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Topological Entropy, Distributional Chaos and the Principal Measure of a Class of Belusov–Zhabotinskii's Reaction Models Presented by García Guirao and Lampart

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ABSTRACT

In this paper, the chaotic properties of the following Belusov-Zhabotinskii's reaction model is explored:

$$a_l^{k+1} = (1-\eta)\theta(a_l^k) + \frac{1}{2}\eta[\theta(a_{l-1}^k) - \theta(a_{l+1}^k)],$$

where k is discrete time index, l is lattice side index with system size M, $\eta \in [0,1)$ is coupling constant and θ is a continuous map on $W = [-1,1]$. This kind of system is a generalization of the chemical reaction model which was presented by García Guirao and Lampart in [Chaos of a coupled lattice system related with the Belusov–Zhabotinskii reaction, *J. Math. Chem*. **48** (2010) 159–164] and stated by Kaneko in [Globally coupled chaos violates the law of large numbers but not the central-limit theorem, *Phys. Rev. Lett.* **65** (1990) 1391–1394], and it is closely related to the Belusov-Zhabotinskii's reaction. In particular, it is shown that for any coupling constant $\eta \in [0, 1/2)$, any $r \in \{1, 2, \dots\}$ and $\theta = Q^r$, the topological entropy of this system is greater than or equal to $r\log(2-2\eta)$, and that this system is Li-Yorke chaotic and distributionally chaotic, where the map Q is defined by $Q(a) = 1 - |1-2a|$, $a \in [0,1]$, and $Q(a) = -Q(-a)$, $a \in [-1,0]$. Moreover, we also show that for any c, d with $0 \le c \le d \le 1$, $\eta = 0$ and $\theta = Q$, this system is distributionally (c, d)-chaotic.

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1. INTRODUCTION

A topological dynamical system (t.d.s. for short) (W, θ) is always assumed to be a compact metric space W together with a continuous map $\theta: W \to W$. Since the term of chaos was first introduced by Li and Yorke in [1], topological dynamical systems were highly considered in the literature [2−3] because they can model many phenomena from biology, physics, chemistry and social sciences. We know that Lattice Dynamical Systems or 1d Spatiotemporal Discrete Systems are generalizations of classical discrete dynamical systems. These kinds of systems have recently appeared as an important subject for investigation. In [4] we can find the importance of these type of systems. To understand when one of these type of systems has complicated dynamical properties or not by the study of one topological dynamical property is an open problem [5]. By using the concept of chaos, Guirao and Lampart characterized the dynamical complexity of a class of coupled lattice systems posed by Kaneko in [6] which is related to the Belusov-Zhabotinskii's reaction [5] and proved that such systems are Devaney chaotic and Li-Yorke chaotic for zero coupling constant. Also, they declared that these systems may be more complicated for non-zero coupling constants. Consequently, to further study the chaotic properties of the systems with non-zero coupling constants are very difficult. Recently, in [7] Wu and Zhu established that the systems with non-zero coupling constant $\eta \in (0, 1)$ are Li-Yorke chaotic and have positive entropy.

Distributional chaos defined by Schweizer and Smtal [8], is very interesting and important. This is because that it is equivalent to positive topological entropy and some other kinds of chaos if the state spaces are restricted to the closed intervals [8] or hyperbolic symbolic spaces [9]. However, we know that this equivalence does not transfer to higher dimensions. For example, positive topological entropy does not imply distributional chaos for triangular maps [10] (such a case can happen for zero-dimensional spaces [11]). It is known from [12] that there is a minimal system which is distributional chaotic. More recently, Wu and Zhu deduced that for $\theta = Q|_{[0,1]}$ and any pair $0 \leq c \leq d \leq 1$, the following coupled lattice system with non-zero coupling constant $\eta \in (0, 1)$ is distributionally (c, d)-chaotic [13]:

$$a_l^{k+1} = (1-\eta)\theta(a_l^k) + \frac{1}{2}\eta[\theta(a_{l-1}^k) + \theta(a_{l+1}^k)], \tag{1}$$

where k is discrete time index, l is lattice side index with system size M (that is, $l \in \{1, 2, \dots, M\}$), $\eta \in (0, 1)$ is coupling constant. Motivated by [14], we will further explore the chaotic properties of the following chemical models which are related to Belusov-Zhabotinskii's reaction:

$$a_l^{k+1} = (1-\eta)\theta(a_l^k) + \frac{1}{2}\eta[\theta(a_{l-1}^k) - \theta(a_{l+1}^k)], \tag{2}$$

where k is discrete time index, l is lattice side index with system size M, $\eta \in [0, 1)$ is coupling constant and θ is a continuous map on $W = [-1, 1]$. In particular, we obtain that for any coupling constant $\eta \in [0, 1/2)$, any $r \in \{1, 2, \ldots\}$ and $\theta = Q^r$, the topological entropy of the above chemical systems are greater than or equal to $r\log(2 - 2\eta)$, and that such systems are Li-Yorke chaotic and distributionally chaotic, where the map Q is defined by $Q(a) = 1 - |1 - 2a|$, $a \in [0, 1]$, and $Q(a) = -Q(-a)$, $a \in [-1, 0]$. At the same time, it is obtained that for any c, d with $0 \le c \le d \le 1$, $\eta = 0$ and $\theta = Q$, this system is distributionally (c, d)-chaotic.

2. PRELIMINARIES

In this article, we always assume that W denotes a compact metric space with metric p, and that (W, θ) denotes a t.d.s..

A pair $(a, b) \in W \times W$ is called a Li-Yorke pair of system (W, θ) if the following are fulfilled:

(1) $\limsup_{l\to\infty} p(\theta^l(a), \theta^l(b)) > 0$.

(2) $\liminf_{l\to\infty} p(\theta^l(a), \theta^l(b)) = 0$.

A subset $D \subset W$ is said to be a LY-scrambled set for θ if the set D has at least two points and any two distinct points in D form a Li-Yorke pair of (W, θ). A system (W, θ) or a map $\theta: W \to W$ is said to be Li-Yorke chaotic if it has an uncountable LY-scrambled set.

Let (W, θ) be a t.d.s.. For any $(a, b) \in W$ and any $l \in \{1, 2, \ldots\}$, the distributional function $\Phi_{ab}^l: [0, +\infty) \to [0, 1]$ is defined by

$$\Phi_{ab}^l(s) = \frac{1}{l}\#\{j \in \{1, 2, \ldots\}: p(\theta^j(a), \theta^j(b)) < s, 1 \le j \le l\},$$

where $\#D$ is the cardinality of the set D. Set

$$\Phi_{ab}(s, \theta) = \liminf_{l\to\infty} \Phi_{ab}^l(s)$$

and

$$\Phi_{ab}^*(s, \theta) = \limsup_{l\to\infty} \Phi_{ab}^l(s).$$

For any $c, d \in [0,1]$ with $c \le d$, a t.d.s. (W, θ) is distributionally (c, d)-chaotic if there exist an uncountable set $D \subset W$ and $\kappa > 0$ such that $\Phi_{ab}(s, \theta) = c$ and $\Phi_{ab}^*(s, \theta) = d$ for any $(a, b) \in D \times D$ with $a \neq b$ and any $s \in (0, \kappa)$. Clearly, (W, θ) is distributionally chaotic if it is distributionally $(0, 1)$-chaotic,s ee [13, 15].

Let G be the diameter of the space W. That is, $G = \sup\{p(a, b): a, b \in W\}$. The principal measure $\nu_p(\theta)$ of a t.d.s. (W, θ) is defined by

$$\nu_p(\theta) = \sup_{a,b\in W} \frac{1}{G}\int_0^{+\infty} (\Phi_{ab}^*(s, \theta) - \Phi_{ab}(s, \theta))ds,$$

see [16]. It is known from [16] that

$$\nu_p(\theta) = \frac{2}{3} + \sum_{l=2}^{\infty} \frac{1}{l}\,\frac{2^{l-1}}{(2^l+1)(2^{l-1}+1)},$$

where θ is the tent map defined by $\theta(a) = 1 - |1 - 2a|$ for any $a \in [0, 1]$.

The state space of the chemical system (1) or (3) is the set

$$\mathcal{A} = \{a: a = \{a_j\}, a_j \in \{-\infty, +\infty\}^t, j \in \{\dots, -1, 0, 1, \dots\}^q, \parallel a_j \parallel < +\infty\}.$$

where $t \geq 1$ is the dimension of the range space of the map of state a_j, $q \geq 1$ is the dimension of the lattice and the norm on $\mathcal{A}$ is defined by

$$\parallel a \parallel_2 = \left(\sum_{j\in\{\dots,-1,0,1,\dots\}^q} |a_j|^2\right)^{\frac{1}{2}},$$

where ($|a_j|$ is the length of the vector a_j) [5].

We will discuss and study the following chemical system related to the model stated by Kaneko in [6] and given by García Guirao and Lampart in [5] which is closely related to the Belusov-Zhabotinskii reaction [5–6, 17–20]:

$$a_l^{k+1} = (1-\eta)\theta(a_l^k) + \frac{1}{2}\eta[\theta(a_{l-1}^k) - \theta(a_{l+1}^k)], \tag{3}$$

where k is discrete time index, l is lattice side index with system size M, $\eta \in [0, 1)$ is coupling constant, and θ is a continuous map on $W = [-1, 1]$.

Generally speaking, for the system (1) or (3), one of the following three assumptions is needed:

1) $a_l^k = a_{l+M}^k$,
2) $a_l^k = a_l^{k+M}$,
3) $a_l^k = a_{l+M}^{k+M}$,

standardly, the first assumption is needed.

3. Main Results

The system (1) was investigated by many authors, mostly experimentally or semi-analytically than analytically. Chen and Liu [21] first obtained analytic results. Especially, they established that this system is Li-Yorke chaotic. In [5] Guirao and Lampart gave an new alternative and simpler proof of this result.

Let p be the product metric on the product space W^M defined by

$$p((a_1, a_2, \dots, a_M), (b_1, b_2, \dots, b_M)) = \left(\sum_{j=1}^{M} (a_j - b_j)^2\right)^{\frac{1}{2}},$$

for any $(a_1, a_2, \dots, a_M), (b_1, b_2, \dots, b_M) \in W^M$, where $W = [-1, 1]$.

Define a map $H: (W^M, p) \to (W^M, p)$ by $H(a_1, a_2, \dots, a_M) = (b_1, b_2, \dots, b_M)$, where $b_j = (1-\eta)\theta(a_j) + \frac{\eta}{2}(\theta(a_{j-1}) - \theta(a_{j+1}^k))$. It is clear that the chemical system (3) is equivalent to the above dynamical system (W^M, H), and that the chemical system (3) is different from the chemical system (1) when $\eta \neq 0$. In [5] Guirao and Lampart claimed that for non-zero couplings constants, this chemical system (3) is more complicated.

Inspired by [14] we have the following theorem.

Theorem 3.1. For any $r \in \{1, 2, \dots\}$, the topological entropy of the chemical system (3) is greater than or equal to $r\log(2-2\eta)$ for any $\eta \in [0, 1/2)$ and $\theta = Q^r$.

Proof. Write

$$\Delta_{[0,1]^M} = \{(a_1, a_2, \dots, a_M): a_1 = a_2 = \cdots = a_M \in [0, 1]\}.$$

As F is a continuous map on W^M, $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ is a subsystem of the dynamical system (W^M, H). Therefore, one has that $h_{top}(H) \geq h_{top}\left(H|_{\Delta_{[0,1]^M}}\right)$.

For any fixed $\eta \in [0, 1/2)$, we let $\theta'(a) = (1-\eta)\theta(a)$ for any $a \in [-1, 1]$. Then we obtain that $\theta'(a) = 2(1-\eta)a$ for any $a \in [0, 1/2)$ and $\theta'(a) = 2(1-\eta)(1-a)$ for any $a \in [1/2, 1]$. Define a map $\varphi: \Delta_{[0,1]^M} \to [0, 1]$ by $\varphi(\overleftarrow{a}) = a$ for any $\overleftarrow{a} = (a, a, \dots, a) \in \Delta_{[0,1]^M}$. It is easily verified that φ is a homeomorphism. Clearly, one gets that

$$\varphi \circ H|_{\Delta_{[0,1]^M}}(\overleftarrow{a}) = \varphi\left(\overleftarrow{\theta'|_{[0,1]}(a)}\right),$$

and

$$\varphi\left(\overleftarrow{\theta'|_{[0,1]}(a)}\right) = \theta'|_{[0,1]}(a) = \theta'|_{[0,1]} \circ \varphi(\overleftarrow{a}).$$

So, one obtains that $\varphi \circ H|_{\Delta_{[0,1]^M}} = \theta'|_{[0,1]} \circ \varphi$. This shows that $(\Delta_{[0,1]^M}, F|_{\Delta_{[0,1]^M}})$ is topologically conjugate to the subsystem $([0, 1], \theta'|_{[0,1]})$. Therefore, one has that

$$h_{top}\left(H|_{\Delta_{[0,1]^M}}\right) = h_{top}(\theta'|_{[0,1]}).$$

By Corollary 4.3.13 from [22] we have

$$h_{top}\left(H|_{\Delta_{[0,1]^M}}\right) = h_{top}(\theta'|_{[0,1]}) = r \cdot h_{top}((1-\eta)Q|_{[0,1]}) = r\log(2-2\eta).$$

Consequently, one gets that $h_{top}(H) \geq r\log(2-2\eta)$. □

Theorem 3.2. For any $r \in \{1, 2, \dots\}$, the chemical system (3) is chaotic in the sense of Li-Yorke for any $\eta \in [0, 1/2)$ and $\theta = Q^r$.

Proof. By Theorem 3.1, we know that if $\eta \in [0, 1/2)$ and $\theta = Q^r$, then the topological entropy of the chemical system (3) is positive. By Proposition 2 in [13], this system (3) is chaotic in the sense of Li-Yorke for any $r \in \{1, 2, \dots\}$ and any $\eta \in [0, 1/2)$. □

Theorem 3.3. For any $r \in \{1, 2, \dots\}$, the chemical system (3) is distributionally chaotic for any $\eta \in [0, 1/2)$ and $\theta = Q^r$.

Proof. From the proof of Theorem 3.1 one can see that for any $\eta \in [0, 1/2)$ and any $r \in \{1, 2, \dots\}$, the system $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ is conjugated with the system $([0, 1], Q^r|_{[0,1]})$. As $Q|_{[0,1]}$ is distributionally chaotic, by Lemmas 2.1 and 2.2 in [23] $Q^r|_{[0,1]}$ is distributionally chaotic for any $r \in \{1, 2, \dots\}$. By the definition and Theorem 2 in

[10] and its proof, the system $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ is distributionally chaotic for any $\eta \in [0, 1/2)$ and any $r \in \{1, 2, \ldots\}$. As $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ is a subsystem of the system (W^M, H), the system (W^M, H) is distributionally chaotic for any $\eta \in [0, 1/2)$ and any $r \in \{1, 2, \ldots\}$. □

Theorem 3.4. For any c, d with $0 \le c \le d \le 1$, the chemical system (3) is distributionally (c, d)-chaotic for $\eta = 0$ and $\theta = Q$.

Proof. By Proposition 3 in [13], $Q|_{[0,1]}$ is distributionally (c, d)-chaotic for any c, d with $0 \le c \le d \le 1$. From the proof of Theorem 3.1 we know that for $\eta = 0$, the subsystem $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ of the system (W^M, H) is topologically conjugated to system $([0, 1], Q|_{[0,1]})$. By Proposition 1.6 in [15], the system $(\Delta_{[0,1]^M}, H|_{\Delta_{[0,1]^M}})$ is distributionally (c, d)-chaotic for any c, d with $0 \le c \le d \le 1$. This means that the system (W^M, H) is distributionally (c, d)-chaotic for $\eta = 0$ and any c, d with $0 \le c \le d \le 1$. □

Theorem 3.5. Let $\theta: W \to W$ be continuous and $\eta \in [0, 1]$ be fixed. Then the principal measure of the chemical system (3) is greater than or equal to $\nu_p((1-\eta)\theta)$ for any $\theta \in [0, 1]$.

Proof. Clearly, for any fixed $\theta \in [0, 1]$ and any $a \in W$, one has that $H(\overleftarrow{a}) = \overleftarrow{\theta'(a)}$, where $\overleftarrow{a} = (a, a, \ldots, a) \in W^M$ and $\theta'(a) = (1-\eta)\theta(a)$ for any $a \in W$. For any $a, b \in W$ with $a \ne b$ and any $s \in (0, \sqrt{M})$, one has that

$$\Phi^*_{\overleftarrow{a}\overleftarrow{b}}(s, H) = \limsup_{i\to\infty} \frac{1}{i} \#\{j \in \mathbb{N}: j \in \{0, 1, \ldots, i-1\}, p(H^j(\overleftarrow{a}), H^j(\overleftarrow{b})) < s\} = \Phi^*_{ab}(\frac{s}{\sqrt{M}}, \theta').$$

By a similar argument, one has that

$$\Phi_{\overleftarrow{a}\overleftarrow{b}}(s, H) = \Phi_{ab}(\frac{s}{\sqrt{M}}, \theta')$$

for any $a, b \in W$ with $a \ne b$. This means that

$$\nu_p(H) \ge \sup_{a,b\in W} \frac{1}{\sqrt{M}} \int_0^{+\infty} \left(\Phi^*_{\overleftarrow{a}\overleftarrow{b}}(s, H) - \Phi_{\overleftarrow{a}\overleftarrow{b}}(s, H)\right) ds = \nu_p(\theta').$$ □

Remark 3.1. Theorem 3.5 completely solves Problem 3.2 given by Li in [24].

Remark 3.2. Roth [25] solved a problem regarding Li-Yorke and distributional chaos and gave the following question: Is there a DC3 chaotic subshift which is not Li-Yorke chaotic? Many definitions of chaos have appeared in the last decades and with them the question if they are equivalent in some more specific spaces. In [26] Roth's focus was distributional chaos, first defined in 1994 and later subdivided into three major types (and even more subtypes). These versions of chaos are equivalent on a closed interval, but distinct in more complicated spaces. As dendrites have much in common with the interval, she explored whether or not she could distinguish these kinds of chaos already on dendrites. She also briefly looked at the correlation with other types of chaos. The relations between concepts of distributional, Li-Yorke and ω chaos were discussed by many authors. In [27] Guirao and Lampart summarized all known connections between these three different types of chaos and fulfilled the results for general compact metric spaces by the construction of a selfmap over a compact perfect set such that this map is ω chaotic, not distributionally chaotic and has zero topological entropy. Among other notions, Li-Yorke chaos and topological entropy belong to basic and widely used notions in the theory of discrete dynamical systems. The question of their mutual relationship is thus very natural. Since 2002, from [28] we know that for continuous maps on compact metric spaces positive topological entropy implies Li-Yorke chaos. Analogical implication between positive topological entropy and distributional chaos of the second type has been obtained by Downarowicz [29]. It is noted that, in both cases, the converse implications do not hold, see [30] and [31], respectively. So, a natural question arises, whether there exists a property connected to positiveness of topological entropy is equivalent to the occurrence of Li-Yorke chaos. This question was solved in [32] by Franzová and Smítal for maps of the compact interval.

Remark 3.3. For some related well-known relations between 2-points-chaos and infinite-points-chaos for all applied notions of chaos, we refer the reader to [26, 27]. From the proofs of our results we can easily see that while increasing the system's dimension all the results in this paper are true.

Problem 3.1. Let c, d with $0 \leq c \leq d \leq 1$ be given and $\theta = Q$. Is the system (3) distributionally (c, d)-chaotic for any $1 \geq \eta > 0$?

Problem 3.2. Is the principal measure of a system or a map invariant under topological conjugacy?

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