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Stochastic Stability and Bifurcation for the Selkov Model with Noise

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ABSTRACT

In this paper, we consider a stochastic Selkov model for the glycolysis process. The stochasticity in the model is introduced by parameter perturbation which is a standard technique in stochastic mathematical modeling. First, we employ polar coordinate transformation and stochastic averaging method to transform the original system into an Itô averaging diffusion system. Next, we investigate the stochastic dynamical bifurcation of the Itô averaging amplitude equation by studying the qualitative changes of invariant measures and explore the phenomenological bifurcation (P-bifurcation) by using the counterpart Fokker-Planck equation. Finally, some numerical simulations are presented to verify our analytic arguments.

1. INTRODUCTION

A process that provided the energy for cellular metabolism by breaking down the glucose is named glycolysis [1, 2]. This process involves several steps where glucose decompose through formations of glucose-6-phosphate followed by fructose-6-phosphate (F6P), which converted to fructose-1,6-diphosphate under the influence of an allosteric enzyme named phosphofructokinase [1].

Higgins in 1964 introduced mathematical equations to interpret the dynamics of the glycolysis [3]. Selkov in 1968 introduced an autocatalytic model for the glycolysis which was simple model. This model has understandable form of a complicated reaction

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[4]. In Selkov model, the activation of phosphofructokinase (PFK) is considered to take place by adenosine diphosphate (ADP) [5]. Selkov model assumes that the initial glycolytic substrate, which is glucose, is injected with a constant rate below some value [5]. This prediction has been verified by many biochemists such as Klitzing, Betz and co-workers [5, 6, 7]. The study of Selkov model from dynamical system perspective has received a significant amount of attention, for example see [8, 9].

In real systems, parameter uncertainty and other sources of noise present everywhere. Moreover, environmental noises are important components in biochemical reactions. Although, we know that the parameters in the deterministic systems are all deterministic irrespective of environmental fluctuations. Therefore, for designing a more realistic model, we need to consider stochastic systems that take into account external influences in complex systems.

On the other hand, in the last decades the study of nonlinear stochastic dynamical systems has received much attention. For instance, in [10], Sarkar examined the linear response of a glycolytic oscillator, driven by a multiplicative coloured noise to an external periodic field.

Li and Zhang [11] studied the stochastic stability and stochastic bifurcation of Brusselator system with multiplicative white noise. Ma and Ning [12] investigated the stochastic P-bifurcation of Van der Pol oscillator with a fractional derivative damping term driven by Gaussian white noise excitation. Xu [13] studied P-bifurcation in a stochastic logistic model with correlated coloured noise. In [14], authors investigated the stability and bifurcation in a stochastic vocal folds model. Kong et al. [15] obtained global stability of a nonlinear oscillator excited by an ergodic real noise and harmonic excitations. For more references about stochastic stability and bifurcation, see [16, 17, 18] and references therein.

Motivated by the above argument we add randomly fluctuating driving force to the deterministic Selkov equations to analysing stochastic model.

Indeed, in this paper, we first add stochastic terms or noise to the deterministic Selkov model and make a stochastic Selkov model. Then, the stochastic Selkov model is reduced to an Itô one dimensional averaged equation by using the stochastic averaging method. Next, the relationship between the qualitative behaviour of the diffusion process and the stationary probability density is studied. Finally, the stochastic dynamics and bifurcation of the model are analysed by some analytical method and numerical simulations. The highlights of this paper can be listed in the following:

- A new stochastic Selkov model is obtained from a deterministic model and the stability of the new model is investigated. In fact, we state some theorems that give necessary and sufficient conditions for stability at equilibrium point.
- By varying some parameters such as noise intensity, we prove the model undergoes P-bifurcation.

- Some numerical simulations are illustrated to verify the established results.

2. PRELIMINARIES

In this section, we present some preliminaries concepts and definitions that will be used in subsequent sections to establish the stochastic stability and bifurcation.

Definition 2.1. (Stability in probability [19, 20]) *The trivial solution* $x(t;t_0,x_0)$ *of stochastic differential equation is said to be stochastically stable or stable in probability if for every pair of* $\epsilon \in (0,1)$ *and* $r > 0$, *there exists a* $\delta = \delta(\epsilon, r, t_0) > 0$ *such that*

$$P\{|x(t;t_0,x_0)| < r \text{ forall } t \geq t_0\} \geq 1-\epsilon,$$

whenever $\| x_0 \| < \delta_0$. Otherwise, it is said to be stochastically unstable.

Definition 2.2. (Asymptotic stability in probability [20]) *The trivial solution* $x(t;t_0,x_0)$ *of stochastic differential equation is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every* $\epsilon \in (0,1)$, *there exists a* $\delta_0 = \delta_0(\epsilon, t_0) > 0$ *such that*

$$P\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} \geq 1-\epsilon,$$

whenever $\| x_0 \| < \delta_0$.

Definition 2.3. (Global asymptotic stability in probability [19,20]) *The equilibrium position is said to be stochastically asymptotically stable in the large if it is stochastically stable and, moreover, for all* $x_0 \in R^d$

$$P\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} = 1.$$

Definition 2.4. (P-Bifurcation [19]) *Phenomenological bifurcation is concerned with the change in the shape of density (stationary probability density) of a family random dynamical systems as the change of the parameter. If there exist a constant* α_0 *satisfying in any neighborhood of* α_D, *there exist other two constant* α_1, α_2 *and their corresponding invariant measures* $p_{\alpha_1}, p_{\alpha_2}$ *satisfying* p_{α_1} *and* p_{α_2} *are not equivalent. Then the constant* α_0 *is a point of phenomenological bifurcation.*

Now, consider the following two-dimensional stochastic differential equations (SDE):

$$\begin{cases} dx = f_1(x,y)dt + g_1(x,y)dW_1(t), \\ dy = f_2(x,y)dt + g_2(x,y)dW_2(t), \end{cases} \tag{2.1}$$

where $f_i \in C^3(R \times R, R)$, $g_i \in C^1(R \times R, R)$ $(i = 1,2)$ and $W_i(t)$ $(i = 1,2)$ are mutually independent standard real-valued Wiener processes on the complete probability space (Ω, F, P). To ensure the existence and uniqueness of the solution of system (2.1), suppose that the functions f and g satisfy the global Lipschitz continuous and linear growth conditions [21]. Moreover, suppose that $f_i(0,0) = 0$ and $g_i(0,0) = 0$ $(i = 1,2)$, i.e. the origin is the fixed point of system (2.1).

In order to investigating the stability and bifurcation analysis of two-dimensional SDE (2.1), we first consider Taylor expansions of f_i and g_i at the point $O(0,0)$ as follows:

$$\begin{cases} dx = & -0.3cm[a_{110}x + a_{101}y + a_{120}x^2 + a_{111}xy + a_{102}y^2 + \mathcal{O}(3)]dt \\ & +[b_{110}x + b_{101}y + \mathcal{O}(2)]dW_1(t), \\ dy = & -0.3cm[a_{210}x + a_{201}y + a_{220}x^2 + a_{211}xy + a_{202}y^2 + \mathcal{O}(3)]dt\text{'} \\ & +[b_{210}x + b_{201}y + \mathcal{O}(2)]dW_2(t), \end{cases}$$

where $\mathcal{O}(3)$ and $\mathcal{O}(2)$ indicate the high order terms with respect to x and y.

Next, we ignore higher order terms and rescaling the system by $a_{ijk} = \varepsilon\bar{a}_{ijk}, b_{imn} = \sqrt{\varepsilon}\bar{b}_{imn}$, where $i = 1,2$, $j,k = 0,1,2,3$ and $m,n = 0,1$ and ε is a sufficiently small positive number. For simplicity, we drop the bars from the new variables and then get the following SDE

$$\begin{cases} dx = \epsilon[a_{110}x + a_{101}y + a_{120}x^2 + a_{111}xy + a_{102}y^2]dt \\ \quad +\sqrt{\epsilon}[b_{110}x + b_{101}y]dW_1(t), \\ dy = \epsilon[a_{210}x + a_{201}y + a_{220}x^2 + a_{211}xy + a_{202}y^2]dt \\ \quad +\sqrt{\epsilon}[b_{210}x + b_{201}y]dW_2(t). \end{cases} \tag{2.2}$$

According to the method presented in [21], we use polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$ with Itô formula and applying stochastic averaging method, we can rewrite the system (2.2) to the following Itô SDE

$$\begin{cases} dr = m_1(r)dt + \sigma_1(\theta)dW_r(t), \\ d\theta = m_2(r)dt + \sigma_2(\theta)dW_\theta(t), \end{cases} \tag{2.3}$$

where, W_r, W_θ are independent and standard Wiener process,

$$m_1(r) = (\mu_1 + \frac{1}{16}\mu_2)r + \frac{1}{8}\mu_3 r^3, \quad m_2(r) = \frac{1}{4}\mu_5 + \frac{1}{8}\mu_6 r^2,$$

are drift coefficients and

$$\sigma_1(\theta) = \sqrt{\frac{\mu_4}{8}}r \text{ and } \sigma_2(\theta) = \sqrt{\frac{\mu_2}{8}}$$

are diffusion coefficients, with the following notations:

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}(a_{110} + a_{201}), \\ \mu_2 &= b_{110}^2 + b_{201}^2 + b_{101}^2 + 3b_{210}^2, \\ \mu_3 &= 3a_{130} + a_{112} + a_{221} + a_{203}, \\ \mu_4 &= 3b_{110}^2 + b_{101}^2 + b_{210}^2 + b_{201}^2, \\ \mu_5 &= -2a_{101} + 2a_{210} + b_{110}b_{101} - b_{210}b_{201}, \end{aligned}$$

$\mu_6 = -a_{103} + a_{212} - a_{121} + 3a_{230}$.

In order to study the stability and bifurcation phenomena of SDE (2.1), it is efficient to consider the averaging modulus equation of system (2.2) [19]. Thus, we investigate the following equation

$$dr = [(\mu_1 + \tfrac{1}{16}\mu_2)r + \tfrac{1}{8}\mu_3 r^3]dt + (\tfrac{\mu_4}{8}r^2)^{\frac{1}{2}} dW_r(t). \qquad (2.4)$$

Note that, to preserve the random factors in the Equation (2.4), we assume $\mu_4 \neq 0$ in the rest of the paper, which implies μ_2 and μ_4 are positive numbers.

We are now in a position to study the stochastic stability of Equation (2.1). Based on the above argument, the local stability of the trivial solution of system (2.1) is equivalent to the stability of the averaging amplitude Equation (2.4) at the origin. For this purpose, the linearized equation of (2.4) at $r = 0$ can be written as

$$dr = [(\mu_1 + \tfrac{1}{16}\mu_2)r]dt + (\tfrac{\mu_4}{8}r^2)^{\frac{1}{2}} dW_r(t), \qquad (2.5)$$

where the solution is

$$r(t) = r(0)\exp\left(\int_0^t \left[\mu_1 + \frac{\mu_2}{16} - \frac{\mu_4}{16}\right] ds + \int_0^t (\tfrac{\mu_4}{8}r^2)^{\frac{1}{2}} dW_r(s)\right). \qquad (2.6)$$

Hence, the associated largest Lyapunov exponent is

$$\lambda = \lim_{t\to+\infty} \frac{\ln\|r(t)\|}{t} = \mu_1 + \frac{\mu_2}{16} - \frac{\mu_4}{16}.$$

From Oseledec multiplicative ergodic theorem [22], we know, the trivial solution of linearized equation is asymptotically stable with probability 1 if and only if the largest Lyapunov exponent is negative.

Now, we can summarize the results in the following theorems.

Theorem 2.5. [21] (i) *When* $\mu_1 + \frac{1}{16}\mu_2 - \frac{1}{16}\mu_4 < 0$, *the trivial solution of the linear Itô stochastic differential equation* (2.4) *is asymptotically stable with probability* 1, *thus the stochastic system* (2.1) *is stable at the equilibrium point* O.
(ii) *When* $\mu_1 + \frac{1}{16}\mu_2 - \frac{1}{16}\mu_4 > 0$, *the trivial solution of the linear Itô stochastic differential equation* (2.4) *is unstable with probability* 1, *which implies that the stochastic system* (2.1) *is unstable at the equilibrium point* O.

Theorem 2.6. [21] *When* $16\mu_1 + \mu_2 - \mu_4 < 0$ *and* $2\mu_3 < \mu_4$, *the stochastic system* (2.1) *is globally stable at the equilibrium point* O.

We can study phenomenological bifurcation by analyzing their steady-state probability density functions $p(r)$. Based on Namachivaya's theory [1] the extreme value of $p(r)$ gives necessary data on the stationary behavior of the Fokker-Planck equation coming from nonlinear SDE. The Fokker-Planck equation associate with Equation (2.4) is

$$\frac{\partial p(r)}{\partial t} = -\frac{\partial}{\partial r}\left[\left((\mu_1 + \frac{\mu_2}{16})r + \frac{\mu_3}{8}r^3\right)p(r)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\frac{\mu_4}{8}r^2 p(r)\right). \qquad (2.7)$$

For more details about Focker-Planck see [16, 1].

3. DETERMINISTIC SELKOV MODEL

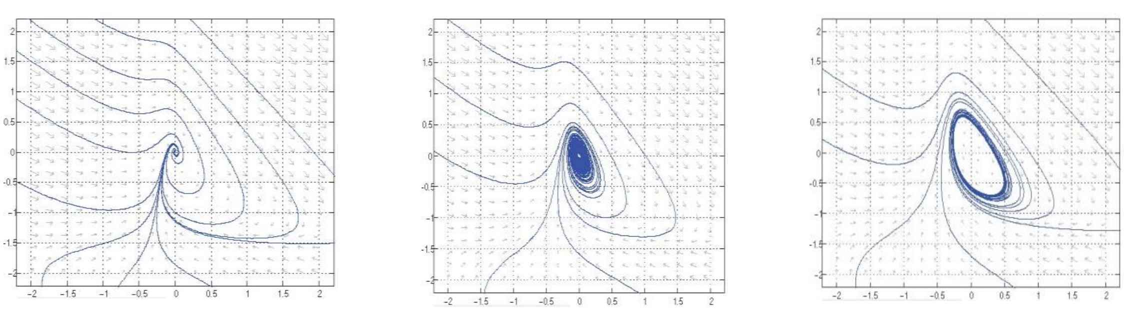
Here, we provide a brief description of the deterministic Selkov model for glycolysis, a process by which living cells break down sugar to obtain energy [8]:

$$\begin{cases} \dot{x} = -x + ay + x^2y, \\ \dot{y} = b - ay - x^2y, \end{cases} \qquad (3.1)$$

where x and y represent the concentrations of **ADP** and **F6P**, respectively, and $a, b > 0$. The parameter b is named phosphofructokinase and the parameter a is called hexokinase which is the activant from all the glycolytic cycle [9]. It is not difficult to see that $p_0 = (b, \frac{b}{a+b^2})$ is the only equilibrium point for this system. If we change coordinates by letting $u = x - b$ and $v = y - \frac{b}{a+b^2}$ the equilibrium is $(0,0)$ and system becomes:

$$\begin{cases} \dot{u} = \frac{b^2-a}{b^2+a}u + (b^2+a)v + \frac{b}{b^2+a}u^2 + 2buv + u^2v, \\ \dot{v} = -\frac{2b^2}{b^2+a}u - (b^2+a)v - \frac{b}{b^2+a}u^2 - 2buv - u^2v, \end{cases} \qquad (3.2)$$

Now, consider system (3.2) and fix $\alpha = 0.1$ and b be the controller parameter. Let $b_1 = 0.419992$ and $b_2 = 0.789688$ as we can see in [8], the origin is stable for $b < b_1$ and there is a stable limit cycle for $b > b_1$. Also, the origin is stable in the region $b > b_2$ and there is a stable limit cycle in the region $b < b_2$. Then in each of the two cases b_1 and b_2 there is a Hopf bifurcation and the origin is unstable in the region $b_1 < b < b_2$, where there exists a unique and stable limit cycle. Phase portrait of system (3.2) for some parameter b is shown in Figure 1.



(a) $b = 0.3$ (b) $b = 0.419992$ (c) $b = 0.5$

Figure 1: Phase portrait of system (3.2) for $a = 0.1$.

4. STOCHASTIC SELKOV MODEL

Here, by adding random terms to nonlinear Selkov model (3.2), we obtain

$$\begin{cases} \dot{u} = (\frac{b^2-a}{b^2+a}u + (b^2+a)v + \frac{b}{b^2+a}u^2 + 2buv + u^2v)dt + \sigma_1 u dW_1(t), \\ \dot{v} = (-\frac{2b^2}{b^2+a}u - (b^2+a)v - \frac{b}{b^2+a}u^2 - 2buv - u^2v)dt + \sigma_2 v dW_2(t), \end{cases} \quad (4.1)$$

where, $W_1(t), W_2(t)$ represent independent, standard Wiener processes and two real constants σ_1, σ_2 representing the size of noises in the system due to the environment.

Let $u = \overline{u}, v = \overline{v}, t = \overline{t}$ and $a_{jis} = \epsilon\overline{a_{jis}}, b_{jis} = \sqrt{\epsilon}\overline{a_{jis}}$ for all j, i, s. Then

$$\begin{cases} du = \epsilon(\frac{b^2-a}{b^2+a}u + (b^2+a)v + \frac{b}{b^2+a}u^2 + 2buv + u^2v)dt + \sqrt{\epsilon}\sigma_1 u dW_1(t), \\ dv = \epsilon\left(-\frac{2b^2}{b^2+a}u - (b^2+a)v - \frac{b}{b^2+a}u^2 - 2buv - u^2v\right)dt + \sqrt{\epsilon}\sigma_2 v dW_2(t). \end{cases} \quad (4.2)$$

Note that for simplicity we remove the bars from the scaled variables.

Now, by using the polar coordinate transformation $u = r\cos\theta$ and $v = r\sin\theta$, and according to the Khasminskii limiting theorem [23], we have the following Itô SDE:

$$\begin{cases} dr = [(\mu_1 + \frac{1}{16}\mu_2)r + \frac{1}{8}\mu_3 r^3]dt + (\frac{\mu_4}{8}r^2)^{\frac{1}{2}}dW_r(t), \\ d\theta = [\frac{1}{4}\mu_5 + \frac{1}{8}\mu_6 r^2]dt + (\frac{\mu_2}{8})^{\frac{1}{2}}dW_\theta(t), \end{cases} \quad (4.3)$$

with the following notations:

$$\begin{aligned} &\beta = \frac{b^2-a}{b^2+a} - (b^2+a), &&\mu_1 = \frac{1}{2}\beta, \\ &\mu_2 = \sigma_1^2 + \sigma_2^2, &&\mu_3 = \mu_6 = -1, \\ &\mu_4 = 3\sigma_1^2 + \sigma_2^2, &&\mu_5 = -2(b^2+a) + 4\frac{b^2}{b^2+a}. \end{aligned}$$

According to the Equation (2.7), we obtain its Fokker-Planck equation as follows:

$$\frac{\partial p(r)}{\partial t} = -\frac{\partial}{\partial r}\left[((\frac{\beta}{2} + \frac{\sigma_1^2+\sigma_2^2}{16})r - \frac{1}{8}r^3)p(r)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}(\frac{3\sigma_1^2+\sigma_2^2}{8}r^2 p(r)).$$

The invariant measure of diffusion process $r(t)$ is the steady-state probability density $p(r)$ which is the solution of the degenerate system:

$$0 = -\frac{\partial}{\partial r}\left[((\frac{\beta}{2} + \frac{\sigma_1^2+\sigma_2^2}{16})r - \frac{1}{8}r^3)p(r)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}(\frac{3\sigma_1^2+\sigma_2^2}{8}r^2 p(r)).$$

By calculation, we obtain

$$P(r)=\begin{cases}\delta(r), & \beta\le\frac{1}{4}\sigma_1^2,\\ \dfrac{r^{\frac{8(\beta)-5\sigma_1^2-\sigma_2^2}{3\sigma_1^2+\sigma_2^2}}\exp(\frac{-1}{3\sigma_1^2+\sigma_2^2}r^2)}{\Gamma(\frac{8(\beta)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2})(3\sigma_1^2+\sigma_2^2)^{\frac{8(\beta)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2}}}, & \beta>\frac{1}{4}\sigma_1^2.\end{cases} \tag{4.4}$$

In the following we need to compare β with some constant value. Hence, we first prove the following lemma.

Lemma 4.1. *Let a, b and k be positive parameters. There are three modes to compare $\beta=\frac{b^2-a}{b^2+a}-(b^2+a)$ and k:*

- If $a>\frac{(k-1)^2}{8}$, then $\beta<k$ for every $b>0$.
- If $a=\frac{(k-1)^2}{8}$, then for $b=\sqrt{\frac{(k+1)(3-k)}{8}}$, we have $\beta=k$, otherwise $\beta<k$ for every $b>0$.
- If $a<\frac{(k-1)^2}{8}$, then for $b=b_1$ and $b=b_2$ we have $\beta=k$, for $b_1<b<b_2$ we obtain $\beta>k$ and otherwise $\beta<k$ for every $b>0$.

Proof. Let $\beta=k$ then

$$\frac{b^2-a}{b^2+a}-(b^2+a)=k$$

$$\Rightarrow \quad \frac{-b^4+b^2(1-2a)-a(1+a)}{b^2+a}=k$$

$$\Rightarrow \quad -b^4+b^2(1-2a)-a(1+a)-kb^2-ka=0$$

$$\Rightarrow \quad -b^4+b^2(1-2a-k)-a(1+a+k)=0$$

$$\Rightarrow \quad b^2=\frac{(1-2a-k)\pm\sqrt{(1-2a-k)^2-4a(1+a+k)}}{2}$$

$$\Rightarrow \quad b^2=\frac{(1-2a-k)\pm\sqrt{(k-1)^2-8a}}{2}.$$

Therefore, this is necessary that $(k-1)^2-8a\ge 0$ and $1-2a-k>0$. Since b is a positive parameter we have the following two roots:

$$b_1=\sqrt{\frac{(1-2a-k)-\sqrt{(k-1)^2-8a}}{2}},\quad b_2=\sqrt{\frac{(1-2a-k)+\sqrt{(k-1)^2-8a}}{2}}. \tag{4.5}$$

□

To determine the stability of the equilibrium point (or trivial solution), we express the following theorem.

Theorem 4.2. (i) *When* $\beta < \frac{1}{4}\sigma_1^2$, *the stochastic system* (4.1) *is globally stable at the equilibrium point* O.
(ii) *When* $\beta > \frac{1}{4}\sigma_1^2$, *the stochastic system* (4.1) *is unstable at the equilibrium point* O.

Proof. The proof is a strightforward consequence of Theorems 2.5, 2.6 and Lemma 4.1. □

5. BIFURCATION ANALYSIS

Intuitively, Phenomenological bifurcation is concerned with the change in the shape of probability density of a family of stochastic dynamical systems as the change of the parameter.

It is easy to seen that the extreme value point of $p(r)$, is $r = 0$ or $r^* = \sqrt{4\beta - \frac{5}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2}$, when $4\beta > \frac{5}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2$. Therefore, we obtain three type of conditions:

Case (1). If $\frac{\sigma_1^2}{4} \leq \beta \leq \frac{5\sigma_1^2+\sigma_2^2}{8}$, then $\lim_{r\to 0^+} P(r) = \infty$ and the random trajectories of system (4.3) centralized in a neighborhood of the point $r = 0$.

Case (2). If $\frac{5\sigma_1^2+\sigma_2^2}{8} < \beta \leq \sigma_1^2 + \frac{\sigma_2^2}{4}$, then $P(r)$ has the minimum value at the origin and the maximum value at the point r^*, but the derivative of $P(r)$ at the origin does not exist. Moreover, the random trajectories of system (4.3) centralized in a neighborhood of the point r^*.

Case (3). then $P(r)$ has the minimum value at the origin and the maximum value at the point r^*. In this case, the probability density function $P(r)$ becomes a smooth function at the point r^*.

These results imply the following theorem.

Theorem 5.1. *System* (4.1) *undergoes stochastic phenomenological bifurcations as the phrase* β *passes through the values of* $\frac{5\sigma_1^2+\sigma_2^2}{8}$ *and* $\sigma_1^2 + \frac{\sigma_2^2}{4}$.

Remark 5.2. *It is note that when the parameter* β *passes through the value of* $\frac{1}{4}\sigma_1^2$, *the probability density function* $P(r)$ *varies from Dirac function* $\delta(r)$ *to the other function in* (4.4), *which means that system* (4.3) *undergoes a P-bifurcation in a generalized sense.*

In the following, we study the joint probability density $\rho(u,v)$ in terms of Cartesian coordinates u and v (for more details see [21, 24]). Hence, we have:

$$\rho(u,v)=\frac{(u^2+v^2)^{\frac{8(\beta)-5\sigma_1^2-\sigma_2^2}{6\sigma_1^2+2\sigma_2^2}}\exp(\frac{-1}{3\sigma_1^2+\sigma_2^2}(u^2+v^2)^2)}{\pi\Gamma(\frac{8(\beta)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2})(3\sigma_1^2+\sigma_2^2)^{\frac{8(\beta)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2}}}. \tag{5.1}$$

Similar to the above argument for $P(r)$, the extremal value point of $\rho(x,y)$ may be obtained. In this way we need to calculate the gradient of $\rho(x,y)$ in $\mathbb{R}^2$. Hence, we reach the following results:

Case (1). If $\beta \leq \frac{1}{4}\sigma_1^2+\sigma_2^2$, then $\rho(u,v)$ goes to infinite as $u\to 0$ and $v\to 0$.

Case (2). If $\frac{1}{4}\sigma_1^2+\sigma_2^2<\beta\leq\frac{11\sigma_1^2+3\sigma_2^2}{8}$, then $\rho(u,v)$ has a minimum value point at the origin, but it's partial derivatives are the origin is not continuous. Moreover, It has a maximum value at the point of the stable limit cycle $u^2+v^2=4\beta-4\sigma_1^2-\sigma_2^2$.

Case (3). If $\frac{11}{8}\sigma_1^2+\frac{3}{8}\sigma_2^2<\beta$, then $\rho(u,v)$ has a minimum at the origin, and a maximum at the point of the stable limit cycle of $u^2+v^2=4\beta-4\sigma_1^2-\sigma_2^2$. Moreover, $\rho(u,v)$ has continuous partial derivatives.

Theses results can be presented as the following theorem.

Theorem 5.3. *The stochastic system* (4.1) *undergoes phenomenological bifurcations as the parameter β passes through the values of* $\frac{1}{4}\sigma_1^2+\sigma_2^2$ *and* $\frac{11}{8}\sigma_1^2+\frac{3}{8}\sigma_2^2$.

5.1 P–BIFURCATION WITH RESPECT TO THE NOISE

In the following, we focus our attention on the qualitative change of the shape in the density of the stochastic Selkov model as the intensity of the noise changes. In other words, we fix β as a constant and investigate the effect of noise intensities on the stationary probability density function. This means by changing values of σ_1 and σ_2 the qualitative behaviour of probability density function changes. If parameters σ_1 and σ_2 choose in the ellipse $E_2:=\sigma_1^2+\frac{1}{4}\sigma_2^2=\beta$, then $P(r)$ is a smooth function that has a maximum value at the point

$$r_1=\sqrt{4\beta-\frac{5}{2}\sigma_1^2-\frac{1}{2}\sigma_2^2}$$

and a minimum value at the point $r = 0$. If σ_1 and σ_2 choose between two ellipses $E_1 := \sigma_1^2 + \frac{1}{5}\sigma_2^2 = \frac{8}{5}\beta$ and E_2, then $P(r)$ has a maximum value at the point r_1 and minimum at the point $r = 0$. But it has not derivative in $r = 0$. If σ_1 and σ_2 choose out of the ellipse E_1, then $\lim_{r\to 0^+} P(r) = \infty$. We can summarize these results as the following theorem.

Theorem 5.4. *The stochastic system* (4.3) *undergoes phenomenological bifurcations as parameters* σ_1 *and* σ_2 *passes through two ellipses* $E_1 = \frac{5}{8}\sigma_1^2 + \frac{1}{8}\sigma_2^2 = \beta$ *and* $E_2 = \sigma_1^2 + \frac{1}{4}\sigma_2^2 = \beta$.

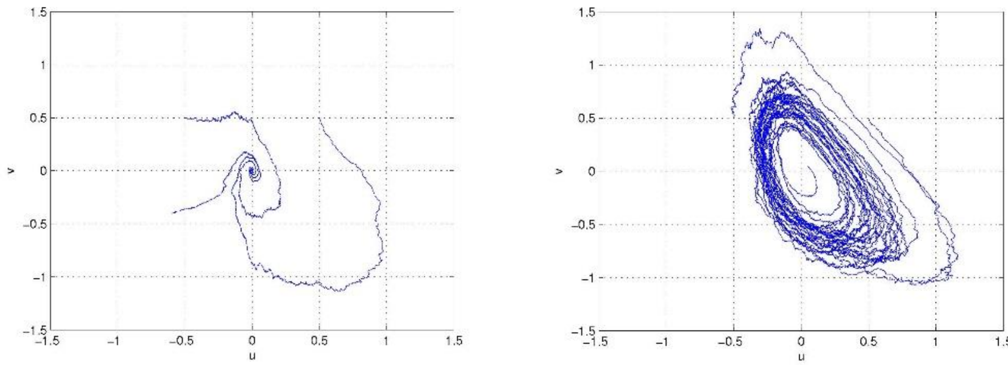
6. Bifurcation Analysis

In this section, we use numerical calculations to verify our analytic results. At the beginning of this section, we illustrate the phase portrait of system (4.1) for different values of parameters. Some discretization scheme may be used to numerically simulate solutions or trajectories for the SDEs. We approximate the solution by using the Euler-Maruyama scheme [25]. The Euler-Maruyama method converges to the Itô solution and it has strong order of convergence $\frac{1}{2}$ [26]. The Euler-Maruyama method applied to Equation (4.1) can be written in the following form

$$\begin{cases} x(i+1) = & x(i) + (\frac{b^2-a}{b^2+a}x(i) + (b^2+a)y(i) + \frac{b}{b^2+a}x(i)^2 + 2bx(i)y(i) \\ & +x(i)^2y(i))h + \sigma_1 x(i)\sqrt{h}N(0,1), \\ y(i+1) = & y(i) + (\frac{-2b^2}{b^2+a}x(i) - (b^2+a)y(i) - \frac{b}{b^2+a}x(i)^2 - 2bx(i)y(i) \\ & -x(i)^2y(i))h + \sigma_2 y(i)\sqrt{h}N(0,1), \end{cases} \tag{6.1}$$

for $i = 0,1,\dots,n$, where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

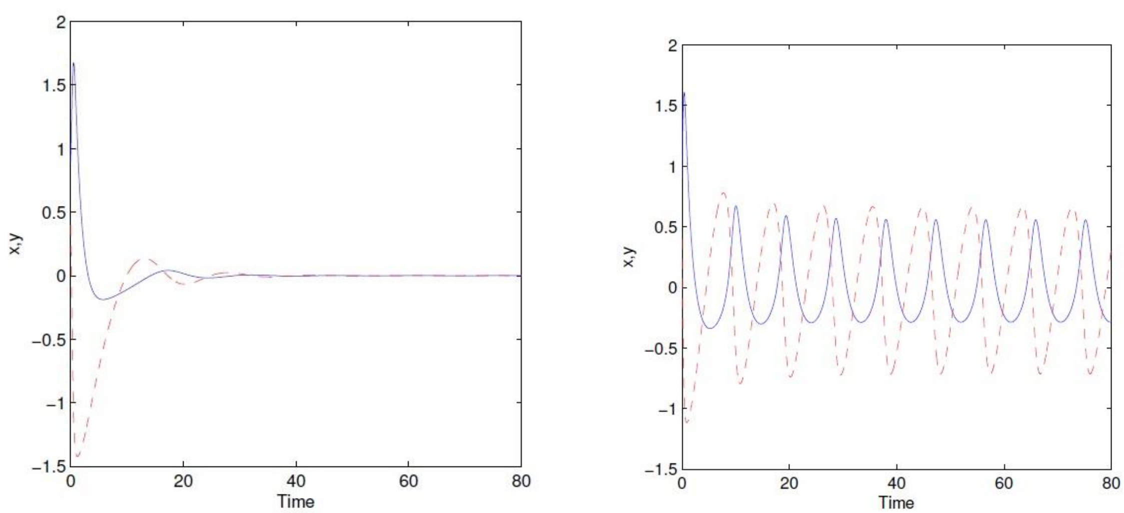
In Figure 2, we study the stability conditions presented in Theorem 4.2. We choose $\sigma_1 = \sigma_2 = 0.1$ and $a = 0.1$. When $b = 0.3$, we have $\beta = -0.2426315790 < \frac{1}{4}\sigma_1^2 = 0.0025$, therefore the origin is stable fixed point and when $b = 0.6$, we have $\beta = 0.1052173913 > 0.0025$ and the origin is unstable. Therefore, Figure 2 verify Theorem 4.2.



(a) $\beta < \frac{1}{4}\sigma_1^2$ (b) $\beta > \frac{1}{4}\sigma_1^2$

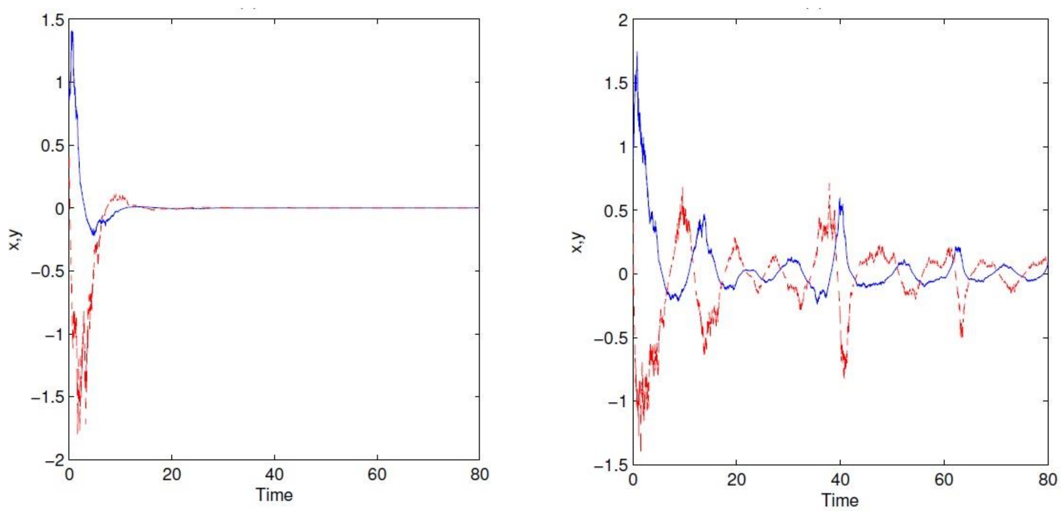
Figure 2: Phase portrait of system (4.1) with $\sigma_1 = \sigma_2 = 0.1, a = 0.1$ and (a) $b = 0.3$, (b) $b = 0.6$.

Figures 3 and 4 represents the time series for the simulation of stochastic Selkov model (4.1). In these simulations $\Delta t = 0.008$ is used, which are repeated 10000 times up to a time $t = 80$. We consider the parameter $a = 0.1$ and initial values $(x_0, y_0) = (0.9,0.4)$. As seen in Figure 4 (b) for the values of $\sigma_1 = 0.3$ and $\sigma_2 = 0.5$, the periodic orbit that exists in the system without noise (see Figure 3), is destroyed and the values of x and y approach to zero. Accordingly, the original Hopf bifurcation is destroyed in the stochastic Selkov model.



(a) $b = 0.3$ (b) $b = 0.6$

Figure 3: Time series of model (4.1) without the noise.



(a) $b = 0.3$ (b) $b = 0.6$

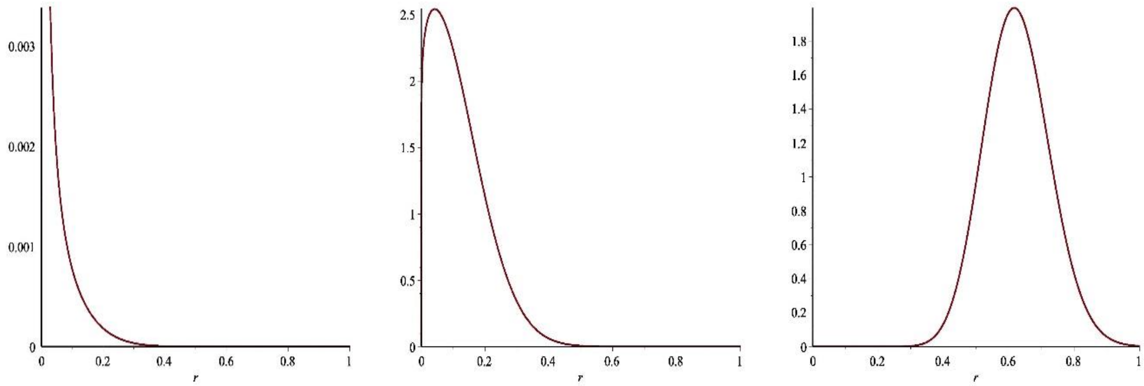
Figure 4: Time series of model (4.1) when the values of noises are $\sigma_1 = 0.3$ and $\sigma_2 = 0.5$.

In the next Examples we illustrate the P-bifurcation. We show that by varying some parameters the shape of probability density function changes.

Example 6.1. *Let* $\sigma_1 = 0.01, \sigma_2 = 0.2$. *By varying parameter* β, *we can see qualitative changes of density function* $P(r)$ *defined by* (4.4). *Simple calculation implies that:*

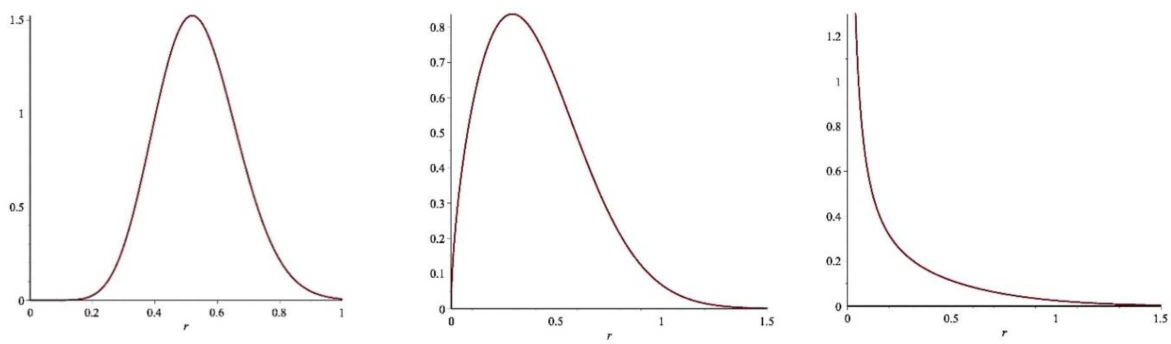
- If $0.000025 < \beta \leq 0.0050625$, then $\lim_{r \to 0^+} P(r) = \infty$, (see Figure 5, case (a)).
- If $0.0050625 < \beta \leq 0.0101$, then $P(r)$ has the minimum value at the the origin and the maximum value at the point r_1, but the derivative of $P(r)$ at the origin does not exist, (see Figure 5, case (b)).
- If $0.0101 < \beta$, then $P(r)$ has the minimum value at the origin and the maximum value at the point r_1, (see Figure 5, case (c)).

Example 6.2. *Let* $a = 0.1, b = 0.5$ *or* $\beta = 0.0785714286$. *By choosing different values for parameters* σ_1 *and* σ_2 *we plot the probability density function* $P(r)$ *in Figure 6. If* $\sigma_1, \sigma_2 \in E_2$ *the probability density is a smooth function with one maximum and one minimum point* (Figure 6 (a)), *if* σ_1, σ_2 *lie between two ellipses* E_1 *and* E_2, *then* $P(r)$ *has one maximum and one minimum point, but it has no derivative in the origin* (Figure 6 (a)). *Finally, if f* σ_1, σ_2 *lie out of two ellipses* $P(r)$ *tend to infinite if* $r \to 0^+$. *This example verifies Theorem* 5.4.



(a) $\beta = 0.000026$ (b) $\beta = 0.0055$ (c) $\beta = 0.1$

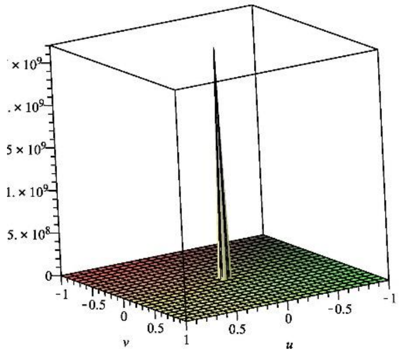
Figure 5: Probability density function $P(r)$ of system (4.3) for parameters $\sigma_1 = 0.01, \sigma_2 = 0.2$.

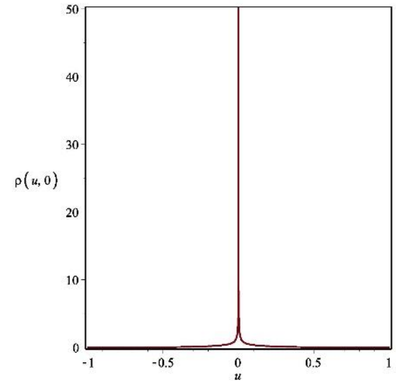


$\sigma_1 = 0.1, \sigma_2 = 0.2.$ $\sigma_1 = 0.3, \sigma_2 = 0.1.$ $\sigma_1 = 0.5, \sigma_2 = 0.2.$

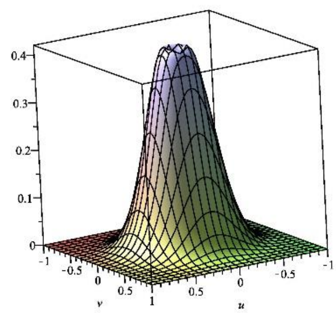
Figure 6: Variations of probability density $P(r)$ of system (4.3) for parameters $\alpha = 0.32$, $\beta = 100, \delta = 0.97$ and $\gamma = 0.78$. (a) $\sigma_1, \sigma_2 \in E_2$, (b) $\sigma_1, \sigma_2 \in E_1 \backslash E_2$, (c) σ_1, σ_2 out of E_1.

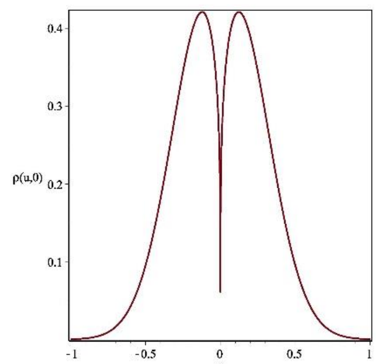
Example 6.3 *As an example, we take* $\sigma_1 = 0.2, \sigma_2 = 0.1$. *By varying parameter* β *for values* 0.015,0.03,0.07, *we plot qualitative changes of the density function* $\rho(x, y)$ *defined by* (5.1) *in Figure* 7.



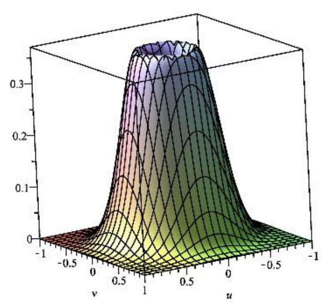


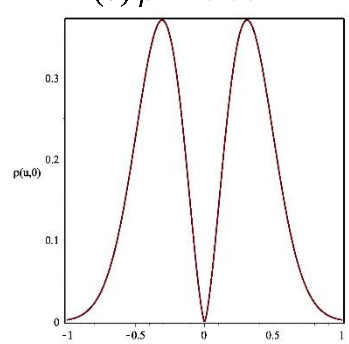
(a) $\beta = 0.015$ (b) $\beta = 0.015$





(c) $\beta = 0.03$ (d) $\beta = 0.03$





(e) $\beta = 0.05$ (f) $\beta = 0.05$

Figure 7: The graph of the joint probability density $\rho(u, v)$ of system (4.1) for $\sigma_1 = 0.2, \sigma_2 = 0.1$ and $\beta = 0.015, 0.03, 0.05$ and it's projection on $v = 0$.

7. CONCLUSION

In this paper stochastic model corresponding to Selkov model is investigated. The stochasticity in this model is presented by parameter perturbation which is the standard technique in SDEs. We discuss the stability and bifuraction of stochastic Selkov model. First, we transform the original model into an Itô limiting diffusion system. Next, we analyze the stability of the fixed point of the averaging system. We also obtain the stochastic bifurcation of the model by investigating the qualitative behaviour of stationary probability density.

It is not difficult to see that noise intensity has an important effect on the systems. Indeed, when the size of environmental noise is non-existent, the system behaves like the deterministic model. It is evident that when the size of the noise parameter is increased, the fluctuations of the stochastic trajectories also increase in an erratic manner.

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