Iranian J. Math. Chem. 10 (4) December (2019) 307 − 318

Iranian Journal of Mathematical Chemistry

Journal homepage: ijmc.kashanu.ac.ir

Some Results on Forgotten Topological Coindex

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ARTICLE INFO

Article History:

Received:10 March 2019
Accepted: 2 May 2019
Published online:30 December 2019
Academic Editor: Tomislav Došlić

Keywords:

Forgotten coindex
Zagreb index
Vertex degree
Bound
Graph invariant

ABSTRACT

The forgotten topological coindex (also called Lanzhou index) is defined for a simple connected graph G as the sum of the terms ${d_u}^2 + {d_v}^2$ over all non-adjacent vertex pairs uv of G, where d_u denotes the degree of the vertex u in G. In this paper, we present some inequalities for the forgotten topological coindex in terms of some graph parameters such as the order, size, number of pendent vertices, minimal and maximal vertex degrees, and minimal non-pendent vertex degree. We also study the relation between this invariant and some well-known graph invariants such as the Zagreb indices and coindices, multiplicative Zagreb indices and coindices, Zagreb eccentricity indices, eccentric connectivity index and coindex, and total eccentricity. Exact formulae for computing the forgotten topological coindex of double graphs and extended double cover of a given graph are also proposed.

1. INTRODUCTION

All graphs considered in this paper are finite, simple and connected. Let G be a graph on n vertices and m edges. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The complement of G, denoted by $\bar{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent if and only if they are not adjacent in G. So $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. It is obvious that, $\bar{m} = |E(\bar{G})| = \binom{n}{2} - m$. The degree $d_G(u)$ of a vertex $u \in V(G)$ is the number of edges incident to u. We denote by

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DOI: 10.22052/ijmc.2019.174722.1432

δ and Δ the minimal and maximal vertex degrees of G, respectively. A vertex of degree one is called a pendent vertex. The distance $d_G(u,v)$ between the vertices $u,v \in V(G)$ is defined as the length of a shortest path in G connecting u and v. The eccentricity $\varepsilon_G(u)$ of a vertex $u \in V(G)$ is the largest distance between u and any other vertex v of G, i.e., $\varepsilon_G(u) = max_{v\in V(G)} d_G(u,v)$. Let r, s and t be positive integers. A graph G is said to be r-regular if all of its vertices have degree r. If the degrees of vertices of G assume only two different values s and t, then G is said to be (s,t)-semiregular, and if its vertex degrees assume exactly three different values r,s, and t, then G is said to be (r,s,t)-triregular.

A *topological index* $Top(G)$ of G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. In organic chemistry, topological indices have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships (SPR), structure-activity relationships (SAR) and pharmaceutical drug design [1].

The *eccentric connectivity index* was introduced by Sharma et al. [2] in 1997. The eccentric connectivity index $\xi^c(G)$ of a graph G is defined as

$$\xi^c(G) = \sum_{u\in V(G)} d_G(u)\varepsilon_G(u) = \sum_{uv\in E(G)}[\varepsilon_G(u) + \varepsilon_G(v)].$$

This graph invariant was successfully used for mathematical models of biological activities of diverse nature [2]. The sum of eccentricities of all vertices of G is called the *total eccentricity* of G and denoted by $\zeta(G)$.

The *second eccentric connectivity index* of G was introduced as [3], $\xi^{(2)}(G) = \sum_{u\in V(G)} d_G(u)\varepsilon_G(u)^2 = \sum_{uv\in E(G)}[\varepsilon_G(u)^2 + \varepsilon_G(v)^2]$. The *eccentric connectivity coindex* has recently been introduced by Hua and Miao [4] as $\bar{\xi}^c(G) = \sum_{uv\notin E(G)}[\varepsilon_G(u) + \varepsilon_G(v)] = \sum_{u\in V(G)}(n-1-d_G(u))\varepsilon_G(u)$.

The *first and second Zagreb eccentricity indices* of G were introduced by Vukičević and Graovac [5] in 2010 as $E_1(G) = \sum_{uv\in E(G)} \varepsilon_G(u)^2$ and $E_2(G) = \sum_{uv\in E(G)} \varepsilon_G(u)\,\varepsilon_G(v)$. The *third Zagreb eccentricity index* of G was proposed by Xu *et al*. [6] in 2016 as a measure for indicating the non-self-centrality extent of graphs,

$$E_3(G) = \sum_{uv\in E(G)}|\varepsilon_G(u) - \varepsilon_G(v)|.$$

The *first Zagreb index* was introduced by Gutman and Trinajstić [7] in 1972 and the *second Zagreb index* was introduced by Gutman *et al*. [8] in 1975. The first and second Zagreb indices of G are respectively defined as $M_1(G) = \sum_{u\in V(G)} d_G(u)^2$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$. The first Zagreb index can also be expressed as a sum over edges of G, $M_1(G) = \sum_{uv\in E(G)}[d_G(u)+d_G(v)]$. Zagreb indices have been used to study molecular complexity, chirality, ZE–isomerism, and hetero–systems. We refer the reader to [9−11] for more information on Zagreb indices.

The *first and second Zagreb coindices* of $G$was put forward in 2008 by Došlić [12] as $\bar{M}_1(G) = \sum_{uv\notin E(G)}[d_G(u)+d_G(v)]$ and $\bar{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v)$. The *third*

Zagreb index [13] (also known as *irregularity index* [14]) of a graph G is defined as $M_3(G) = \sum_{uv\in E(G)}|d_G(u)-d_G(v)|$. The *third Zagreb coindex* was introduced in 2016 by Veylaki and Nikmehr [15] as $\bar{M}_3(G) = \sum_{uv\notin E(G)}|d_G(u)-d_G(v)|$.

The *first and second multiplicative Zagreb indices* of G were introduced in 2010 by Todeschini and Consonni [16] as $\Pi_1(G) = \prod_{u\in V(G)} d_G(u)^2$ and $\Pi_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v)$. The second multiplicative Zagreb index can also be expressed as a product over vertices of G [17], $\Pi_2(G) = \prod_{u\in V(G)} d_G(u)^{d_G(u)}$.

Multiplicative versions of Zagreb coindices were introduced by Xu et al. [18] in 2013. The *first and second multiplicative Zagreb coindices* of G are respectively defined as $\bar{\Pi}_1(G) = \prod_{uv\notin E(G)}[d_G(u)+d_G(v)]$ and $\bar{\Pi}_2(G) = \prod_{uv\notin E(G)} d_G(u)d_G(v)$. The second multiplicative Zagreb coindex can also be expressed as [18], $\bar{\Pi}_2(G) = \prod_{u\in V(G)} d_G(u)^{n-1-d_G(u)}$.

The *hyper–Zagreb index* was put forward in 2013 by Shirdel et al. [19] as $HM(G) = \sum_{uv\in E(G)}[d_G(u) + d_G(v)]^2$. The *hyper–Zagreb coindex* was defined in 2017 by Gutman [20] as $\overline{HM}(G) = \sum_{uv\notin E(G)}[d_G(u) + d_G(v)]^2$.

In [7], Gutman and Trinajstić introduced another invariant defined as the sum of cubes of vertex degrees. This invariant was then received no attention until 2015, when it was revived under the name *forgotten topological index* (*F-index*) by Furtula and Gutman [21]. They showed that the predictive ability of this invariant is almost similar to the predictive ability of the first Zagreb index and both of these invariants yield correlation coefficients more than 0.95 for the entropy and acentric factor. The forgotten topological index is defined as $F(G) = \sum_{u\in V(G)} d_G(u)^3 = \sum_{uv\in E(G)}[d_G(u)^2 + d_G(v)^2]$.

The *forgotten topological coindex* (*F-coindex*) was put forward in 2016 by De et al. [22] as $\bar{F}(G) = \sum_{uv\notin E(G)}(d_G(u)^2 + d_G(v)^2)$. The F-coindex can also be expressed as, $\bar{F}(G) = \sum_{u\in V(G)}(n-1-d_G(u))d_G(u)^2$. The same index was also proposed in a paper by Vukičević et al. [23] under the name *Lanzhou index* in which the authors showed that this invariant performs better than a number of existing indices on several benchmark datasets proposed by the International Academy of Mathematical Chemistry. We refer the reader to [24−27] for more information on forgotten topological index and coindex.

In this paper, we present several inequalities for F-coindex in terms of some graph parameters and investigate the relation between this invariant and some well-known invariants. We also present exact formulae for computing F-coindex for double graphs and extended double cover of a given graph. We refer the reader to [28−30] for more information on computing bounds on topological indices of graphs.

2. PRELIMINARIES

In this section, we recall some basic lemmas which will be used throughout the paper.

Lemma 2.1. [31] *Let G be a graph of order n and size m. Then*

$$\overline{M}_1(G) = 2m(n-1) - M_1(G), \tag{2.1}$$

$$\overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G). \tag{2.2}$$

Lemma 2.2. [20] *Let G be a graph of ordern and size m. Then*

$$\overline{HM}(G) = 4m^2 + (n-2)M_1(G) - HM(G). \tag{2.3}$$

Lemma 2.3. [18] *Let G be a graph of order n. Then*

$$\overline{\Pi}_2(G) = \frac{\Pi_1(G)^{\frac{n-1}{2}}}{\Pi_2(G)}. \tag{2.4}$$

Lemma 2.4. [32] *Let G be a graph of order n. Then*

$$\bar{\xi}^c(G) = (n-1)\zeta(G) - \xi^c(G). \tag{2.5}$$

Lemma 2.5. [3] *For any graph G,*

$$\xi^{(2)}(G) = E_3(G) + 2E_2(G). \tag{2.6}$$

Lemma 2.6. [33] *Let G be a nontrivial connected graph of order n. For each vertex $u \in V(G)$, $\varepsilon_G(u) \le n - d_G(u)$, with equality if and only if $G \cong P_4$ or $G \cong K_n - iK_2$, $0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, where $K_n - iK_2$ denotes the graph obtained from the complete graph K_n by removing $i$independent edges.*

3. SOME INEQUALITIES FOR F−COINDEX

In this section, we present several inequalities for F-coindex in terms of some structural graph parameters and well-known graph invariants. Throughout this section, let G be a nontrivial simple connected graph of order n and size m. We denote by $\overline{m}$ the number of edges in $\bar{G}$. For convenience, we suppose that $d_u = d_G(u)$ and $\varepsilon_u = \varepsilon_G(u)$, for each $u \in V(G)$.

Theorem 3.1. *For any graph G,*

$$\bar{F}(G) = \overline{HM}(G) - 2\overline{M}_2(G). \tag{3.1}$$

Proof. Using definition of F-coindex, we have

$$\bar{F}(G) = \sum_{uv \notin E(G)} \left({d_u}^2 + {d_v}^2\right) = \sum_{uv \notin E(G)} [(d_u + d_v)^2 - 2 d_u d_v]$$
$$= \overline{HM}(G) - 2\bar{M}_2(G). \qquad \blacksquare$$

By plugging Eqs. (2.2) and (2.3) to Eq. (3.1), we easily arrive at:

Corollary 3.2. *Let G be a graph of order n. Then*

$$\bar{F}(G) = (n-1)M_1(G) + 2M_2(G) - HM(G).$$

Theorem 3.3. *Let G be a graph of order n and size m. Then*

$$\bar{F}(G) \leq 2\bar{m}(n-2)^2, \tag{3.2}$$

with equality if and only if $G \cong K_n$ or G is $(n-2)$-regular or $(n-2, n-1)$-semiregular.

Proof. It is clear that for each $uv \notin E(G)$, $d_u, d_v \leq n-2$. Hence

$$\bar{F}(G) = \sum_{uv \notin E(G)} \left({d_u}^2 + {d_v}^2\right) \leq 2\bar{m}(n-2)^2.$$

The equality holds in (3.2) if and only if $d_u = d_v = n-2$, for each $uv \notin E(G)$. This implies that each vertex of G has degree $n-1$ or $n-2$, that is $G \cong K_n$ or G is $(n-2)$-regular or $(n-2, n-1)$-semiregular. $\blacksquare$

Theorem 3.4. *Let G be a graph of order $n \geq 3$ with p pendent vertices and minimal non-pendent vertex degree δ_1. Then*

$$2\bar{m}{\delta_1}^2 - p(n-2)\left({\delta_1}^2 - 1\right) \leq \bar{F}(G) \leq 2\bar{m}\Delta^2 - p(n-2)(\Delta^2 - 1). \tag{3.3}$$

The left hand side equality holds in (3.3) if and only if G is regular or $(1, \delta_1)$ –semiregular or $(\delta_1, n-1)$–semiregular or $(1, \delta_1, n-1)$–triregular and the right hand side equality holds if and only if G is regular or $(1, \Delta)$ –semiregular.

Proof. From the definition of the F-coindex,

$$\bar{F}(G) = \sum_{\substack{uv \notin E(G) \\ d_u, d_v \neq 1}} \left({d_u}^2 + {d_v}^2\right) + \sum_{\substack{uv \notin E(G) \\ d_u = 1, d_v \neq 1}} \left(1 + {d_v}^2\right) + \sum_{\substack{uv \notin E(G) \\ d_u = d_v = 1}} (1+1)$$
$$\geq \sum_{\substack{uv \notin E(G) \\ d_u, d_v \neq 1}} ({\delta_1}^2 + {\delta_1}^2) + \sum_{\substack{uv \notin E(G) \\ d_u = 1, d_v \neq 1}} (1 + {\delta_1}^2) + 2\binom{p}{2}$$
$$= 2{\delta_1}^2\left(\bar{m} - p(n-p-1) - \binom{p}{2}\right) + p(n-p-1)(1 + {\delta_1}^2) + 2\binom{p}{2}$$
$$= 2\bar{m}{\delta_1}^2 - p(n-2)\left({\delta_1}^2 - 1\right).$$

The above equality holds if and only if $d_u = d_v = \delta_1$, for each $uv \notin E(G)$, with $d_u, d_v \neq 1$, and $d_v = \delta_1$, for each $uv \notin E(G)$, with $d_u = 1$, $d_v \neq 1$. Hence, each vertex of G has degree 1, δ_1, or $n-1$, that is G is $(1, \delta_1)$–semiregular or $(1, \delta_1, n-1)$–triregular if $p > 0$, and G is regular or $(\delta_1, n-1)$–semiregular if $p = 0$. Similarly,

$$\begin{aligned}\bar{F}(G) &\leq \sum_{\substack{uv\notin E(G)\\ d_u,d_v\neq 1}} (\Delta^2 + \Delta^2) + \sum_{\substack{uv\notin E(G)\\ d_u=1,d_v\neq 1}} (1 + \Delta^2) + 2\binom{p}{2}\\ &= 2\bar{m}\Delta^2 - p(n-2)(\Delta^2 - 1).\end{aligned}$$

The above equality holds if and only if $d_u = d_v = \Delta$, for each $uv \notin E(G)$, with $d_u, d_v \neq 1$, and $d_v = \Delta$, for each $uv \notin E(G)$, with $d_u = 1$, $d_v \neq 1$. This implies that, each vertex of G has degree 1 or Δ, that is G is $(1, \Delta)$–semiregular if $p > 0$, and G is regular if $p = 0$. ∎

As a direct consequence of Theorem 3.4, we get the following corollary.

Corollary 3.5. *Let G be a graph of order n and size m without pendent vertices. Then*

$$2\bar{m}\delta^2 \leq \bar{F}(G) \leq 2\bar{m}\Delta^2, \tag{3.4}$$

The left hand side equality holds in (3.4) *if and only if the graph G is regular or $(\delta, n-1)$–semiregular and the right hand side inequality holds in* (3.4) *if and only if G is regular.*

Theorem 3.6. *Let G be a graph of order n. Then*

$$(n-1-\Delta)M_1(G) \leq \bar{F}(G) \leq (n-1-\delta)M_1(G). \tag{3.5}$$

The equality holds in both sides if and only if G is a regular graph.

Proof. Since $\delta \leq d_u \leq \Delta$, for each $u \in V(G)$, we have

$$\begin{aligned}\sum_{u\in V(G)} (n-1-\Delta)d_u^2 \leq \bar{F}(G) &= \sum_{u\in V(G)} (n-1-d_u)d_u^2\\ &\leq \textstyle\sum_{u\in V(G)}(n-1-\delta)d_u^2,\end{aligned}$$

from which we straightforwardly arrive at Eq. 3.5. The left hand side inequality holds in (3.5) if and only if $d_u = \Delta$ for each $u \in V(G)$ and the right hand side inequality holds if and only if $d_u = \delta$ for each $u \in V(G)$, which implies that G is a regular graph. ∎

Theorem 3.7. *For any non-complete graph G,*

$$\bar{F}(G) \geq \tfrac{1}{\bar{m}}\bar{M}_3(G)^2 + 2\bar{M}_2(G), \tag{3.6}$$

with equality if and only if $|d_u - d_v|$ is constant for each $uv \notin E(G)$.

Proof. Using definition of F-coindex and Cauchy–Schwarz inequality, we have

$$\begin{aligned}\bar{F}(G) &= \textstyle\sum_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\\ &= \textstyle\sum_{uv\notin E(G)}[{d_u}^2 - 2d_u d_v + {d_v}^2] + 2\sum_{uv\notin E(G)} d_u d_v\\ &= \textstyle\sum_{uv\notin E(G)} |d_u - d_v|^2 + 2\bar{M}_2(G)\\ &\geq \frac{1}{\bar{m}}\left(\textstyle\sum_{uv\notin E(G)}|d_u - d_v|\right)^2 + 2\bar{M}_2(G)\\ &= \frac{1}{\bar{m}}\bar{M}_3(G)^2 + 2\bar{M}_2(G),\end{aligned}$$

and Eq. 3.6 holds. By Cauchy–Schwarz inequality, the equality holds in (3.6) if and only if $|d_u - d_v|$ is constant for each $uv \notin E(G)$. ■

By plugging Eqs. 2.2 and Eq. 3.6, we easily arrive at:

Corollary 3.8. *For any non-complete graph* G, $\bar{F}(G) \geq 4m^2 - M_1(G) - 2M_2(G) + \frac{1}{\bar{m}}\bar{M}_3(G)^2$, *with equality if and only if* $|d_u - d_v|$ *is constant for each* $uv \notin E(G)$.

Theorem 3.9. *Let* G *be a graph of order* n *and size m. Then*

$$\bar{F}(G) \geq \delta(\xi^c(G) - 2m), \tag{3.7}$$

with equality if and only if $G \cong K_n$ *or the graph obtained from* K_n *by removing a perfect matching.*

Proof. Using definition of F-coindex and Lemma 2.6, we obtain

$$\begin{aligned}\bar{F}(G) &= \textstyle\sum_{u\in V(G)}(n-1-d_u)d_u^2\\ &\geq \textstyle\sum_{u\in V(G)}[(n-d_u)-1]d_u\delta\\ &\geq \delta\textstyle\sum_{u\in V(G)}(\varepsilon_u - 1)d_u\\ &= \delta\textstyle\sum_{u\in V(G)}[d_u\varepsilon_u - d_u]\\ &= \delta(\xi^c(G) - 2m).\end{aligned}$$

The above first equality holds if and only if $d_u = \delta$ for each $u \in V(G)$, which implies that G is a regular graph. The above second equality holds if and only if $\varepsilon_u = n - d_u$, for each $u \in V(G)$, which by Lemma 2.6 implies that $G \cong P_4$ or $G \cong K_n - iK_2$, $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. So, the equality in (3.7) holds if and only if $G \cong K_n$ or the graph obtained from K_n by removing a perfect matching. ■

Theorem 3.10. *Let* G *be a graph of order* n *and size m. Then*

$$\bar{F}(G) \leq 2n^2\bar{m} + (n-1)E_1(G) - 2n\bar{\xi}^c(G) - \xi^{(2)}(G), \tag{3.8}$$

with equality if and only if $G \cong P_4$ *or* $G \cong K_n - iK_2$, $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Using definition of F-coindex and Lemma 2.6, we obtain

$$\begin{aligned}\bar{F}(G) &= \textstyle\sum_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\\ &\le \textstyle\sum_{uv\notin E(G)}[(n-\varepsilon_u)^2 + (n-\varepsilon_v)^2]\\ &= \textstyle\sum_{uv\notin E(G)}(2n^2 - 2n(\varepsilon_u + \varepsilon_v) + ({\varepsilon_u}^2 + {\varepsilon_v}^2))\\ &= 2n^2\bar{m} - 2n\bar{\xi}^c(G) + (n-1)E_1(G) - \xi^{(2)}(G).\end{aligned}$$

The equality holds in (3.8) if and only if $\varepsilon_u = n - d_u$, $\varepsilon_v = n - d_v$, for each $uv \notin E(G)$. If $\varepsilon_u = n - d_u$, for each $u \in V(G)$, then the equality in (3.8) holds trivially. Now let the equality holds in (3.8) and let $u \in V(G)$. If $\varepsilon_u = 1$, then $d_u = n-1$ and the equality $\varepsilon_u = n - d_u$ holds. If $\varepsilon_u > 1$, then there exists a vertex $v \in V(G)$ such that $uv \notin E(G)$, which implies that $\varepsilon_u = n - d_u$. Hence, the equality holds in (3.8) if and only if $\varepsilon_u = n - d_u$ for each $u \in V(G)$. This by Lemma 2.6 implies that $G \cong P_4$ or $G \cong K_n - iK_2$, $0 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$. ■

By plugging Eqs. (2.5) and (2.6) to Eq. (3.8), we easily arrive at:

Corollary 3.11. *Let G be a graph of order n and size m. Then $\bar{F}(G) \le 2n^2\bar{m} - 2n(n-1)\zeta(G) + 2n\xi(G) + (n-1)E_1(G) - 2E_2(G) - E_3(G)$, with equality if and only if $G \cong P_4$ or $G \cong K_n - iK_2$, $0 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$.*

Theorem 3.12. *Let G be a non-complete graph of order n and size m. Then*

$$\bar{F}(G) \ge \delta\bar{m}(\bar{\Pi}_1(G))^{\frac{1}{\bar{m}}}, \tag{3.9}$$

with equality and only if G is regular or $(\delta, n-1)$ –semiregular.

Proof. Using the definition of F-coindex and arithmetic-geometric mean inequality, we get

$$\begin{aligned}\bar{F}(G) &= \textstyle\sum_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\\ &\ge \bar{m}\left(\textstyle\prod_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\right)^{\frac{1}{\bar{m}}}\\ &\ge \bar{m}\left(\textstyle\prod_{uv\notin E(G)}(\delta d_u + \delta d_v)\right)^{\frac{1}{\bar{m}}} = \delta\bar{m}(\bar{\Pi}_1(G))^{\frac{1}{\bar{m}}}.\end{aligned}$$

By arithmetic-geometric mean inequality, the above first equality holds if and only if ${d_u}^2 + {d_v}^2$ is constant for each $uv \notin E(G)$. The above second equality holds if and only if $d_u = d_v = \delta$, for each $uv \notin E(G)$. So, the equality holds in (3.9) if and only if $d_u = d_v = \delta$, for each $uv \notin E(G)$. This implies that each vertex of G has degree δ or $n-1$ which shows that G is regular or $(\delta, n-1)$–semiregular. ■

Theorem 3.13. *Let G be a non-complete graph of order n and size m. Then*

$$\bar{F}(G) \ge 2\bar{m}(\bar{\Pi}_2(G))^{\frac{1}{\bar{m}}}, \tag{10}$$

with equality if and only if G is regular or $(\delta, n-1)$–semiregular.

Proof. Using the definition of F-coindex and arithmetic-geometric mean inequality, we get

$$\begin{aligned}\bar{F}(G) &= \sum_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\\ &\geq \bar{m}\left(\prod_{uv\notin E(G)}\left({d_u}^2 + {d_v}^2\right)\right)^{\frac{1}{\bar{m}}}\\ &\geq \bar{m}\left(\prod_{uv\notin E(G)} 2\sqrt{{d_u}^2{d_v}^2}\right)^{\frac{1}{\bar{m}}}\\ &= 2\bar{m}(\bar{\Pi}_2(G))^{\frac{1}{\bar{m}}}.\end{aligned}$$

By arithmetic-geometric mean inequality, the above first equality holds if and only if ${d_u}^2 + {d_v}^2$ is constant for each $uv \notin E(G)$ and the above second equality holds if and only if ${d_u}^2 = {d_v}^2$, for each $uv \notin E(G)$. So, the equality holds in (3.10) if and only if $d_u = d_v = \delta$, for each $uv \notin E(G)$, which implies that G is regular or $(\delta, n-1)$ −semiregular. ■

By plugging Eq. (2.4) to Eq. (3.10), we easily arrive at:

Corollary 3.14. *Let G be a non-complete graph of order n and size m. Then*

$$\bar{F}(G) \geq 2\bar{m}\left(\frac{\Pi_1(G)^{\frac{n-1}{2}}}{\Pi_2(G)}\right)^{\frac{1}{\bar{m}}},$$

with equality and only if G is regular or $(\delta, n-1)$–semiregular.

4. F−Coindex of Double Graph and Extended Double Cover

In this section, we present exact formulae for computing the forgotten topological coindex of double graphs and extended double cover of a given graph.

4.1. Double Graph

Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The *double graph* G^* of G is formed by making two copies $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ of G (including the initial edge set of each of them) by adding edges $x_i y_j$ and $x_j y_i$ for every edge $v_i v_j \in E(G)$. From definition of double graph, $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$, for $1 \leq i \leq n$.

Theorem 4.1 *For any graph G, $\bar{F}(G^*) = 16\bar{F}(G) + 8M_1(G)$.*

Proof. By definition of F-coindex,

$$\begin{aligned}\bar{F}(G^*) &= \textstyle\sum_{uv\notin E(G^*)}(d_{G^*}(u)^2 + d_{G^*}(v)^2)\\
&= \textstyle\sum_{v_iv_j\notin E(G)}[(d_{G^*}(x_i)^2 + d_{G^*}(x_j)^2) + (d_{G^*}(y_i)^2 + d_{G^*}(y_j)^2)\\
&+(d_{G^*}(x_i)^2 + d_{G^*}(y_j)^2) + (d_{G^*}(x_j)^2 + d_{G^*}(y_i)^2)]\\
&+ \textstyle\sum_{i=1}^{n}(d_{G^*}(x_i)^2 + d_{G^*}(y_i)^2)\\
&= 16\textstyle\sum_{v_iv_j\notin E(G)}[d_G(v_i)^2 + d_G(v_j)^2] + 8\sum_{i=1}^{n} d_G(v_i)^2\\
&= 16\bar{F}(G) + 8M_1(G),\end{aligned}$$

Proving the result. ■

4.2. EXTENDED DOUBLE COVER

Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The *extended double cover* G^{**} of G is a bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ in which x_i and y_j are adjacent if and only if either v_i and v_j are adjacent in G or $i = j$. From definition of extended double cover, $d_{G^{**}}(x_i) = d_{G^{**}}(y_i) = d_G(v_i) + 1$, for $1 \le i \le n$.

Theorem 4.2. *Let G be a graph of order n and size m. Then*

$$\bar{F}(G^{**}) = 2\bar{F}(G) + 4\bar{M}_1(G) + 2(n-1)M_1(G) + 4n(n-1) + 4m(2n-3). \tag{4.1}$$

Proof. By definition of F–coindex,

$$\begin{aligned}\bar{F}(G^{**}) &= \textstyle\sum_{uv\notin E(G^{**})}(d_{G^{**}}(u)^2 + d_{G^{**}}(v)^2)\\
&= \textstyle\sum_{v_iv_j\notin E(G)}[(d_{G^*}(x_i)^2 + d_{G^*}(y_j)^2) + (d_{G^*}(x_j)^2 + d_{G^*}(y_i)^2)]\\
&+ \textstyle\sum_{1\le i<j\le n}[(d_{G^{**}}(x_i)^2 + d_{G^{**}}(x_j)^2) + (d_{G^{**}}(y_i)^2 + d_{G^{**}}(y_j)^2)]\\
&= 2\textstyle\sum_{v_iv_j\notin E(G)}[(d_G(v_i) + 1)^2 + (d_G(v_j) + 1)^2]\\
&+ 2\textstyle\sum_{1\le i<j\le n}[(d_G(v_i) + 1)^2 + (d_G(v_j) + 1)^2]\\
&= 2\textstyle\sum_{v_iv_j\notin E(G)}\left[\left(d_G(v_i)^2 + d_G(v_j)^2\right) + 2\left(d_G(v_i) + d_G(v_j)\right) + 2\right]\\
&+ 2\textstyle\sum_{1\le i<j\le n}\left[\left(d_G(v_i)^2 + d_G(v_j)^2\right) + 2\left(d_G(v_i) + d_G(v_j)\right) + 2\right]\\
&= 2[\bar{F}(G) + 2\bar{M}_1(G) + 2\bar{m}] + 2\left[(n-1)M_1(G) + 4m(n-1) + 2\binom{n}{2}\right]\\
&= 2\bar{F}(G) + 4\bar{M}_1(G) + 2(n-1)M_1(G) + 4n(n-1) + 4m(2n-3)\end{aligned}$$

which completes the proof. ■

By plugging Eq. (2.1) to Eq. (4.1), we easily arrive at:

Corollary 4.3. *Let G be a graph of order n and size m. Then*

$$\bar{F}(G^{**}) = 2\bar{F}(G) + 2(n-3)M_1(G) + 4n(n-1) + 4m(4n-5).$$

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