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Some Properties of the Leap Eccentric Connectivity Index of Graphs

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ABSTRACT

The leap eccentric connectivity index of G is defined as $L\xi^C(G) = \sum_{v \in V(G)} d_2(v|G)e(v|G)$, where $d_2(v|G)$ be the second degree of the vertex v and $e(v|G)$ be the eccentricity of the vertex v in G. In this paper, we give some properties of the leap eccentric connectivity index of the graph G.

1. INTRODUCTION

In this paper, we only consider simple, undirected, finite graphs. Let G denote a graph with n vertices and m edge sets. Denoted by $d_G(u,v)$ the shortest path of length connecting u and v in G, for vertices u, $v \in V(G)$. For a vertex v and a positive integer k, we let $N_k(v|G)$ denote the open k-neighborhood of vertex v in G and defined as $N_k(v|G) = \{u \in V(G) | d_G(u,v) = k\}$. Let $d_k(v|G)$ denote the k degree of the vertex in G, expressed as the number of vertices in the open k-neighborhood of vertex v in G, that is, $d_k(v|G) = |N_k(v|G)|$. We can see that for any vertex v in G there are $d_1(v|G) = |N_1(v|G)|$ and $d_2(v|G) = |N_2(v|G)|$. The graph invariant $d_2(v|G)$ is also known as the connection number of v [19].

The eccentricity is defined as $e(v|G)$, for a vertex v in G, which represents the maximum distance from vertex v to other vertices in the graph, that is, $e(v|G) = max\{d_G(u,v) | u \in V(G)\}$. For any vertex in the graph, we define the maximum eccentricity value as the diameter $diam(G)$ and the minimum eccentricity value as the

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radius $rad(G)$. If the eccentricity of a vertex is equal to the radius of graph G, we call this vertex the center. If the eccentricity of all vertices in graph G is equal to the radius, we call G a self-centered graph. We let $V_e^{\alpha}(G) \subseteq V(G)$ denote the set of vertices in G where the eccentricity is equal to α, where $\alpha = 1,2, \dots, diam(G)$, obviously $V_e^1(G)$ represents the set of vertices in G that have a eccentricity of 1 to other vertices, the degree of these vertices is $n-1$, we call these vertices full vertices.

Let $H \subseteq V(G)$ denote any subset of vertices of G, then the induced subgraph $\langle H \rangle$ of G is the graph that the vertex set is H, and the edge set is the edge in graph G with the vertex in H as the endpoint. If there are no graphs isomorphic to graph F in all induced subgraphs of graph G, we call graph G the F-free graph. In [1], the Moore graph with a diameter of 2 is a pentagon, a Petresen graph, a Huffman-Singleton graph, or a 57 regular graph with 57^2+1 vertices. For other terms and symbols that are not defined here, please refer to [2].

Structure descriptors based on molecular graphs are often called topological indices and have very important meanings. In 1972, Gutman and Trinajestic [3] introduced the classical topological indices, namely the first and second Zagreb indices, and elaborated them in [4]. The definition is $M_1(G) = \sum_{v\in V(G)} d_1^2(v|G)$ and $M_2(G) = \sum_{uv\in E(G)} d_1(u|G)d_1(v|G)$. For the properties of these two indices, please refer to [5–7]. In recent years, some new invariants about Zagreb index have been proposed, such as Zagreb coindices [8–9], leap Zagreb index [10] and so on. The leap Zagreb indices are defined as $LM_1(G) = \sum_{v\in V(G)} d_2^2(v|G)$ and $LM_2(G) = \sum_{uv\in E(G)} d_2(u|G)d_2(v|G)$, and these indices were also studied independently under the name Zagreb connection indices [20].

In addition to above mentioned degree-based topological indexes, some distance-based topological indexes have also caused extensive research. In 2004, Dankelmann introduced the eccentricity sum index [11], defined as $\theta(G) = \sum_{v\in V(G)} e(v|G)$.

In 2012, Ghorbani proposed the Zagreb eccentricity index [12], defined as $E_1(G) = \sum_{v\in V(G)} e^2(v|G)$, $E_2(G) = \sum_{uv\in E(G)} e(u|G)e(v|G)$. Sharma proposed the eccentric connectivity index [13], defined as $\xi^C(G) = \sum_{v\in V(G)} d_1(v|G)e(v|G)$. Recently, Naji proposed the leap eccentric connectivity index [14], defined as $L\xi^C(G) = \sum_{v\in V(G)} d_2(v|G)e(v|G)$

In this paper, we investigate the leap eccentric connectivity index and give some properties of the leap eccentric connectivity index of graph G.

2. PRELIMINARY

In this section, we introduce some lemmas that will be useful in later proofs of this article. Firstly we introduce some properties of the second degree.

Lemma 2.1. [16] Let G be a connected graph with n vertices and m edges, then

$$d_2(v|G) \leq \sum_{u\in N_1(v|G)} d_1(u|G) - d_1(v|G), \quad (1)$$

with equality if and only if G is a $\{C_3, C_4\}$-free graph.

Note that $\sum_{v\in V(G)} \sum_{u\in N_1(v|G)} d_1(u|G) = M_1(G)$, see [18]. Thus, Lemma 2.1 has the following corollary.

Corollary 2.2. [15] Let G be a connected graph with n vertices and m edges, then

$$\sum_{v\in V(G)} d_2(v|G) \leq M_1(G) - 2m, \quad (2)$$

with equality if and only if G is a $\{C_3, C_4\}$-free graph.

Lemma 2.3. [16] Let G be a connected graph and $|V(G)| = n$, then for any vertex v in G

$$d_2(v|G) \leq n + 1 - d_1(v|G) - e(v|G). \quad (3)$$

Lemma 2.4. [10] Let G be a connected graph and $|V(G)| = n$, then for any vertex v in G

$$d_2(v|G) \leq d_1(v|\bar{G}) = n - 1 - d_1(v|G), \quad (4)$$

with equality if and only if the diameter of G is at most 2.

Next we introduce some properties on graph G and its complement $\bar{G}$.

Lemma 2.5. [17] Let G and the complement $\bar{G}$ be connected, then

i. If $diam(G) > 3$, then $diam(\bar{G}) = 2$;
ii. If $diam(G) = 3$, then $\bar{G}$ has a induced subgraph as a double star graph.

Lemma 2.6. [10] Let G be a connected graph of n vertices, then for any vertex v in the complement $\bar{G}$ of G

$$d_2(v|\bar{G}) = d_1(v|G), \quad (5)$$

with equality if and only if the diameter of the complement $\bar{G}$ of G is at most 2 or the diameter of G is at least 4 or G is a regular graph with a diameter of at least 2 or $G = K_1$.

Finally, we introduce some properties of the bound of $M_1(G)$.

Lemma 2.7. [5] Let G be a connected graph with $n \geq 2$ vertices and m edges, then $M_1(G) \geq \frac{4m^2}{n}$ and with equality if and only if G is a regular graph.

Proof. In the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

We set $a_i = d_1(v_i|G)$, $b_i = 1$, then

$$M_1(G)n = \left(d_1^2(v_1|G) + d_1^2(v_2|G) + \cdots + d_1^2(v_n|G)\right)(1^2 + 1^2 + \cdots + 1^2)$$
$$\geq (d_1(v_1|G) \cdot 1 + d_1(v_2|G) \cdot 1 + \cdots + d_1(v_n|G) \cdot 1)$$

$= (2m)^2 = 4m^2.$

Therefore, $M_1(G) \geq \frac{4m^2}{n}$, with equality if and only if $d_1(v_1|G) = d_1(v_2|G) = \cdots = d_1(v_n|G)$ that is G is a regular graph. ∎

3. SOME PROPERTIES OF LEAP ECCENTRIC CONNECTIVITY INDEX

Firstly we give the upper bound of the leap eccentric connectivity index.

Theorem 3.1. Let G be a connected graph with n vertices and m edges, then

$$L\xi^C(G) \leq (n-1)\theta(G) - \xi^C(G), \tag{6}$$

with equality if and only if the diameter of G is at most 2.

Proof. According to Lemma 2.4 and the definition of $L\xi^C(G)$, we can get

$$\begin{aligned} L\xi^C(G) &= \sum_{v\in V(G)} d_2(v|G)e(v|G) \\ &\leq \textstyle\sum_{v\in V(G)}[n-1-d_1(v|G)]e(v|G) \\ &\leq \textstyle\sum_{v\in V(G)}(n-1)e(v|G) - \sum_{v\in V(G)} d_1(v|G)e(v|G) \\ &= (n-1)\theta(G) - \xi^C(G). \end{aligned}$$

Assuming that the diameter of G is at most 2, we need to explain the following two cases:

Case 1. If $diam(G) = 1$, and because G is a connected graph, it means that for every vertex v in G there is $e(v|G) = 1$, so for each vertex v is connected to other vertices, we can get $d_2(v|G) = 0$, $d_1(v|G) = n-1$. Then,

$$\theta(G) = \textstyle\sum_{v\in V(G)} e(v|G) = n,\ \xi^C(G) = \sum_{v\in V(G)} d_1(v|G)e(v|G) = n(n-1).$$

So, $L\xi^C(G) = (n-1)\theta(G) - \xi^C(G) = (n-1)n - n(n-1) = 0.$

Case 2. If $diam(G) = 2$, we can get from Lemma 2.4

$$d_2(v|G) \leq d_1(v|\bar{G}) = n-1-d_1(v|G),$$

for any vertex v in G. Then,

$$\begin{aligned} L\xi^C(G) &= \sum_{v\in V(G)} d_2(v|G)e(v|G) = \sum_{v\in V(G)} [n-1-d_1(v|G)]e(v|G) \\ &= \textstyle\sum_{v\in V(G)}(n-1)e(v|G) - \sum_{v\in V(G)} d_1(v|G)e(v|G) \\ &= (n-1)\theta(G) - \xi^C(G). \end{aligned}$$

In the following we assume $diam(G) \geq 3$.

If $diam(G) \geq 3$, there is at least a vertex v in G that satisfies $e(v|G) \geq 3$. Therefore, for the vertex v, we have

$$d_2(v|G) < d_1(v|\bar{G}) = n-1-d_1(v|G).$$

Then, $\xi^C(G) < (n-1)\theta(G) - \xi^C(G)$.

This completes the proof. ■

Theorem 3.2. Let G be a connected graph with n vertices and m edges, and ≥ 3 , then

$$L\xi^C(G) \leq n\theta(G) - E_1(G) - n + 1, \qquad (7)$$

with equality if and only if $G \cong S_n$.

Proof. Let $v \in V(G)$, we have that

$$\begin{aligned} n - 1 &= d_1(v|G) + d_2(v|G) + \cdots + d_{e(v|G)}(v|G) \\ &\geq 1 + d_2(v|G) + e(v|G) - 2 \\ &= d_2(v|G) + e(v|G) - 1. \end{aligned}$$

Then $d_2(v|G) \leq n - e(v|G)$, with equality if and only if $e(v|G) = 2$ and $d_1(v|G) = 1$ or $e(v|G) \geq 3$ and $d_1(v|G) = d_3(v|G) = \cdots = d_{e(v|G)}(v|G) = 1$. Then we will prove the two cases when the equality holds.

Case 1. When $e(v|G) = 2$ and $d_1(v|G) = 1$. Assuming that all vertices v in G have a eccentricity of 2 and a degree of 1 holds, then $d_2(v|G) = n - 2$. Let $N(v|G) = \{u\}$, then $d_1(u|G) = n - 1$, contradict. Therefore, we can only have a eccentricity of 2 and a degree of 1 for some vertices in G. According to the above analysis, we can get $|V_e^1(G)| \geq 1$. Then,

$$\begin{aligned} L\xi^C(G) &= \sum_{v\in V(G)} d_2(v|G)e(v|G) \\ &= \sum_{v\in V_e^1(G)} d_2(v|G)e(v|G) + \sum_{v\notin V_e^1(G)} d_2(v|G)e(v|G) \\ &= \sum_{v\notin V_e^1(G)} d_2(v|G)e(v|G) = \sum_{v\notin V_e^1(G)}(n - e(v|G))\, e(v|G) \\ &= \sum_{v\notin V_e^1(G)} n\, e(v|G) - \sum_{v\notin V_e^1(G)} e^2(v|G) \\ &= n\big(\sum_{v\in V(G)} e(v|G) - \sum_{v\in V_e^1(G)} e(v|G)\big) - (\sum_{v\in V(G)} e^2(v|G) \\ &\quad - \sum_{v\in V_e^1(G)} e^2(v|G)) \\ &= n\theta(G) - E_1(G) - n|V_e^1(G)| + |V_e^1(G)| \\ &= n\theta(G) - E_1(G) + |V_e^1(G)|(1 - n) \leq n\theta(G) - E_1(G) - n + 1, \end{aligned}$$

with equality if and only if $|V_e^1(G)| = 1$, then there is only a vertex u in G, which satisfies $d_1(u|G) = n - 1$, and all other vertices v satisfy $e(v|G) = 2$ and $d_1(v|G) = 1$. Obviously, $G \cong S_n$.

Case 2. When all vertices in G have $e(v|G) \geq 3$ and $d_1(v|G) = d_3(v|G) = \cdots = d_{e(v|G)}(v|G) = 1$, we assume $diam(G) = r$ and let $P(G) = u_1u_2 \ldots u_r$ be a diameter path in G. We found that $d_1(u_2|G) = 2 > 1$, contradiction. Therefore, there is no connected graph G satisfying such a condition.

The following assumes that when $d_2(v|G) \leq n - e(v|G)$ is not equal, we compare the bound of the leap eccentric connectivity index obtained at this time is smaller than the bound when equal. Set $L\xi_*^C(G) = n\theta(G) - E_1(G) - n + 1$. We still discuss it in two cases.

Case A. When $e(v|G) = 2$ and $d_1(v|G) \geq 2$, $n-1 = d_1(v|G) + d_2(v|G) \geq 2 + d_2(v|G)$, then $d_2(v|G) \leq n-3$, so, $\sum_{v\in V(G)} d_2(v|G)e(v|G) \leq \sum_{v\in V(G)}(n-3)e(v|G) = (n-3)\theta(G) = L\xi_1^C(G)$, after making a difference:

$$\begin{aligned} L\xi_*^C(G) - L\xi_1^C(G) &= n\theta(G) - E_1(G) - n + 1 - (n-3)\theta(G) \\ &= 3\theta(G) - E_1(G) - n + 1 \\ &= 3\sum_{v\in V(G)} e(v|G) - \sum_{v\in V(G)} e^2(v|G) - n + 1. \\ &\geq n + 1 > 0 \end{aligned}$$

Case B. When $e(v|G) \geq 3$ and at least one of $d_1(v|G), d_3(v|G), \ldots, d_{e(v|G)}(v|G)$ is greater than 1, and the others are 1, we have

$$\begin{aligned} n - 1 &= d_1(v|G) + d_2(v|G) + \cdots + d_{e(v|G)}(v|G) \\ &\geq 1 + d_2(v|G) + e(v|G) - 1 = d_2(v|G) + e(v|G). \end{aligned}$$

At this time, $d_2(v|G) \leq n - 1 - e(v|G)$. Then,

$$\begin{aligned} \sum_{v\in V(G)} d_2(v|G)e(v|G) &\leq \sum_{v\in V(G)}\big(n - 1 - e(v|G)\big)e(v|G) \\ &= (n-1)\theta(G) - E_1(G) = L\xi_2^C(G). \end{aligned}$$

The difference between $L\xi_*^C(G)$ and $L\xi_2^C(G)$ can be obtained,

$$\begin{aligned} L\xi_*^C(G) - L\xi_2^C(G) &= n\theta(G) - E_1(G) - n + 1 - (n-1)\theta(G) + E_1(G) \\ &= \theta(G) - n + 1 = \sum_{v\in V(G)} e(v|G) - n + 1 \\ &\geq n - n + 1 = 1 > 0. \end{aligned}$$

The proof is completed. ■

Theorem 3.3. Let G be a connected graph and $|V(G)| = n$, $|E(G)| = m$, then

$$L\xi^C(G) \leq \sqrt{LM_1(G)E_1(G)}, \tag{8}$$

with equality if one of the following conditions is satisfied:

i. G is a regular graph and $diam(G) \leq 2$,
ii. G is a $\{C_3, C_4\}$-free regular self-centered graph.

Proof. In the Cauchy-Schwartz inequality $(\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$, we set $a_i = d_2(v|G)$ and $b_i = e(v|G)$. Then

$$\sum_{v\in V(G)} d_2(v|G)e(v|G) \leq \sqrt{\sum_{v\in V(G)} d_2^2(v|G) \sum_{v\in V(G)} e^2(v|G)},$$

We can get,

$$\begin{aligned} L\xi^C(G) = \sum_{v\in V(G)} d_2(v|G)e(v|G) &\leq \sqrt{\sum_{v\in V(G)} d_2^2(v|G) \sum_{v\in V(G)} e^2(v|G)} \\ &= \sqrt{LM_1(G)E_1(G)}. \end{aligned}$$

In the following, we assume that G is a k-regular graph and $diam(G) = r \leq 2$. We will discuss the following cases.

Case 1. When $diam(G) = r \leq 1$, G is a complete graph, so for each vertex v of G there are $d_2(v|G) = 0$ and $e(v|G) = 1$. Then $L\xi^C(G) = 0$, $LM_1(G) = 0$, and equation $L\xi^C(G) = \sqrt{LM_1(G)E_1(G)}$ holds.

Case 2. When $diam(G) = r = 2$, $d_2(v|G) = n - 1 - d_1(v|G)$. For the vertex v of G, $e(v|G) = 1$ or $e(v|G) = 2$. If $e(v|G) = 1$, then $d_1(v|G) = n - 1$, and G is a regular graph, then the degrees of other vertices should be $n - 1$, which contradicts $diam(G) = 2$. So for all vertices of G there is $e(v|G) = 2$, then $L\xi^C(G) = 2n(n - 1 - k)$, $LM_1(G) = n(n - 1 - k)^2$, $E_1(G) = 4n$, then the equation $L\xi^C(G) = \sqrt{LM_1(G)E_1(G)}$ holds.

Now suppose G is a $\{C_3, C_4\}$-free k-regular graph. For each vertex of G, there is $d_2(v|G) = k(k - 1)$, then $L\xi^C(G) = k(k - 1)\sum_{v\in V(G)} e(v|G)$, $LM_1(G) = nk^2(k - 1)^2$, $E_1(G) = \sum_{v\in V(G)} e^2(v|G)$.

According to the Cauchy-Schwartz inequality,

$$\left(\sum_{v\in V(G)} e(v|G)\right)^2 \leq \left(\sum_{v\in V(G)} 1^2\right)\left(\sum_{v\in V(G)} e^2(v|G)\right).$$

Then $\sum_{v\in V(G)} e(v|G) \leq \sqrt{n\sum_{v\in V(G)} e^2(v|G)}$, with equality if and only if the eccentricity is equal for each vertex v.

At this time equation $L\xi^C(G) = \sqrt{LM_1(G)E_1(G)}$ holds. Therefore, when G is a $\{C_3, C_4\}$-free regular graph and the eccentricity of each vertex of graph G is equal, inequality (8) takes equal. When the eccentricity of each vertex of G is equal, the eccentricity of all vertices of G is equal to the radius of graph G, then these vertices are the center, then G is a self-centered graph, so when G is a $\{C_3, C_4\}$-free regular self-centered graph, the inequality (8) is equal.

The proof is completed. ■

In general, Theorem 3.3 is not true in reverse. For example, in Figure 1, we can see that the equation $L\xi^C(G) = \sqrt{LM_1(G)E_1(G)}$ in Figure 1 is true, because for each vertex v, there is $d_2(v|G) = e(v|G) = 4$, but it does not satisfy any of the conditions (i) and (ii) in Theorem 3.3.

Next we give the lower bound on the leap eccentric connectivity index.

Theorem 3.4. Let G be a $\{C_3, C_4\}$-free graph with n vertices, m edges, and a radius of $rad(G)$, and satisfy $d_2(v_1|G) \geq d_2(v_2|G) \geq \cdots \geq d_2(v_n|G)$ and $e(v_1|G) \geq e(v_2|G) \geq \cdots \geq e(v_n|G)$, then

$$L\xi^C(G) \geq \frac{2m}{n} rad(G)(2m - n). \tag{9}$$

The bound attains on $\{C_3, C_4\}$-free regular self-centered graph.

Proof. In Chebyshev sum inequality, if $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \geq (\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} b_i).$$

By Corollary 2.2, if G is a $\{C_3, C_4\}$-free graph, then $\sum_{v \in V(G)} d_2(v|G) \leq M_1(G) - 2m$. And according to Lemma 2.7, $M_1(G) \geq \frac{4m^2}{n}$ and with equality if and only if G is a regular graph. Then

$$\begin{aligned} nL\xi^C(G) &= n \sum_{v \in V(G)} d_2(v|G) e(v|G) \geq \left(\sum_{v \in V(G)} d_2(v|G)\right)(\sum_{v \in V(G)} e(v|G)) \\ &= (M_1(G) - 2m)(\sum_{v \in V(G)} e(v|G)) \geq (M_1(G) - 2m) n \cdot rad(G) \\ &\geq \left(\frac{4m^2}{n} - 2m\right) n \cdot rad(G) = 4m^2 rad(G) - 2mn \cdot rad(G) \\ &= 2m \cdot rad(G)(2m - n). \end{aligned}$$

So, we have $L\xi^C(G) \geq \frac{2m}{n} rad(G)(2m - n)$.

When inequality (9) is equal, G is required to be a $\{C_3, C_4\}$-free regular graph, and for each vertex v of G, the eccentricity is equal to the radius. When the eccentricity of all vertices of G is equal to the radius, graph G is a self-centered graph. Thus, we can get that when G is a $\{C_3, C_4\}$-free regular self-centered graph, the inequality (9) is equal.

The proof is completed. ■

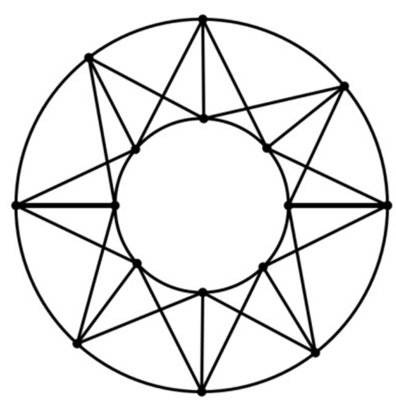

Figure 1. A graph for which $L\xi^C(G) = \sqrt{LM_1(G)E_1(G)}$.

Next, we will prove some bounds of graph G and its complements $\bar{G}$.

Theorem 3.5. Let G be a connected graph of n vertices and m edges, the complement of G is $\bar{G}$ and $|E(\bar{G})| = \bar{m}$, then

$$L\xi^C(G) \leq 2\bar{m}\, diam(G), \tag{10}$$

with equality if and only if the diameter of G is at most 2.

Proof. From Lemma 2.4, we have

$$\begin{aligned} L\xi^C(G) &= \textstyle\sum_{v\in V(G)} d_2(v|G)e(v|G) \\ &= \textstyle\sum_{v\in V_e^1(G)} d_2(v|G)e(v|G) + \sum_{v\notin V_e^1(G)} d_2(v|G)e(v|G) \\ &= 0 + \textstyle\sum_{v\notin V_e^1(G)} d_2(v|G)e(v|G) \le \sum_{v\in V(G)} d_1(v|\bar{G})\, diam(G) \\ &= diam(G) \textstyle\sum_{v\in V(\bar{G})} d_1(v|\bar{G}) \\ &= 2\bar{m}\, diam(G). \end{aligned}$$

with equality if and only if the diameter of G is at most 2.

Conversely, when the diameter of G is at most 2, it can also be proved that inequality (10) will be changed to equality. The proof is completed. ■

Corollary 3.6. Let G be a connected graph of n vertices and m edges, the number of edges in the complement $\bar{G}$ of G is $\bar{m}$, and $|V_e^1(G)| = 0$, then

$$n \cdot rad(G) \le L\xi^C(G) \le 2\bar{m}\, diam(G)$$

Theorem 3.7. Let G be a connected graph of n vertices and m edges and $diam(G) \ge 4$, the complement $\bar{G}$ of G is also connected, then $L\xi^C(\bar{G}) = 4m$.

Proof. When $diam(G) \ge 4$, and both G and $\bar{G}$ are connected, according to Lemma 2.5, $diam(\bar{G}) = 2$. So in $\bar{G}$, for each vertex v, $e(v|\bar{G}) = 2$, and $d_2(v|\bar{G}) = d_1(v|G)$, so we can get, $L\xi^C(\bar{G}) = \sum_{v\in V(\bar{G})} d_2(v|\bar{G})e(v|\bar{G}) = 2\sum_{v\in V(G)} d_1(v|G) = 4m$, as desired. ■

According to Lemma 2.6, Theorem 3.5 and 3.7, we have the following results.

Corollary 3.8. Let G be a connected graph of n vertices and m edges and the complement $\bar{G}$ of G is also connected, then

$$L\xi^C(\bar{G}) \le 2m\, diam(\bar{G}), \tag{11}$$

with equality if and only if the diameter of G is at least 4 or G is a regular graph with a diameter of at least 2 or $G = K_1$.

Theorem 3.9. Let G be a connected graph of n vertices and m edges, and the complement $\bar{G}$ of G is also connected, where $|V(\bar{G})| = \bar{n}$, $|E(\bar{G})| = \bar{m}$, then $L\xi^C(G) \le 2n\bar{m} - LM_1(G)$.

Proof. In the proof of Theorem 3.2, we have $d_2(v|G) \le n - e(v|G)$, that is, $e(v|G) \le n - d_2(v|G)$, with equality if and only if $e(v|G) = 2$ and $d_1(v|G) = 1$ or $e(v|G) \ge 3$ and $d_1(v|G) = d_3(v|G) = d_4(v|G) = \cdots = d_{e(v|G)}(v|G) = 1$. Then we have

$$\begin{aligned} L\xi^C(G) &= \sum_{v\in V(G)} d_2(v|G)e(v|G) \le \sum_{v\in V(G)} d_2(v|G)(n - d_2(v|G)) \\ &= n\textstyle\sum_{v\in V(G)} d_2(v|G) - \sum_{v\in V(G)} d_2^2(v|G) \\ &\le n\textstyle\sum_{v\in V(G)} d_1(v|\bar{G}) - \sum_{v\in V(G)} d_2^2(v|G) \\ &= 2n\bar{m} - LM_1(G). \end{aligned}$$

The proof is completed. ■

Next, we will give the Nordhaus-Gaddum-type result of the leap eccentric connectivity index.

Theorem 3.10. Let G be a connected graph of n vertices and m edges, and the complement $\bar{G}$ of G is a connected graph of $\bar{n}$ vertices and $\bar{m}$ edges, then

$$0 \leq L\xi^C(G) + L\xi^C(\bar{G}) \leq 2n(n-1) - 4m + 2m\, diam(\bar{G}),$$

with left equality if $G \cong K_n$, with right equality if G is a Moore graph with a diameter of 2.

Proof. The left equality is obvious. In the following we prove the equation on the right. According to Theorem 3.5 and Corollary 3.8, we can get

$$L\xi^C(G) + L\xi^C(\bar{G}) \leq 2\bar{m}\, diam(G) + 2m\, diam(\bar{G}).$$

In Theorem 3.5, with equality if $diam(G) \leq 2$, In Corollary 3.8, with equality if the diameter of G is at least 4 or G is a regular graph with a diameter of at least 2 or $G = K_1$. So we can get that the equation is equal when G is a regular graph and the diameter is 2. This is the definition of the Moore graph, so when G is a Moore graph with a diameter of 2, the above equation is equal. At this time

$$\begin{aligned} L\xi^C(G) + L\xi^C(\bar{G}) &= 2\bar{m}\, diam(G) + 2m\, diam(\bar{G}) \\ &= 4\bar{m} + 2m\, diam(\bar{G}) \\ &= 4\left(\frac{n(n-1)}{2} - m\right) + 2m diam(\bar{G}) \\ &= 2n(n-1) - 4m + 2m\, diam(\bar{G}). \end{aligned}$$

The proof is completed. ■

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REFERENCES

1. R. M. Damerell, On Moore graphs, *Math. Proc. Cambridge Phil. Soc.* **74** (1973) 227–236.
2. J. A. Bondy and U. S. R. Murty, Graph Theory, Graduate Texts in Mathematics, Springer, Berlin, 2008.
3. I. Gutman and N. Trinajstić, Graph theory and molecular orbitals, Total φ-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
4. I. Gutman, B. Ruščić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.* **62** (1975) 3399–3405.

5. B. Borovicanin, K. C. Das, B. Furtula and I. Gutman, Bounds for Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 17–100.
6. K. C. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.* **52** (2004) 103–112.
7. I. Gutman and K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.
8. A. R. Ashrafi, T. Došlić and A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.* **158** (2010) 1571–1578.
9. I. Gutman, B. Furtula, Z. K. Vukićević and G. Popivoda, On Zagreb indices and coindices, *MATCH Commun. Math. Comput. Chem*. **74** (2015) 5–16.
10. I. Gutman, A. M. Naji and N. D. Soner, On leap Zagreb indices of graphs, *Comm. Comb. Optim.* **2** (2017) 99–117.
11. P. Dankelmann, W. Goddard and C. S. Swart, The average eccentricity of a graph and its subgraphs, *Util. Math*. **65** (2004) 41–52.
12. M. Ghorbani and M. A. Hosseinzadeh, A new version of Zagreb indices, *Filomat* **26** (2012) 93–100.
13. V. Sharma, R. Goswami and A. K. Madan, Eccentric connectivity index: A novel highly discriminating topological descriptor for structure-property and structure-activity studies, *J. Chem. Inf. Comput. Sci*. **37** (1997) 273–282.
14. A. M. Naji, P. Shiladhar and N. D. Soner, On leap eccentric connectivity index of graphs, *Iranian J. Math. Sci. Inf.*, under review.
15. N. D. Soner and A. M. Naji, The k-distance neighborhood polynomial of a graph, *Int. J. Math. Comput. Sci*. **3** (2016) 2359–2364.
16. S. Yamaguchi, Estimating the Zagreb indices and the spectral radius of triangle-and quadrangle-free connected graphs, *Chem. Phys. Lett*. **458** (2008) 396–398.
17. L. Zhang and B. Wu, The Nordhaus-Gaddum-type inequalities for some chemical indices, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 189–194.
18. T. Réti, A. Ali, P. Varga and E. Bitay, Some properties of the neighborhood first Zagreb index, *Discrete Math. Lett*. **2** (2019) 10–17.
19. R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley–VCH, Weinheim, (2000).
20. A. Ali and N. Trinajstić, A novel/old modification of the first Zagreb index, *Mol. Inform.* **37** (2018) 1800008.