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Turbulence, Erratic Property and Horseshoes in a Coupled Lattice System related with Belusov–Zhabotinsky Reaction

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ABSTRACT

In this paper we continue to study the chaotic properties of the following lattice dynamical system $b_j^{i+1} = \alpha_1 g(b_{j-1}^i) + \alpha_2 g(b_j^{i+1}) + \alpha_3 g(b_{j+1}^i)$, where i is discrete time index, j is lattice side index with system size L, g is a selfmap on $[0,1]$ and α_1,α_2, $\alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ are coupling constants. In particular, it is shown that if g is turbulent (resp. erratic) then so is the above system, and that if there exists a g-connected family G with respect to disjointed compact subsets $D_1, D_2, \ldots, D_m$, then there is a compact invariant set $K' \subset D'$ such that $F|_{K'}$ is semi-conjugate to m-shift for any coupling constants $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, where $D' \subset I^L$ is nonempty and compact. Moreover, an example and two problems are given.

1. INTRODUCTION AND PRELIMINARIES

A topological dynamical system (G, g) is a compact metric space G together with a continuous map $g: G \to G$ on G. Since Li and Yorke [1] first defined chaos in 1975, many dynamical properties in topological dynamical systems were highly discussed in the literatures (see [2−3]). For the importance of the lattice dynamical systems, we refer the reader to [4]. Many authors (see [5−13]) studied the following lattice dynamical system:

$$b_j^{i+1} = \alpha_1 g(b_{j-1}^i) + \alpha_2 g(b_j^{i+1}) + \alpha_3 g(b_{j+1}^i),$$

where i is discrete time index, j is lattice side index with system size L, α_1,α_2,$\alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, are coupling constants and g is a continuous self map on $I = [0,1]$.

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To analyze whether such a system is complicated or not by the observation of some topological dynamic property of this system is an open problem (see [5]). In [5], the authors characterized the dynamical complexity of a coupled lattice system stated by Kaneko in [14] which is related to the Belousov−Zhabotinsky reaction and deduced that this kind of system is Li−Yorke chaotic and Devaney chaotic for the case of $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 =0$. Also, some problems on the dynamics of the system (1) with $\alpha_1 < 1$ and $\alpha_2 > 0$ (resp. $\alpha_3 > 0$) are presented by them. Recently, in [13] the authors established that for $\alpha_2 = \alpha_3 = \frac{1}{2}\alpha > 0$ and $\alpha_1 = 1 - \alpha$ and the unimodal map g on I, this system (1) is Li−Yorke chaotic and has positive entropy.

The notion of distributional chaos was first introduced by Schweizer and Smítal in [15]. It is very important, because it is equivalent to the concept of positive topological entropy and some other kinds of chaos for compact intervals [15] or hyperbolic symbolic spaces [16]. But this equivalence is not true for higher dimensional spaces [17], the same happens if the dimension is zero [18]). In [19] the authors constructed a distributional chaotic minimal system. More recently, in [20] Wu and Zhu established that for $\alpha_2 = \alpha_3 = 1/2\alpha > 0$ and $\alpha_1 = 1 - \alpha$ and the tent map Λ defined by $\Lambda(y) = 1 - |1 - 2y|$, the system (1) is distributionally (a, b)-chaotic for any $0 \le a \le b \le 1$ and obtained that principal measure of such a system is not less than

$$\frac{2}{3} + \sum_{j=2}^{\infty} \frac{1}{j} \frac{2^{j-1}}{(2^j+1)(2^{j-1}+1)}$$

for the map Λ. Inspired by the above results, we will continue to consider some new chaotic properties of the system (1). In particular, we prove that for the system (1) and any $\alpha_1, \alpha_2, \alpha_3 \in I$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, if g turbulent or erratic, then so is the system (1). Moreover, we present an example shows that there is a turbulent and erratic coupled lattice system related with Belousov−Zhabotinsky reaction. For some problems, some properties and notions of dynamical systems, we refer the reader to [21].

Let (G, ρ) be a metric space and (G, g) be a dynamical system. A pair $(a, b) \in G \times G$ is a Li−Yorke pair of the system (G, g) if

$$\lim_{j\to\infty} \sup \rho(g^j(a), g^j(b)) > 0$$

and

$$\lim_{j\to\infty} \inf \rho(g^j(a), g^j(b)) = 0$$

A subset $E \subset G$ is a Li−Yorke set of g if any pair of distinct points in E is a Li−Yorke pair of the system (G, g). A dynamical system (G, g) or a map $g: G \to G$ is Li−Yorke chaotic if the space G contains an uncountable Li−Yorke set of g.

For any topological dynamical system (G, g), any $a, b \in G$ and any integer $j > 0$, the distributional function $\Phi_{ab}^j: \mathbb{R}^+ \to [0, 1]$ is given by

$$\Phi_{ab}^{j}(t) = \frac{1}{j} \sharp \{i \in \mathbb{N} : \rho(g^{i}(a), g^{i}(b)) < t, 1 \leq i \leq j\},$$

where $\mathbb{R}^{+}=[0,+\infty)$ and $\sharp A$ is the cardinality of the set A. Set

$$\Phi_{ab}(t,g) = \liminf_{j\to\infty} \Phi_{ab}^{j}(t)$$

and

$$\Phi_{ab}^{*}(t,g) = \limsup_{j\to\infty} \Phi_{ab}^{j}(t)$$

For any $(a,c) \in I$ with $a \leq b$, a topological dynamical system (G,g) or a selfmap $g: G \to G$ is distributionally (a,b)-chaotic if the space G contains an uncountable subset E such that for some $\tau > 0$, $\Phi_{xy}(t,g) = a$ and $\Phi_{xy}^{*}(t,g) = b$ for any $x,y \in E$ with $x \neq y$ and any $t \in (0,\tau)$. Clearly, if the system (G,g) or the map g is distributionally $(0,1)$-chaotic then it is distributional chaotic (see [20, 22]).

The principal measure $\mu_p(g)$ of a topological dynamical system (G,g) or a selfmap $g: G \to G$ is given by

$$\mu_p(g) = sup_{a,b\in G}\ \frac{1}{l}\int_0^{+\infty} (\Phi_{ab}^{*}(t,g) - \Phi_{ab}(t,g))dt$$

where $l = diam\,(G)$ denotes the diameter of the space G (see [23]). From [23] one can know that

$$\mu_p(\Lambda) = \frac{2}{3} + \sum_{j=2}^{\infty} \frac{1}{j} \frac{2^{j-1}}{(2^{j}+1)(2^{j-1}+1)}$$

where Λ is the tent map given by $\Lambda(y) = 1 - |1 - 2y|$ for any $y \in I$.

Let $g: G \to G$ be a continuous map on a topological space G. The map g is called a turbulent map (see [24]) if there are two nonempty closed subsets $J, K \subset G$ with $J \cap K = \emptyset$ such that $J \cup K \subset g(J) \cap g(K)$. The map g is said to be erratic (see [24]) if there is a nonempty closed set $B \subset G$ such that $B \cap g(B) = \emptyset$ and $B \cup g(B) \subset g^2(B)$.

The state space of the system (1) is the set

$$\boldsymbol{G} = \{\boldsymbol{b} : \boldsymbol{b} = \{\boldsymbol{b_j}\}, \boldsymbol{b_j} \in \mathbb{R}^{\boldsymbol{p}}, \boldsymbol{j} \in \mathbb{Z}^{\boldsymbol{q}}, \| \boldsymbol{b_j} \| < \infty\},$$

where $p \geq 1$ is the dimension of the range space of the map of state b_j, $|b_j|$ is the length of the vector b_j, $q \geq 1$ is the dimension of the lattice and the l^2 norm

$$\| b \|_2 = \left(\sum_{j\in\mathbb{Z}^q} |b_i|^2\right)^{\frac{1}{2}} \tag{1}$$

is usually taken (see [5]). In general, one of the following periodic boundary conditions of the system (1) is needed:

1. $b_j^i = b_{j+L}^i$,
2. $b_j^i = b_j^{j+L}$,
3. $b_j^i = b_{j+L}^{i+L}$,

where standardly, the first case of the boundary conditions is used.

2. MAIN RESULTS

The system (1) with $\alpha_1 = 1 - \alpha$ and $\alpha_2 = \alpha_3 = \alpha/2$ and $\alpha \in I$ was explored by lots of authors, mostly experimentally or semi-analytically than analytically. The first paper having analytic results is [25], where the authors got that the system (1) with $\alpha_1 = 1 - \alpha$, $\alpha_2 = \alpha_3 = \alpha/2$ and $\alpha \in I$ is Li–Yorke chaotic. In [5] the authors presented an alternative and easier proof of the above result and claimed that the system (1) with $\alpha_1 = 1 - \alpha < 1$, $\alpha_2 = \alpha_3 = \alpha/2 > 0$ and $\alpha \in I$ is more complicated.

Let $\tilde{\rho}$ be the product metric on the product space I^L, i.e.,

$$\tilde{\rho}\big((x_1, x_2, \dots, x_L), (y_1, y_2, \dots, y_L)\big) = \left(\sum_{i=1}^{L}(x_i - y_i)^2\right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \dots, x_L)$, $(y_1, y_2, \dots, y_L) \in I^L$. Define the map $F: (I^L, d) \to (I^L, d)$ by $F(x_1, x_2, \dots, x_L) = (y_1, y_2, \dots, y_L)$ where $y_i = \alpha_1 g(x_i) + \alpha_2 g(x_{i-1}) + \alpha_3 g(x_{i+1})$, α_1, α_2, $\alpha_3 \in I$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. It is obvious that the system (1) is equivalent to the system (I^L, F). In [20] the authors got that if $g = \Lambda$, then the system (1) with $\alpha_1 = 1 - \alpha$, $\alpha_2 = \alpha_3 = \alpha/2$ and $\alpha \in I$ is distributionally (a, b)-chaotic for any $(a, b) \in I \times I$ with $a \leq b$ and any $\alpha \in (0, 1)$. Inspired by the above results we have the following theorems.

Theorem 2.1. For any coupling constants $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and any selfmap g on $[0, 1]$, if g is turbulent, then so is the system (1).

Proof. Fix $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. By hypothesis and the definition, there are two nonempty closed subsets $J, K \subset G$ with $J \cap K = \emptyset$ such that $J \cup K \subset g(J) \cap g(K)$. Let

$$J' = \{(x, x, \dots, x) \in I^L : x \in J\}$$

and

$$K' = \{(y, y, \dots, y) \in I^L : y \in K\}.$$

Then we have that $J', K' \subset I^L$ are two nonempty closed subsets with $J' \cap K' = \emptyset$ and

$$J' \cup K' \subset \underbrace{g \times g \times \cdots \times g}_{L}(J') \cap \underbrace{g \times g \times \cdots \times g}_{L}(K').$$

By the definition, the system (1) is turbulent.

Problem 2.1. Let $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and the system (1) be turbulent. Is g turbulent?

Theorem 2.2. For any coupling constants $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and any selfmap g on $[0, 1]$, if g is erratic, then so is the system (1).

Proof. Fix $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. By hypothesis and the definition, there is a nonempty closed subset $J \subset G$ with $J \cap g(J) = \emptyset$ such that $J \cup g(J) \subset g_2(J)$. Let $J' = \{(x, x, \ldots, x) \in I^L : x \in J\}$. Then we have that $J' \subset I^L$ is a nonempty closed subset with $J' \cap \underbrace{g \times g \times \cdots \times g}_{L}(J') = \emptyset$ and

$$J' \cup \underbrace{g \times g \times \cdots \times g}_{L}(J') \subset (\underbrace{g \times g \times \cdots \times g}_{L})^2(J').$$

By definition, the system (1) is erratic.

Problem 2.2. Let α_1,α_2,$\alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and the system (1) be erratic. Is g erratic?

Example 2.1. Let g be the tent map defined by $g(y) = 1 - |1 - 2y|$, $y \in [0,1]$. Then g is erratic and turbulent. Consequently, for the tent map and any coupling constants α_1,α_2, $\alpha_3 \in [0,1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the system (1) is erratic and turbulent.

Proof. From Example 1 in [24] we know that the the tent map g is erratic. By the definition and Example 1 in [24] one can see that the tent map g is turbulent. By Theorems 2.1 and 2.2, the system (1) is erratic and turbulent.

Now we recall some aspects of symbolic dynamics (see [26−28]). Let $S_m = \{0, 1, \ldots, m-1\}$, $Z_+ = \{0, 1, \ldots\}$, $\sum_m^+ = \sum_+^{S_m}$ and $\sum_m = \sum^{S_m}$. For any sequences $s = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots\}, \bar{s} = \{\ldots, \bar{s}_{-n}, \ldots, \bar{s}_{-1}, \bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n, \ldots\} \in \sum_m$ (or $s = \{s_0, s_1, \ldots, s_n, \ldots\}, \bar{s} = \{\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n, \ldots\} \in \sum_m^+$), the distance between s and $\bar{s}$ is defined by $d(s, \bar{s}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}$ in case of bi-infinite sequences (or $d(s, \bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^{|i|}} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}$ in case of one-sided sequences). With the distance defined as above, $\sum_m$ (or $\sum_m^+$) is a perfect, totally disconnected and compact metric space (see [26−27]). A m-shift map σ: $\sum_m \to \sum_m$ (or $\sigma : \sum_m^+ \to \sum_m^+$) is defined by $\sigma(s)_i = s_{i+1}$ for any $s \in \sum_m$ (or $s \in \sum_m^+$).

Definition 2.1. (see [26−27]) Let X be a metric space and $g: X \to X$ be a continuous map. Let $K \subset X$ be a compact invariant set of g. If there is a continuous surjective map $h: K \to \sum_m$ (or $h: K \to \sum_m^+$) such that $h \circ g = \sigma \circ h$, then $g|_K$ is said to be semi-conjugate to σ. Let X be a metric space, $D \subset X$ be compact and D_1, D_2,…, D_m be mutually disjoint compact subsets of D. Suppose that $g : D_i \to X$ is a continuous map for every $i \in \{1, 2, \ldots, m\}$.

Definition 2.2. (see [26–27]) Let D_1, D_2,…, D_m be mutually disjoint compact subsets of D. Assume that γ is a compact connected subset of D such that for each $i \in \{1, 2, \ldots, m\}$, $\gamma_i = \gamma \cap D_i$ is nonempty and compact. Then we say that γ is a connection with respect to D_1, D_2,…, D_m . Let G be a family of connections γs with respect to D_1, D_2,…, D_m such that $\gamma \in G$ implies $g(\gamma_i) \in G$ for each $i \in \{1, 2, \ldots, m\}$. Then we say that G is a g-connected family with respect to D_1, D_2,…, D_m.

Horseshoe Lemma (see [26–27]) Suppose that there exists a g-connected family G with respect to disjointed compact subsets D_1, D_2,…, D_m. Then there is a compact invariant set $K \subset D$ such that $g|_K$ is semi-conjugate to m-shift. Now we establish the following result.

Theorem 2.3. For any coupling constants $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ $with$ $\alpha_1 + \alpha_2 + \alpha_3 = 1$, if there exists a g-connected family G with respect to disjointed compact subsets D_1, D_2,…, D_m then, for the system (1) there is a compact invariant set $K' \subset D'$ such that $F|_{K'}$ is semiconjugate to m-shift, where $D' \subset I^L$ is nonempty and compact.

Proof. Write $\Delta_{[0,1]^L} = \{(x_1, x_2, \ldots, x_L) : x_1 = x_2 = \cdots = x_L \in [0, 1]\}$. Since F is a continuous selfmap of I^L, $(\Delta_{[0,1]^L}, F|_{\Delta_{[0,1]^L}})$ is a subsystem of the system (I^L, F). Define a map $h: \Delta_{[0,1]^L} \to [0, 1]$ by $h(\vec{a}) = a$ for any $\vec{a} = (a, a, \ldots, a) \in \Delta_{[0,1]^L}$. It is easily verified that h is a homeomorphism. Clearly, we have $h \text{ o } F|_{\Delta_{[0,1]^L}}(\vec{x}) = h(\overrightarrow{g(x)})$ and $h(\overrightarrow{g(x)}) = g(x) = g \text{ o } h(\vec{x})$. So, $h \text{ o } F|_{\Delta_{[0,1]^L}} = g \text{ o } h$. This shows that $(\Delta_{[0,1]^L}, F|_{\Delta_{[0,1]^L}})$ is topologically conjugate to the system $([0, 1], g)$. Clearly, horseshoe chaos is invariant under topological conjugation. Consequently, Theorem 3.1 holds. Thus, the proof is ended.

3. REMARK

Similar to the above discussion, one can prove that the results in this paper hold for the following coupled map lattice:

$$b_{m+1,n} = \alpha_{-s}g(b_{m,n-s}) + \alpha_{-s+1}g(b_{m,n-s+1}) + \cdots + \alpha_0 g(b_{m,n}) + \cdots + \alpha_t g(b_{m,n+t}) \quad (2)$$

where $s, t \in \mathbb{Z}, s, t \geq 0$, $\alpha_{-s}, \ldots, \alpha_0, \ldots, \alpha_t > 0$, $\alpha_{-s} + \cdots + \alpha_0 + \cdots + \alpha_t = 1$.

In fact, by $\alpha_{-s} + \cdots + \alpha_0 + \cdots + \alpha_t = 1$, $F(\vec{b}) = \overrightarrow{g(b)}$, for any $\vec{b} \in I^L$. This implies that the the system (2) has the same dynamical properties as the system (1).

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