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Some Topological Indices Related to Paley Graphs

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ABSTRACT

Let $\mathrm{GF}(q)$ denote the finite field with q elements. The Paley graph $\mathrm{Paley}(q)$ is a graph with vertex set $\mathrm{GF}(q)$ such that two vertices a and b form an edge if $a-b$ is a non-zero square. If we assume $q \equiv 1(\mathrm{mod}4)$, then this graph is undirected. In this paper, our aim is to compute the topological indices of $\mathrm{Paley}(q)$ such as Wiener index, PI index and Szeged index.

1. INTRODUCTION

Let $G=(V,E)$ be a simple connected graph with vertex set V and edge set E. For $u,v \in V$, the edge joining u to v is denoted by uv and the distance between u and v is denoted by $d(u,v)$. The Weiner index of G is denoted by $W(G)$ and is defined by $W(G)=\frac{1}{2}\sum_{u,v\in V} d(u,v)$.

For $v \in V$, let d(v) denote the sum of distances between v and all other vertices x of V, i.e. $d(v)=\sum_{x\in V} d(v,x)$. Then we have $W(G)=\frac{1}{2}\sum_{v\in V} d(v)$.

The Wiener index is one of the oldest descriptors concerned with the molecular graphs. This index appeared in a paper by H. Weiner [8]. Weiner's original definition was different, but equivalent to the formula we have written before. There are many papers on calculation of Weiner indices of several graphs [2]. Another indices that we are interested to find them are Szeged and PI-indices of graphs.

Let $e=uv$ be an edge of the graph G. By $n_u(e|G)$ we mean the number of vertices of G lying closer to u than v, and $n_u(e|G)$ is defined similarly. Let us define the following sets

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$$N_u(e|G) = \{w \in V | d(w,u) < d(w,v)\},$$
$$N_v(e|G) = \{w \in V | d(w,v) < d(w,u)\}.$$

We set $n_u(e|G) = |N_u(e|G)|$ and $n_v(e|G) = |N_v(e|G)|$. The Szeged index of G defined by the following formula:

$$Sz(G) = \sum_{e=uv\in E} n_u(e|G)\, n_v(e|G).$$

The PI-index of a graph G defined as follows. Let the number of edges in the graphs induced by $N_u(e|G)$ and $N_v(e|G)$ be denoted by $n_{eu}(e|G)$ and $n_{ev}(e|G)$, respectively. Then the PI-index of G is defined by:

$$PI(G) = \sum_{e\in E}(n_{eu}(e|G) + n_{ev}(e|G)).$$

Paley graph and its automorphism group is of great interest to those who study algebraic graph theory. This graph was introduced in [6] and has many important properties. It is one of the two families of self-complementary arc-transitive graphs [7]. Paley graphs are also distance-transitive graphs, strongly regular and conference graphs [3]. Its automorphism group acts transitively on both its vertices and edges. Using this latter property of the Paley graph, we will find some recently defined topological indices of this graph.

2. PRELIMINARIES

Let $\mathrm{GF}(q)$ denote the Galois field with q elements, where q is a power of the prime p and $q \equiv 1(\mathrm{mod}\ 4)$. Let S denote the set of non-zero squares in $\mathrm{GF}(q)$, i.e. $S = \{x^2 |\ 0 \neq x \in GF(q)\}$. The Paley graph denoted by $\mathrm{Paley}(q)$, is the graph with vertex set $\mathrm{GF}(q)$ and two vertices x and y are joined by an edge if and only if $x - y \in$ S. Since $q \equiv 1(\mathrm{mod}\ 4)$, -1 is a square in $\mathrm{GF}(q)$, hence if (x,y) is an edge, (y,x) is also an edge; therefore $\mathrm{Paley}(q)$ is an undirected graph. In fact, $\mathrm{Paley}(q)$ is a Cayley graph with the additive group of $\mathrm{GF}(q)$ and the connecting set S. It is a regular graph of degree $(q-1)/2$ with q vertices and $q(q-1)/4$ edges. Since the additive group of S generates $\mathrm{GF}(q)$, we deduce that $\mathrm{Paley}(q)$ is a connected graph. The following lemma is taken from [5].

Lemma 2.1. The automorphism group of $\mathrm{Paley}(q)$ is isomorphic to:

$$A\Sigma L_1(q) = \left\{ t_{a,b,\sigma}: GF(q) \to GF(q) \,\middle|\, \begin{array}{c} t_{a,b,\sigma}(x) = ax^\sigma + b \\ a \in S, b \in GF(q), \sigma \in Aut\big(GF(q)\big) \end{array} \right\}.$$

Proof. The semi-linear affine group in dimension 1 is defined by:

$$A\Gamma L_1(q) = \left\{ t_{a,b,\sigma}: GF(q) \to GF(q) \,\middle|\, \begin{array}{c} t_{a,b,\sigma}(x) = ax^\sigma + b\,, a \neq 0 \\ a, b \in GF(q), \sigma \in Aut\big(GF(q)\big) \end{array} \right\}.$$

and it is clear that $A\Sigma L_1(q) \leq A\Gamma L_1(q)$. Let A=Aut(Paley($q$)). It can be verified that $A\Sigma L_1(q) \leq A$, and that $A\Sigma L_1(q)$ acts transitively on the set of arcs of Paley(q), and we will prove that Aut(Paley(q)) = A. Let f be any automorphism of Paley(q). By transitivity of $A\Sigma L_1(q)$ on arcs of Paley(q) and composing f with suitable elements of $A\Sigma L_1(q)$, we may assume that $f(0) = 0, f(1) = 1$. Now let us define the function $\chi: \mathrm{GF}(q) \to \mathrm{GF}(q)$ by

$$\chi(x) = \begin{cases} 0, & if\ x = 0, \\ 1, & if\ x\ is\ a\ square, \\ -1, & if\ x\ is\ a\ non-square \end{cases}.$$

Since σ is an automorphism of the graph Paley(q), we obtain

$$\chi\big(\sigma(a) - \sigma(b)\big) = \chi(a - b)$$

for all $a, b \in GF(q)$. Now by a result of [1], the mapping σ must be of the form $\sigma(x) = x^{pi}$, for some i and the lemma is proved. ■

For $a \in S$ and $b \in GF(q)$, and $\sigma \in Aut\big(GF(q)\big)$, we define

$$t_b, f_a: GF(q) \to GF(q)$$

by $t_b(x) = x + b$, $f_a(x) = ax^{\sigma}$, then $T = \{t_b \mid b \in GF(q)\}$ is a normal subgroup of $A\Sigma L_1(q)$ and $W = \{f_a \mid a \in S\}$ is its subgroup such that $A\Sigma L_1(q) = \mathrm{T} \rtimes \mathrm{W}$, the semi-direct product of T with W.

From above it is easily verified that the Paley graph is a vertex and edge transitive graph.

3. SOME TOPOLOGICAL INDICES OF THE PALEY GRAPH

Let Paley(q) be the Paley graph defined as the Cayley graph defined on the finite field GF(q) with connecting set $S = \{x^2 \mid 0 \neq x \in GF(q)\}, q \equiv 1 \pmod 4$. Then Paley$(q)$ is connected graph with q vertices and $q(q-1)/4$ edges.

Proposition 4.1. The Wiener index of Paley(q) is:

$$W\big(Paley(q)\big) = \frac{3q}{4}(q-1).$$

Proof. Since Paley(q) is vertex transitive by [2] we have $W(G) = |V|d(v)/2$, for $v \in V$, where $G = (V, E)$ is the graph in question. Note that $|V| = q$ and $d(v) = \sum_{x \in V} d(v, x)$. We may take $v = 0$ and find the sum of distances of vertices from the vertex 0. An easy observation shows that

$$d(0, x) = \begin{cases} 1 & if\ x\ is\ a\ non-zero\ square \\ 2 & if\ x\ is\ a\ non-square. \end{cases}$$

Therefore,

$$d(0) = \frac{q-1}{2} + \frac{q-1}{2} \times 2 = \frac{3}{2}(q-1).$$

By the formula

$$W(G) = \frac{1}{2}(q)\left(\frac{3}{2}(q-1)\right) = \frac{3}{4}q(q-1),$$

proving the result. ■

Proposition 4.2. The Szeged index of Paley(q) is

$$Sz\big(Paley(q)\big) = \frac{1}{64}(q(q-1)(3q+1)(q+3)).$$

Proof. Since Paley(q) is edge-transitive, by [2] we have

$$Sz(G) = |E| n_u(e|G) n_v(e|G).$$

where $e = uv$ is any vertex of the graph $\mathrm{G} = (V, E)$. Here, we have $V = GF(q)$ and $e = uv$ is an edge if and only if $u - v$ is a square in $GF(q)$. We may take $e = 01$, a certain edge of G. For $n_u(e|G)$, we must count the number of $w \in V$ such that $d(w,0) < d(w,1)$.

First we show that the diameter of $Paley(q)$ is 2. Let a and b be two elements of $GF(q)$. If $a - b$ is a square, then $d(a,b) = 1$, otherwise $a - b$ is a non-square. By [4, p. 237] $a - b$ is written as the sum of two square elements of $GF(q)$, say $a - b = c^2 + d^2$, where c and d are non-zero elements of $GF(q)$. Now $b + c^2$ is joined to both a and b, implying $d(a,b) = 2$. Therefore, the diameter of $Paley(q)$ is 2.

Next we count the number of $w \in GF(q)$ such that $d(w,0) < d(w,1)$. One of the choices for w is 0. If w is a non-zero square, then we must count the number of w such that $d(w,1) > 1$, hence $d = 2$. Since $w0$ and 01 are edges of $Payley(q)$, hence $d(w,1) > 1$. Therefore, the number of vertices w is equal to $(q-1)/2$. If w is non-square, then w is not connected to 1 and in this case the distance between w and 1 would be 2. Since the number of non-square w's that are not connected to 1 is $(q-1)/4$, $n_u(e|G) = 1 + \frac{q-1}{2} + \frac{q-1}{4} = \frac{3q+1}{4}$.

To compute $n_v(e|G)$, we must find the number of w such that $d(w,0) < d(w,1)$. One choice for w is $w = 1$. If w is a non-zero square, then $d(w,0) = 1$, hence $d(w,1) < 1$, a contradiction. Hence w should be a non-zero square, $d(w,1) < 2$ implying $d(w,0) = 1$. But the number of non-square w's joining to 1 is $(q-1)/4$ and we obtain $n_v(e|G) = 1 + \frac{1}{2}(q-1) = \frac{1}{4}(q+3)$. Therefore, $Sz\big(Paley(q)\big) = \frac{q(q-1)}{4} \times \frac{3q+1}{4} \times \frac{q+3}{4} = \frac{q(q-1)(3q+1)(q+3)}{64}$. ■

Proposition 4.3. The PI-index of the graph $Paley(q)$ is $PI(Paley(q)) = \frac{1}{16}q(q-1)(q^2+q+2)$.

Proof. Again, by edge-transitivity of $Paley(q)$ and by [2] we have:

$$PI\big(Paley(q)\big) = |E|(n_{eu}(e|G) + n_{ev}(e|G)),$$

where $e = uv$ is any edge of $G = Paley(q)$. Hence $n_{eu}(e|G)$ and $n_{ev}(e|G)$ are the number of edges in graphs induced by $N_u(e|G)$ and $N_v(e|G)$, respectively. We may take $u = 0, v = 1$ and $e = 01$. First, we count the number of edges in $N_u(e|G)$. In this case, we must count the number of vertices w such that

$$d(w,u) < d(w,v), \text{ i.e. } d(w,0) < d(w,1).$$

Case 1. w is a non-zero square: therefore $d(w,1) > 1$ and since diameter of $Paley(q)$ is 2 we obtain $d(w,1) = 2$. Now $w01$ is path of length 2 from w to 1, hence, $\frac{q-1}{2} + 1 = \frac{q+1}{2}$ edges appear in this case. But $wt1, t \neq 0$, is another possibility of a path of length 2 from w to 1. But by [5] the number of common neighbors of w and 1, where $w - 1$ is a non-square, is equal with $(q-1)/4$. In this way we obtain $\frac{q-1}{2} \times \frac{q-1}{4} \times 2 = \frac{(q-1)^2}{4}$ edges inside of $N_u(e|G)$. Therefore, the total edges equals $\frac{(q-1)^2}{4} + \frac{q+1}{2}$.

Case 2. w is a non-square: therefore $d(w,0) < d(w,1)$, hence $d(w,1) > 2$, a contradiction. Next, we count the number of edges inside $N_v(e|G)$. To do this, we must count the number of w such that $d(w,v) < d(w,u)$ i.e. $d(w,1) < d(w,0)$. Again, we consider two cases:

Case a. w is a non-zero square: $d(w,1) < 1$ which implies $w = 0$ and we obtain the edge $e = 01$.

Case b. w is a non-square: $d(w,1) < d(w,0) = 2$ which implies that $d(w,1) = 1$. But in this case the number of w's is $(q-1)/4$ and the number of edges is $(q-1)/4$. Therefore,

$$PI\big(Paley(q)\big) = \frac{q(q-1)}{4}\left(\frac{q+1}{2} + \frac{(q-1)^2}{6} + \frac{q-1}{4}\right).$$
$$= \frac{q(q-1)(q^2+q+2)}{16}.$$

This completes the proof. ■

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