Iranian J. Math. Chem. 11(2) July (2020) 83 − 93

Iranian Journal of Mathematical Chemistry

Journal homepage: ijmc.kashanu.ac.ir

Pseudospectrum Energy of Graphs

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ARTICLE INFO

Article History:

Received: 24 February 2020
Accepted: 19 May 2020
Published online: 30 July 2020
Academic Editor: Ali Reza Ashrafi

Keywords:

energy of graph
resolvent matrix
resolvent energy of graph
pseudospectrum

ABSTRACT

Let G be a simple graph of order N. The concept of resolvent energy of graph G, i.e. $ER = \sum_{i=1}^{N} (N-\lambda_i)^{-1}$ where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues, was established in: [Resolvent energy of graphs, *MATCH Commun. Math. Comput. Chem.* 75 (2016), 279–290]. In this paper, the notion of pseudospectrum energy of graphs will be introduced. This is the set defined as

$$PE = \{\sum_{i=1}^{N} |(N-\lambda_i)^{-1}| > \varepsilon_j^{-1}, \varepsilon > 0, j = 1,2,\ldots\}$$

and we establish a number of properties of PE. In addition, the pseudospectrum energy of some graphs with non-normal adjacency matrices were computed.

1. INTRODUCTION

The resolvent matrix of a given matrix $\mathcal{M}$ of finite (or infinite) order is defined as $R_\mu(\mathcal{M}) := (\mu I - \mathcal{M})^{-1}$ where μ is a complex variable and I is the unit matrix. The matrix $R_\mu(\mathcal{M})$ has many applications in different areas, as well, operator algebra [4], spectral analysis [1] and singularities [7]. We refer the interested readers to consult papers [2,3,5], for more details on this topic. Gutman et al. in [10], investigated the applications of resolvent matrix in graph theory and introduced the so-called "resolvent energy of graphs". In the following, we consider the graph G of order N with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and adjacency matrix $A = (A_{ij}(G))$, where

$$A_{ij}(G) = \begin{cases} 1 & if\ v_j\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}.$$

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DOI: 10.22052/ijmc.2020.221182.1488

The set of eigenvalues of $A(G)$ is said to be the *spectrum* of G and the *energy* is the sum of absolute values of them, i.e. $E = E(G) = \sum_{i=1}^{N} |\lambda_i|$. Moreover, the kth spectral moment of a graph G is given by the following formula:

$$M_k = M_k(G) = \sum_{i=1}^{N} (\lambda_i)^k \tag{1}$$

with $M_0 = N,\ M_1 = 0,\ M_2 = 2m$, and $M_k = 0$ for all odd values of k if and only if G is bipartite, see [17] for details. Since Ivan Gutman established the phenomena of a spectrum–based graph invariant in 1978 [11]; many authors studied it and a number of graph energies have been introduced according to adjacency matrices [12,13,14,15]. The resolvent matrix of $A(G)$ is defined as $R_\mu(A(G)) = (\mu I - A(G))^{-1}$ and its eigenvalues are

$$\frac{1}{\mu-\lambda_i},\ \ i = 1,2,\dots,N. \tag{2}$$

In [10], the resolvent energy of G was defined as

$$ER = ER(G) = \sum_{i=1}^{N} |\frac{1}{N-\lambda_i}| \tag{3}$$

where G is a graph of order N, $\mu = N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues. The relationship between (1) and (3) was given in [10] by

$$ER = ER(G) = \frac{1}{N}\sum_{k=0}^{\infty} \frac{M_k(G)}{N^k}.$$

The resolvent energy of graph of order N is outside the spectra as well, outside kth spectral moment, i.e. $R_\mu(A(G))$ exists for all complex values μ which does not coincides with an eigenvalues of $A(G)$ and it is easy to check that the right-hand side of (3), is always positive-valued. For large resolvent energy value, it is natural to ask about more information may hidden there. From analysis point of view, to understand more about an object represented by a matrix, we have to analyze not only the eigenvalues and spectrum, but also the pseudospectrum. In language of graph theory we define the pseudospectrum energy of graphs as:

Definition 1.1 *Let $A(G)$ be the adjacency matrix of graph G of order N with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ and ε be a positive integer. The pseudospectrum energy of G is the following set:*

$$PE = PE(G) = \{\sum_{i=1}^{N} |\frac{1}{N-\lambda_i}| > \varepsilon_j^{-1},\ j = 1,2,\dots\}. \tag{4}$$

Obviously, the set (4) is located between the eigenvalues of $A(G)$ and $R_\mu(A(G))$ shown in (2). The phenomena of pseudospectrum have a number of applications in diffident objects: dynamical systems, hydrodynamic stability, Markov chains and non-Hermitian quantum mechanics. We refer to an interesting research work by Trefethen and Embree [8] which gave a wide studies and applications to the mentioned concept. In Section 2, we will focus on simple graphs, that is graphs without directed, multiple, or weighted edges, and without self-loops and investigate their pseudospectrum energy of such graphs when the eigenvalues are real and the adjacency matrix is normal, i.e.

$A(G)A^T(G) = A^T(G)A(G)$. In Section 3, we will give some examples of pseudospectrum energy of directed and multidirected graphs which the adjacency matrices are non-normal, i.e. $A(G)A^T(G) \neq A^T(G)A(G)$ and the eigenvalues are complex numbers.

2. BASIC PROPERTIES OF PSEUDOSPECTRUM ENERGY OF GRAPHS

Theorem 2.1 *For any graph G of order N, $M_k(G) \subset PE(G)$.*

Proof. Let $0 < j \leq j+1$ and $\Delta_\varepsilon = \varepsilon_{j+1} - \varepsilon_j$. Then,

$$(\mu I - A(G))^{-1} < \Delta_\varepsilon \tag{5}$$

which implies that $\sum_{i=1}^{N} \frac{1}{N-\lambda_i} < \Delta_\varepsilon$, where $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues. By Theorem 2 in [10], for $|\lambda_i/N| < 1$, we have

$$\sum_{i=1}^{N} \frac{1}{N-\lambda_i} = \frac{1}{N}\sum_{k=0}^{\infty} \frac{1}{N^k}\sum_{i=1}^{N} (\lambda_i)^k < \Delta_\varepsilon. \tag{6}$$

The Inequality (6) means that the pesudospectrum energy graph set contains the set of all eigenvalues of $A(G)$ and hence $M_k(G) \subset PE(G)$. ■

By removing an edge e from a graph G, we will have another graph, denoted by $G - e$, which is subgraph of G.

Corollary 2.1. $PE(G - e) \subset PE(G)$.

Proof. The proof follows from Corollary 3 in [10] and from fact that the set of eigenvalues of $G - e$ is less then the set of eigenvalues of G. ■

It is well known that the spectra of complete graph K_N of order N and $\overline{K}_N$ the edgeless graph; are $\{N-1, -1, \dots, -1\}$ and $\{0,0,\dots,0\}$, respectively [11,16]. In [10], it was obtained that for any graph G of order N different from K_N and $\overline{K}_N$, we have

$$ER(\overline{K}_N) \leq ER(G) \leq ER(K_N),$$

and

$$1 \leq ER(G) \leq \gamma_N, \tag{7}$$

where $\gamma_N = \frac{2N}{N+1}$.

Corollary 2.2. *Let G be a graph of order N and different from $\overline{K}_N$ and K_N. Then*

$$\lim_{N\to\infty} PE(G) \simeq \frac{2}{\varepsilon_j}, \quad j = 1,2,3,\dots.$$

Proof. Its easy to check that $PE(G) = (0,1)$ if and only if $G \cong \overline{K}_N$ and $PE(G) = (1, \frac{2N}{N+1})$ if and only if $G \cong K_N$. From (7), we can put

$$\frac{1}{\varepsilon_j} < PE(G) < \frac{\gamma_N}{\varepsilon_j}, \tag{8}$$

where $\gamma_N = \frac{2N}{N+1}$ and $j = 1,2,3, \dots$ Now, by taking the limit to the right-hand side of (8), we have

$$\lim_{N\to\infty} PE(G) \simeq \lim_{N\to\infty} \frac{\gamma_N}{\varepsilon_j} \simeq \frac{2}{\varepsilon_j}, \quad j = 1,2,3, \dots .$$

■

In [10, Observation 15], it was observed that among connected unicyclic graphs of order N, $N \geq 4$, the cyclic C_N has the smallest and graph C_N^* second-smallest resolvent energy, respectively. While the graphs X_N and X_N^* $(N \geq 5)$, have respectively, maximum and second-maximum resolvent energy, see Figure 1. Now, it is easy to check that $PE(C_N^*) \subset PE(C_N)$ and $PE(X_N) \subset PE(X_N^*)$.

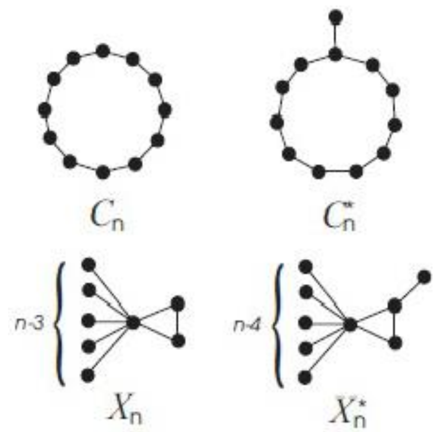

Figure1. Graphs C_N^*, C_N, X_N and X_N^*.

3. COUNTEREXAMPLES OF PSEUDOSPECTRUM ENERGY GRAPHS OF NON-NORMAL ADJACENCY MATRICES

In this section, we show some examples of graphs which the adjacency matrices are non-normal and the eigenvalues are complex numbers. Obviously, the computation of pseudospectrum energy of graphs in this case will be more complicated. By using MATLAB platform "EigTool" [9], we show the pseudospectrum energy of graphs around the spectra in a complex Banach space X. So, we rewrite the definition of the pseudospectrum energy of graphs as

$$PE = PE(A(G)) = \{\mu \in \mathbb{C}: \| (\mu I - (A(G))^{-1} \| > \varepsilon_j^{-1}, \quad \varepsilon > 0, j = 1,2, \dots\}.$$

where $\|.\|$ is the norm. For more details about pseudospectrum in complex Banach space see [8].

In the next examples, we consider directed and multidirected graphs and 2-norm which can be defined by $\| x \|_2 = (\sum |x_j|^2)^{\frac{1}{2}}$ for $x \in X$.

Example 3.1. *Let $G(V,E)$ be a non-simple graph with the following scheme (Figure.2):*

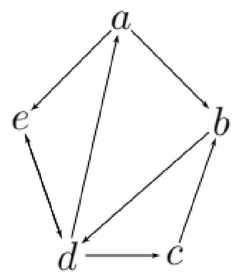

Figure 2. Graph $G(V,E)$.

The adjacency matrix of $G(V,E)$ is non-normal which is given by:

$$A\big(G(V,E)\big) = \begin{bmatrix} & a & b & c & d & e \\ a & 0 & 1 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 0 & 0 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(G(E,V))$ are: $\lambda_1 = 1.6, \lambda_{2,3} = -0.8 + 1.04i, \lambda_{4,5} = 0$ and the energy is

$$E(A(G(E,V))) \approx 3.2 + 2.08i.$$

In Figures 3 and 4, we show the computation of pesudospectrum energy of $G(V,E)$ outside the spectrum-based invariant for $\varepsilon = -2.5, -2.25, -2, -1.75, -1.5, -1.25, -1, 0.75, 0.5, 0.25$ and iteration $n = 78$:

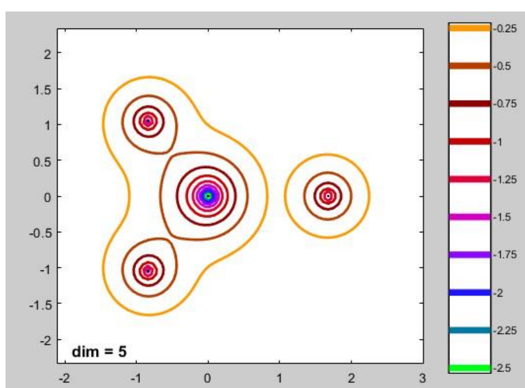

Figure 3. Pseudospectrum energy of $A\big(G(V,E)\big)$.

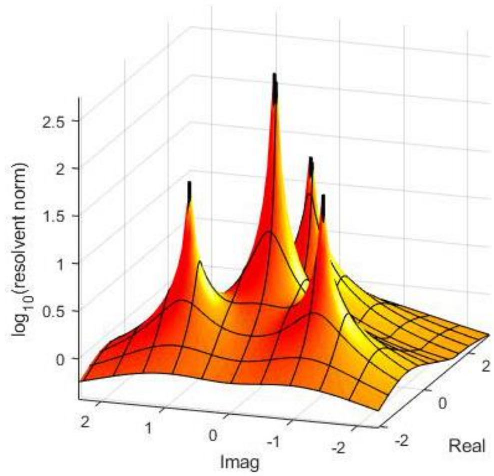

Figure 4. 3D pseudospectrum energy of $A(G(V,E))$.

Example 3.2. *Let us consider the following Multidigraph $M(V,E)$ as in Figure 5:*

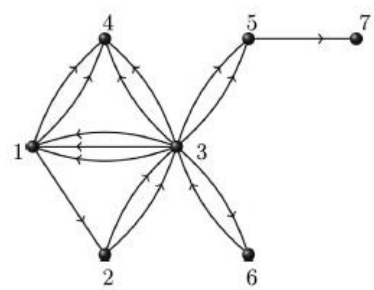

Figure 5. The graph $M(V,E)$.

The adjacency matrix of multidigraph G containing n vertices, has been defined in [17] as the $n \times n$ matrix $A(G) = [a_{ij}]$, whose ij-th entry a_{ij} is equal to the number of directed edges originating from the vertex i and ending at the vertex j. Thus, the adjacency matrix of the graph $M(V,E)$ is non-normal and can be given by:

$$A_M = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 2 & 2 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A_M are: $\lambda_{1,2} \approx -1 \pm 1.4i\lambda_3 = 2, \lambda_{4,5,6,7} = 0$ and the energy is

$$E(M) \approx 4 + 2.8i.$$

In Figures 6 and 7, we show the computation of pesudospectrum energy of the Multidigraph M outside the spectrum-based invariant for $\varepsilon = -2.25, -2, -1.75, -1.5, -1.25, -1, 0.75, 0.5, 0.25$ and iteration $n = 100$ as:

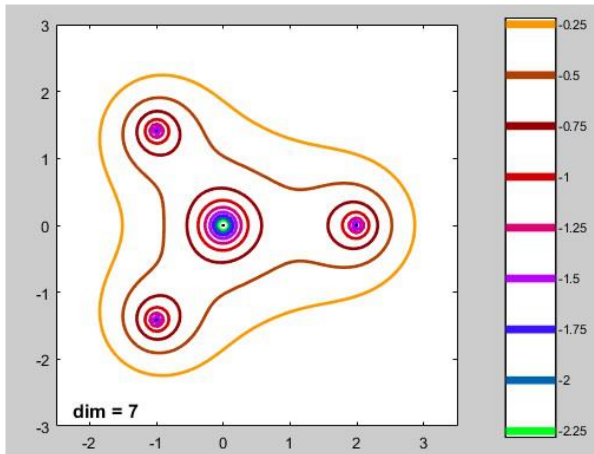

Figure.6. Pseudospectrum energy of the graph of A_M.

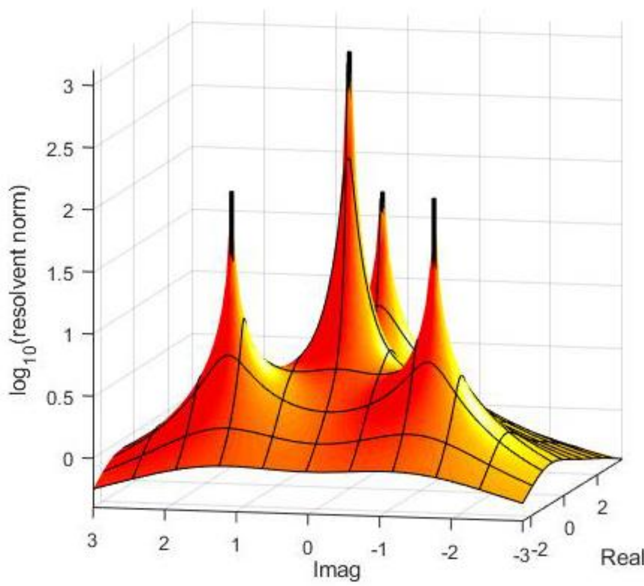

Figure 7. 3D Pseudospectrum energy of the graph of A_M.

Conjecture 3.1. *Throughout this paper, we observed that may only digraphs (except cycle digraphs) and multigraphs have non-normal adjacency matrix.*

4. OPEN QUESTION

In particular, normal matrices are less interested in calculating the pseudospectrum phenomena in Banach spaces but the next example of normal adjacency matrix shows a special case:

Example 4.1. *Let us consider the monad graph. The monad graph G is a discrete dynamical system contains (Γ, f, G), where Γ is a finite set and $f: \Gamma \to \Gamma$ is a map scheming every vertices of elements of the corresponding Γ with its image by directed connected edge. For more details about monad graphs we refer to* [18]. *In particular, we consider the following system: $\Gamma = \mathbb{Z}_5$ with addition operation and $f_+(x) = x + 1$, for all $x \in \mathbb{Z}_5$. The monad graph of $(\mathbb{Z}_5, f_+)$ is shown in Figure 7:*

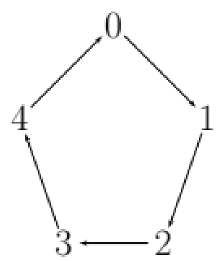

Figure 7. Graph of $(\mathbb{Z}_5, f_+)$.

Now, the adjacency matrix of $A(\mathbb{Z}_5, f_+)$ is given by:

$$A(\mathbb{Z}_5, f_+) = \begin{bmatrix} + & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the adjacency matrix of $A(\mathbb{Z}_5, f_+)$ is normal but the eigenvalues are complex numbers. The eigenvalues of $A(\mathbb{Z}_5, f_+)$ are: $\lambda_{1,2} = -0.8090 \pm 0.5878i$, $\lambda_{3,4} = 0.3090 \pm 0.9511i$, $\lambda_5 = 1 + i$ and the energy is

$$E(A(\mathbb{Z}_5, f_+)) \approx 3.2 + 3.8i.$$

In Figures 9 and 10, we show the computation of the pesudospectrum energy of graph with adjacency matrix $A(\mathbb{Z}_5, f_+)$ outside the spectrum-based invariant for $\varepsilon = 0.1, -0.1, -0.3, -0.5, -0.7, -0.9, -1.1, -1.3, -1.5$ and iteration $n = 78$:

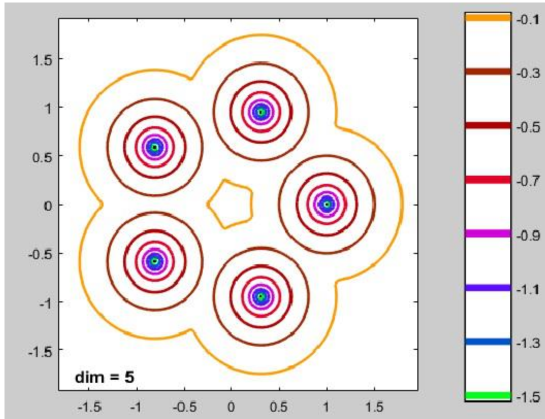

Figure 9. Pseudospectrum energy of a graph with adjacency matrix $A(\mathbb{Z}_5, f_+)$.

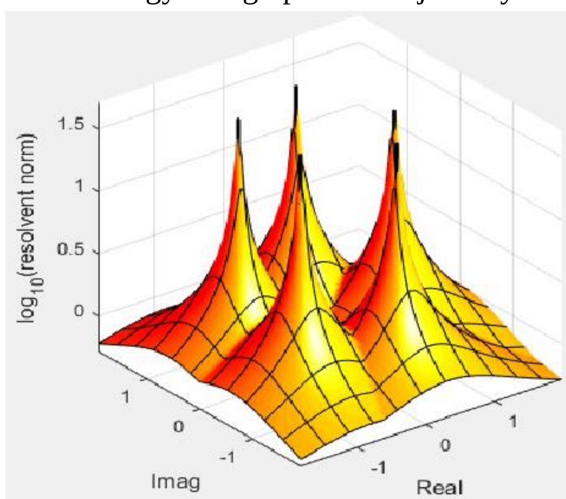

Figure 10. 3D pseudospectrum energy of graph.

Obviously, the pesudospectrum energy of graph constructed by $(\mathbb{Z}_5, f_+)$ in example 4.1 is related to quasicrystallography which discovered by Nobel laureate Dan Shechtman in 1982, see [6]. This observation lead us to ask about the explanation of the relationship between pesudospectrum energy of graph of system $(\mathbb{Z}_5, f_+)$ and quasicrystallography as an open question.

5. CONCLUSION

Eigenvalues are one of most talented instrument of mathematics in applied sciences. The set of all eigenvalues is the spectrum of a given matrix and it is a nonempty set while the resolvent set is outside the spectrum. Significant information has been provided by analyzing the eigenvalues for 100 yeas. Moreover, the limitation of eigenvalue analysis was determined in the second half of the twentieth century by several mathematicians, engineers and others, than non-normal matrices caused new evaluation to understanding the behavior of matrices and in this connection elaborated the notion of the pseudospectrum. In this paper, we introduced the pseudospectrum energy of graph and established a number of its properties. As additional results, by using MATLAB programming platform, we computed the pesudospectrum energy of two graphs represented by non-normal adjacency matrices.

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