Let $z_j = |z_j|(cos(\theta_j) + i\ sin(\theta_j))$, for $j = 1, \ldots, s$. Then

$$z_j^k = |z_j|^k(cos(k\theta_j) + i\ \ sin(k\theta_j)), \qquad j = 1, \dots, s.$$

It follows that

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n} z_j^k = \sum_{j=1}^{s} z_j^k + \sum_{j=s+1}^{n} z_j^k \\
&= 2\textstyle\sum_{j=1}^{\frac{s}{2}} |z_j|^k cos(k\theta_j) + \sum_{j=s+1}^{n} z_j^k \\
&= \textstyle\sum_{j=1}^{n} Re\left(z_i^k\right) = M_k(G).
\end{aligned}$$

This completes the proof. ∎

In the following examples, we will show that by deleting an arc, the resolvent energy may increase, decrease or not change.

Example 4.1. *Let G be a digraph with the vertex set $V = \{v_1, \dots, v_5\}$, arc set*

$$E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_1, v_2v_4, v_3v_2, v_3v_4, v_3v_5, v_4v_2, v_5v_1, v_5v_4\}.$$

The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $ER(G) = 1.076097$. If $G_1 = G\backslash\{v_5v_4\}$, then $ER(G_1) = 1.076491$.

Example 4.2. *Let G be a digraph with the vertex set $V = \{v_1, \dots, v_5\}$, arc set*

$$E = \{v_1v_2, v_1v_4, v_2v_1, v_2v_3, v_2v_4, v_2v_5, v_3v_2, v_3v_5, v_4v_2, v_4v_5, v_5v_4\}.$$

The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $ER(G) = 1.087597$. If $G_2 = G\backslash\{v_2v_5\}$, then $ER(G_2) = 1.080998$.

An out tree (in tree) of order n, denoted by $\overrightarrow{\mathbb{T}}_n$, is a directed tree such that for every vertex v of vertex set, we have $deg^-(v) \leq 1$ ($deg^+(v) \leq 1$). Moreover, an out star (in star) is a directed star graph such that for every vertex v, $deg^-(v) \leq 1$ ($deg^+(v) \leq 1$).

Example 4.3. *The resolvent energy of out star* (*in star*) *or out tree* (*in tree*) *digraphs of order n is equal to* 1*, and after deleting any edge does not change.*

The next theorem is very useful in calculating the resolvent energy of a graph.

Theorem 4.1. *Let G be a digraph and $G_1, \dots, G_k$ its strong components. Then*

$$ER(G) = \sum_{i=1}^{k} ER(G_i).$$

Proof. Since $G_1, \ldots, G_k$ are strong components of G, $\Phi_G(x) = \Phi_{G_1}(x) \ldots \Phi_{G_k}(x)$, then
$$Spec(G) = \bigcup_{i=1}^{k} Spec(G_i),$$
this completes the proof. ■

Let G be an acyclic digraph with n vertices. Since its strong components are isolated vertices, so the characteristic polynomial of G is $\Phi_G(x) = x^n$, by Theorem 4.1. Therefore, the resolvent energy of an acyclic digraph is equal to 1.

A linear digraph is a digraph such that every vertex has indegree and outdegree equal to 1. Clearly, a linear digraph consists of cycles. By the following theorem (the Coefficient theorem for Digraphs) we can calculate the characteristic polynomial for digraph.

Theorem 4.2. [4, Theorem 1.2] *Let G be a digraph with characteristic polynomial*
$$\Phi_G(x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n. \tag{4}$$
Then for $k = 1, \ldots, n$
$$b_k = \sum_{L \in \mathcal{L}_k} (-1)^{comp(L)},$$
where $\mathcal{L}_k$ is the set of all linear subdigraphs L of G with exactly k vertices; $comp(L)$ denotes the number of components of L.

4.1 Resolvent Energy of the Cyclic Digraph

Now we consider cyclic digraph. Let $\vec{C}_n$ be the cycle of n vertices. By Theorem 4.2, one can easily see, the characteristic polynomial of $\vec{C}_n$ is $\Phi_{\vec{C}_n}(x) = x^n - 1$ and so the eigenvalues of $\vec{C}_n$ are the n-th roots of unity. Therefore, the adjacency spectrum of cycle $\vec{C}_n$ is
$$Spec(\vec{C}_n) = \{cos(\tfrac{2k\pi}{n}) + i\ sin(\tfrac{2k\pi}{n}),\quad k = 0, \cdots n-1\}, \tag{5}$$
and the resolvent energy of $\vec{C}_n$ is $ER(\vec{C}_n) = \sum_{k=0}^{n-1} \frac{1}{n - cos(\frac{2k\pi}{n})}$.

In order to compare the resolvent energy of the circles and gain better view of the properties of resolvent energy, we have done extensive computer-assisted studies. The results were as follows:

$$\begin{array}{llll}
ER(\vec{C}_3) & = 1.071429 & ER(\vec{C}_9) & = 1.006231 \\
ER(\vec{C}_4) & = 1.033333 & ER(\vec{C}_{10}) & = 1.005038 \\
ER(\vec{C}_5) & = 1.020642 & ER(\vec{C}_{100}) & = 1.000050 \\
ER(\vec{C}_6) & = 1.014186 & ER(\vec{C}_{200}) & = 1.000013 \\
ER(\vec{C}_7) & = 1.010363 & ER(\vec{C}_{900}) & = 1.000001 \\
ER(\vec{C}_8) & = 1.007905 & &
\end{array}$$

We did a lot of tests with MATLAB 2020, which are shown in Figure 2. The results show the energy resolvent of $\vec{C}_n$ decreases with increasing n. Finally we state the following conjecture:

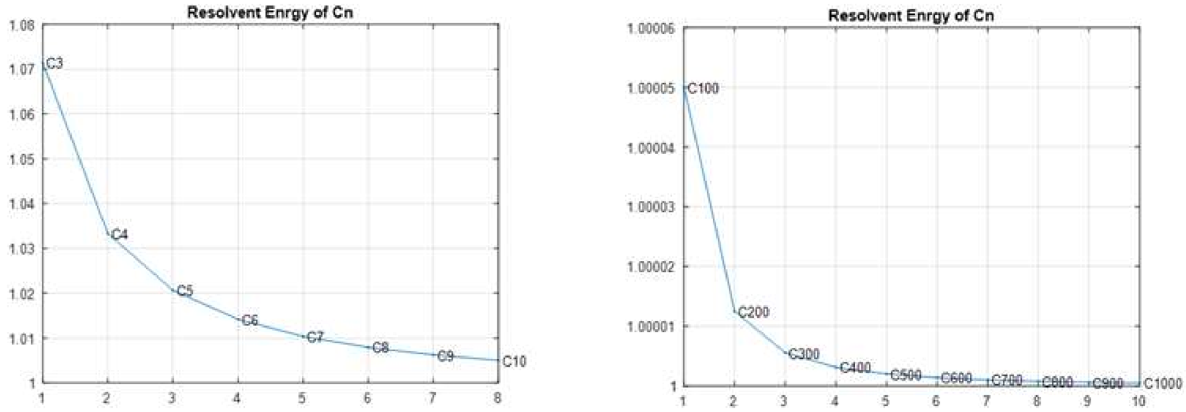

Figure 4. The resolvent energy of cyclic digraph.

Conjecture 4.1. *The resolvent energy of* $\vec{C}_n$ *decreases with increasing* n, *in other word*

$$ER(\vec{C}_3) > ER(\vec{C}_4) > \cdots.$$

4.2 RESOLVENT ENERGY OF THE UNICYCLIC DIGRAPH

In the sequel of this section, we denote by $\vec{U}_{m,n}$ the connected unicyclic digraph of girth m with $n \geq 3$ vertices and unique cycle of order m, where $3 \leq m \leq n$. We consider that the vertices in the cycle are $v_1, v_2, \ldots, v_m$. It is clear that

$$\vec{U}_{m,n} = \vec{C}_m \cup \vec{T}_1 \cup \ldots \cup \vec{T}_m,$$

where $\vec{T}_i$ is a directed pendant tree of order a_i hangs of v_i, $1 \leq i \leq m$. Hence $a_i \geq 1$ and so $\sum_{i=1}^{m} a_i = n$.

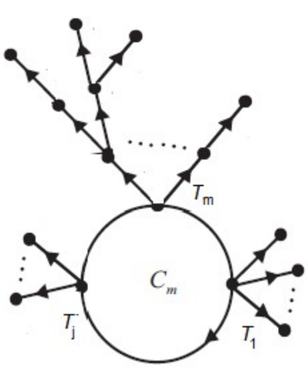

Figure 5. The unicyclic digraph of order n and girth m.

Now we want to calculate the resolvent energy of $\vec{U}_{m,n}$. By Theorem 4.1, we have the next result.

Theorem 4.3. *Let* $G = \vec{U}_{m,n}$. *Then*

$$ER(G) = \sum_{k=1}^{m} \frac{1}{n - cos(\frac{2k\pi}{m})} + \frac{\sum_{i=1}^{m}(a_i - 1)}{n}$$

$$= \sum_{k=1}^{m} \frac{1}{n - cos(\frac{2k\pi}{m})} + \frac{n-m}{n}.$$

In the following, we compare the resolvent energy of unicyclic digraphs. As you can see in Table 1, we show the resolvent energy of unicyclic digraph of order $n \leq 10$.

Table 1. The resolvent energy of unicyclic digraph of order $n \leq 10$.

n	$ER(\overleftarrow{U}_{n,n})$	$ER(\overleftarrow{U}_{n-1,n})$	$ER(\overleftarrow{U}_{n-2,n})$	$ER(\overleftarrow{U}_{n-3,n})$	$ER(\overleftarrow{U}_{n-4,n})$	$ER(\overleftarrow{U}_{n-5,n})$	$ER(\overleftarrow{U}_{n-6,n})$	$ER(\overleftarrow{U}_{n-7,n})$
3	1.071429							
4	1.033333	1.027778						
5	1.020642	1.016667	1.013636					
6	1.014186	1.011828	1.009524	1.007692				
7	1.010363	1.008883	1.007405	1.005952	1.004762			
8	1.007905	1.006917	1.005929	1.004942	1.003968	1.003951		
9	1.006231	1.005538	1.004846	1.004154	1.003462	1.002778	1.002193	
10	1.005038	1.004534	1.004030	1.003526	1.003023	1.002519	1.002020	1.001586

Also we did a lot of tests with MATLAB 2020, which are shown in Figure 4. The results show that the resolvent energy of $\vec{U}_{m,n}$ decreases with decreasing m for the fix n. Finally we state the following conjecture:

Conjecture 4.2. *For a fixed* n, *the resolvent energy of* $\vec{U}_{m,n}$ *decreases with decreasing* m, *in other word*

$$ER(\vec{U}_{3,n}) \leq \cdots \leq ER(\vec{U}_{n-1,n}) \leq ER(\vec{U}_{n,n}). \tag{6}$$

4.3 Lower Bound on Resolvent Energy of Directed Cycle

Now, we present a lower bound on $ER(\vec{C}_n)$. First, consider the following trigonometric identity.

Remark 4.1. *We have*

$$\sum_{i=0}^{k-1} \cos(a + ib) = \frac{\sin(\frac{kb}{2})}{\sin(\frac{b}{2})} \cos(a + (k-1)\frac{b}{2}).$$

In this case, we have the following results:

1. Let $n = 4k$, so
$$\sum_{i=1}^{k-1} \cos\frac{4i\pi}{n} = 0.$$

2. Let $n = 4k + 2$, so
$$\sum_{i=1}^{k} \cos\frac{4i\pi}{n} = -1/2.$$

3. Let $n = 4k + 1$ or $n = 4k + 3$. We have
$$\sum_{i=1}^{k} \cos\frac{2i\pi}{n} = \frac{1}{4\sin(\frac{\pi}{2n})} - \frac{1}{2},$$
$$\sum_{i=1}^{k} \cos\frac{(2i-1)\pi}{n} = \frac{1}{4\sin(\frac{\pi}{2n})}.$$

Theorem 4.4. *Let $\vec{C}_n$ be a cycle of order n. Then*
$$ER(\vec{C}_n) \geq \begin{cases} \frac{2n}{n^2-1} + \frac{2}{n} + \frac{n(n-4)}{n^2-\frac{1}{2}} & n = 4k, \\ \frac{2n}{n^2-1} + \frac{n(n-2)}{n^2-\frac{1}{2}+\frac{1}{n-2}} & n = 4k+2, \\ \frac{1}{n-1} + \frac{(n-1)(n^2-n+1)}{n(n^2-n+2)} & n = 4k+1, \\ \frac{1}{n-1} + \frac{(n-3)(n^2-3n+1)}{n(n^2-3n+2)} & n = 4k+3, \end{cases}$$
where k is a positive integer.

Proof. We consider the following two cases:

• Let n be even, hence $n = 2(2k + l)$, where $l \in \{0,1\}$. By (5), the set of real part of spectrum of $\vec{C}_n$ is
$$\{\pm 1, \pm(cos\frac{2\pi}{n})^{(2)}, \pm(cos\frac{4\pi}{n})^{(2)}, \dots \pm (cos\frac{2((k+l)-1)\pi}{n})^{(2)}, \pm(l\ \ cos\frac{2k\pi}{n})^{(2)}\}.$$
Therefore,
$$ER(\vec{C}_n) = \frac{1}{n-1} + \frac{1}{n+1} + 2\sum_{i=1}^{k+l-1} [\frac{1}{n-\cos\frac{2i\pi}{n}} + \frac{1}{n+\cos\frac{2i\pi}{n}}] + \frac{2(1-l)}{n}$$
$$= \frac{2n}{n^2-1} + 2\sum_{i=1}^{k+l-1} \frac{2n}{n^2-\cos^2\frac{2i\pi}{n}} + \frac{2(1-l)}{n}$$
Now by the Cauchy-Schwarz inequality, we get that
$$ER(\vec{C}_n) \geq \frac{2n}{n^2-1} + \frac{4n(k+l-1)^2}{n^2(k+l-1)-\sum_{i=1}^{k+l-1}\cos^2\frac{2i\pi}{n}} + \frac{2(1-l)}{n}$$
$$= \frac{2n}{n^2-1} + \frac{4n(k+l-1)^2}{n^2(k+l-1)-\frac{1}{2}\sum_{i=1}^{k+l-1}(1+\cos\frac{4i\pi}{n})} + \frac{2(1-l)}{n}$$

$$= \frac{2n}{n^2-1} + \frac{4n(k+l-1)^2}{n^2(k+l-1)-\frac{1}{2}(k+l-1)-\frac{1}{2}\sum_{i=1}^{k+l-1}\cos\frac{4i\pi}{n}} + \frac{2(1-l)}{n}.$$

We denote

$$M = \sum_{i=1}^{k+l-1} cos\frac{4i\pi}{n},$$

which implies that

$$ER(\vec{C}_n) \geq \frac{2n}{n^2-1} + \frac{4n(k+l-1)^2}{n^2(k+l-1)-\frac{1}{2}(k+l-1)-\frac{1}{2}M} + \frac{2(1-l)}{n}.$$

On the other hand, by Remark 4.1, if $l = 0$, then $M = 0$, otherwise $M = -\frac{1}{2}$ so the proof is complete.

• Let n be odd, hence $n = 2(2k + l) + 1$, where $l \in \{0,1\}$. By (5), the set of real part of spectrum of $\vec{C}_n$ is

$$\{1, \underbrace{(\cos\frac{2\pi}{n})^{(2)}, (\cos\frac{4\pi}{n})^{(2)}, \dots, (\cos\frac{2k\pi}{n})^{(2)}}_{k},$$
$$\underbrace{(-\cos\frac{\pi}{n})^{(2)}, (-\cos\frac{3\pi}{n})^{(2)}, \dots, (-\cos\frac{(2(k+l)-1)\pi}{n})^{(2)}}_{k+l}\}.$$

Therefore,

$$ER(\vec{C}_n) = \frac{1}{n-1} + 2\sum_{i=1}^{k}\left[\frac{1}{n-\cos\frac{2i\pi}{n}} + \frac{1}{n+\cos\frac{(2i-1)\pi}{n}}\right] + \frac{2l}{n+\cos\frac{(2k+1)\pi}{n}}.$$

Now by the Cauchy-Schwarz inequality, we get that

$$ER(\vec{C}_n) \geq \frac{1}{n-1} + \frac{2k^2}{nk\ \ \sum_{i=1}^{k}\cos\frac{2i\pi}{n}} + \frac{2k^2}{nk\ \ \sum_{i=1}^{k}\cos\frac{(2i-1)\pi}{n}}.$$

We denote

$$X = \sum_{i=1}^{k} \cos\frac{2i\pi}{n}, \quad Y = \sum_{i=1}^{k} \cos\frac{(2i-1)\pi}{n}.$$

So

$$ER(\vec{C}_n) \geq \frac{1}{n-1} + \frac{2k^2}{nk-X} + \frac{2k^2}{nk+Y}.$$

On the other hand, by Remark 4.1, we get that $X = \frac{1}{4\sin(\frac{\pi}{2n})} - \frac{1}{2}$ and $Y = \frac{1}{4\sin(\frac{\pi}{2n})}$.

Consequently,

$$ER(\vec{C}_n) \geq \frac{1}{n-1} + \frac{2k^2}{nk-\frac{1}{4\sin(\frac{\pi}{2n})}+\frac{1}{2}} + \frac{2k^2}{nk\ \ \frac{1}{4\sin(\frac{\pi}{2n})}}$$
$$= \frac{1}{n-1} + \frac{2k^2(2nk+\frac{1}{2})}{n^2k^2+\frac{nk}{2}+\frac{1}{8\sin(\frac{\pi}{2n})}-\frac{1}{16\sin^2(\frac{\pi}{2n})}}.$$

Since $\sin(\frac{\pi}{2n}) \leq \frac{1}{2}$, so

$$ER(\vec{C}_n) \geq \frac{1}{n-1} + \frac{2k^2(2nk+\frac{1}{2})}{n^2k^2+\frac{nk}{2}}.$$

■

By Theorems 4.1 and 4.4, we get the next corollary.

Corollary 4.2. *Let $\vec{U}_{m,n}$ be an unicyclic graph. Then*

$$ER(\vec{U}_{m,n}) \geq \begin{cases} \frac{2n}{n^2-1} + \frac{2}{n} + \frac{n(n-4)}{n^2-\frac{1}{2}} + \frac{n-m}{n} & n = 4k; \\ \frac{2n}{n^2-1} + \frac{n(n-2)}{n^2-\frac{1}{2}+\frac{1}{n-2}} + \frac{n-m}{n} & n = 4k+2; \\ \frac{1}{n-1} + \frac{(n-1)(n^2-n+1)}{n(n^2-n+2)} + \frac{n-m}{n} & n = 4k+1; \\ \frac{1}{n-1} + \frac{(n-3)(n^2-3n+1)}{n(n^2-3n+2)} + \frac{n-m}{n} & n = 4k+3, \end{cases}$$

where k is a positive integer.

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