Iranian J. Math. Chem. 10 (4) December (2019) 295 − 305

Iranian Journal of Mathematical Chemistry

Journal homepage: ijmc.kashanu.ac.ir

On the Graovac–Ghorbani Index of Graphs

MODJTABA GHORBANI•,1, SHAGHAYEGH RAHMANI1 AND OTTORINO ORI2

[1]Department of Mathematics, Faculty of Science, Shahid Rajaee, Teacher Training University, Tehran, 16785 – 136, I. R. Iran
[2]Actinium Chemical Research, Via Casilina, Rome, Italy

ARTICLE INFO

Article History:

Received: 26 January 2019
Accepted: 9 May 2019
Published online: 30 September 2019
Academic Editor: Ali Reza Ashrafi

Keywords:

Atom bond connectivity index
Molecular graphs
Topological index.

ABSTRACT

For the edge $e = uv$ of a graph G, let $n_u = n(u|G)$ be the number of vertices of G lying closer to the vertex u than to the vertex v and $n_v = n(v|G)$ can be defined similarly. Then the ABC_{GG} index of G is defined as $ABC_{GG}(G) = \Sigma_{e=uv}\sqrt{f(u,v)}$, where $f(u,v) = (n_u + n_v - 2)/n_u n_v$. The aim of this paper is to give some new results on this graph invariant. We also calculate the ABC_{GG} of an infinite family of fullerenes.

1. INTRODUCTION

All graphs G considered in this paper are undirected and finite on n vertices, loops and multiple edges do not occur. If Λ denotes the class of all finite graphs then a topological index is a function Top from Λ into real numbers with this property that Top(G) = Top(H), if G and H are isomorphic, see [14, 15, 16]. The Wiener index [17] is the first reported distance based topological index defined as half sum of the distances between all the pairs of vertices in a molecular graph. Topological indices are abundantly being used in the *QSPR* and *QSAR* researches. So far, many various types of topological indices have been described. Estrada et al. [2, 3, 4, 5] introduced atom-bond connectivity (ABC) index, which it has been applied up until now to study the stability of alkanes and the strain energy of cyclo-alkanes. This index is defined as follows:

$$\mathrm{ABC}(G) = \sum_{e=uv\in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}}, \qquad (1)$$

•Author to whom correspondence should be addressed (E-mail: mghorbani@sru.ac.ir).

DOI: 10.22052/ijmc.2019.169508.1420

where $d_G(u)$ stands for the degree of the vertex u. Graovac and Ghorbani [6] introduced a distance-based analog of this index and Furtula [9] used the name Graovac-Ghorbani index for this graph invariant.

For arbitrary vertices u and v, the distance $d(u,v)$ is defined as the length of a minimal path connecting u and v. For $e = uv \in E(G)$, let $n_u = n(u|G)$ and $n_v = n(v|G)$ be respectively the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. Then the ABC_{GG} index is defined as follows:

$$\mathrm{ABC}_{GG}(G) = \sum_{e=uv\in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u n_v}}. \tag{2}$$

The normalized ABC_{GG} index, NGG index, is defined by Dimitrov et al. [1] as follows:

$$NGG(G) = \sum_{e=uv\in E(G)} \frac{1}{\sqrt{n_u n_v}}. \tag{3}$$

Furtula in [6] showed that the ABC_{GG} index gave better prediction in the case of entropy and acentric factor than ABC index. However, ABC_{GG} is a distance-based topological descriptor, while ABC is a degree-based one.

Gutman in [10] introduced a graph invariant, named as the Szeged index, defined by

$$Sz(G) = \sum_{e\in E(G)} n_u n_v. \tag{4}$$

Also, the vertex PI index of G is defined in [11] as

$$PI_v(G) = \sum_{e\in E(G)} n_u + n_v. \tag{5}$$

A fullerene is a molecule composed of carbon atoms in the form of many shapes such as hollow sphere, ellipsoid, tube, etc. The most important fullerenes are spherical fullerenes or buckyballs. Carbon nanotubes or buckytubes are cylindrical fullerenes, see [12,13] for details. Fullerenes are similar in structure to graphite, but they may also contain pentagonal and hexagonal rings. In mathematical language, a fullerene graph on n vertices is a cubic 3-connected graph with exactly 12 pentagons and $(n/2-10)$ hexagons, where $n \neq 20$.

The aim of this paper is to present some bounds on ABC_{GG} index and then we compute the Graovac-Ghorbani index of an infinite class of fullerene graphs.

2. MAIN RESULTS AND DISCUSSION

For every edge $e = uv$ of a graph G, we define $N_{uv} = \{x \in V :\ d(u,x) = d(v,x)\}$ and $n_{uv} = |N_{uv}|$. It is clear that

$$\mathrm{ABC}_{GG}(G) = \sum_{e=uv\in E(G)} \sqrt{\frac{n - n_{uv} - 2}{n_u n_v}}.$$

Theorem 1. Let G be a connected graph with n vertices. Then

$$\lim_{n\to\infty}\left|\frac{\mathrm{ABC}_{GG}(G)}{\sum_{e=uv\in E(G)}\sqrt{n_u+n_v-2}}-\frac{n+2}{2n}\right|\to\frac{1}{2}.$$

Proof. Since for every edge $e= uv$, $n_u+n_v\le n$, $n_un_v\le n/2\times n/2=n^2/4$. Then we can conclude that

$$\mathrm{ABC}_{GG}(G)=\sum_{e=uv\in E(G)}\sqrt{\frac{n_u+n_v-2}{n_un_v}}\ge\frac{2}{n}\sum_{e=uv\in E(G)}\sqrt{n-n_{uv}-2}.$$

On the other hand, for every vertex u, we have $n_u \ge 1$ and thus

$$\mathrm{ABC}_{GG}(G)\le\sum_{e=uv\in E(G)}\sqrt{n-n_{uv}-2}.$$

Since for integers a, b, x where $a\le x\le b$, we have

$$\frac{a}{2}-\frac{b}{2}\le x-\frac{a+b}{2}\le\frac{b}{2}-\frac{a}{2}$$

we can conclude that

$$\left|\frac{\mathrm{ABC}_{GG}(G)}{\sum_{e=uv\in E(G)}\sqrt{n-n_{uv}-2}}-\frac{n+2}{2n}\right|\le\frac{n-2}{2n}.$$

This completes the proof.

Theorem 2. Let G be a connected graph and m is the number of edges. Then

$$\mathrm{ABC}_{GG}(G)\ge\sqrt{\frac{PI_v(G)-2m}{Sz(G)}}.$$

Proof. It is not so difficult to see that

$$\mathrm{ABC}_{GG}(G)=\sum_{e=uv\in E(G)}\sqrt{\frac{n_u+n_v-2}{n_un_v}}\ge\sqrt{\sum_{e=uv\in E(G)}\frac{n_u+n_v-2}{n_un_v}}\ge\sqrt{\frac{\sum_{e=uv\in E(G)}n_u+n_v-2}{\sum_{e=uv\in E(G)}n_un_v}}.$$

3. ABC_{GG} INDEX OF FULLERENES

An automorphism of a graph G is a permutation σ on the set of vertices which preserves the edge set, namely $e = uv$ is an edge of G if and only if $\sigma(u)\sigma(v)$ is an edge in G. The set of all automorphisms of G is denoted by Aut(G) which under composition of mappings forms a group. Now consider the molecular graph of fullerene C_{12n} ($n \ge 2$), Figure 10. This class of fullerenes has exactly $12n$ carbon atoms and the first member of this class of fullerenes can be obtained by putting n = 2, namely C_{24}. In [7] it is proved that the symmetry group of this class of fullerenes is isomorphic to the dihedral group of order 24. Also, Ghorbani et al. in [8] proved the following result.

Theorem 3 [8]. Let C_{12n} (n is even) is the fullerene graph. There are $n + 1$ orbits under the action of automorphism group on the set of edges.

Edges	n_u, n_v, n_{uv}	***Number***
e_1	18,18,12n−36	12
e_2	15,12n−31,16	12
e_3	24,12n−28,4	24
e_4	18,12n−18,0	12
e_5	32,12n−34,2	24
e_6	30,12n−30,0	12
e_7	40,12n−42,2	24
e_8	42,12n−42,0	12
e_9	50,12n−51,1	24
e_{10}	54,12n−54,0	12
e_{11}	60,12n−61,1	24
e_{12}	66+0×6,12n−(66+0×6),0	12
e_{13}	66+1×6,12n−(66+1×6),0	24
e_{14}	66+2×6,12n−(66+2×6),0	12
e_{15}	66+3×6,12n−(66+3×6),0	24
⋮	⋮	⋮
e_{n-1}	66+(n−13)×6,12n−(66+(n−13)×6),0	24
e_n	66+(n−12)×6,12n−(66+(n−12)×6),0	12
e_{n+1}	66+(n−11)×6,12n−(66+(n−11)×6),0	12

Table 1. Computing n_u, n_v and n_{uv} for edges in the different orbits of fullerene graph C_{12n}, (n is even and $n \geq 12$).

Theorem 4. Let $E_1, \ldots, E_r$ be the orbits of G under the action of Aut(G) on $E(G)$. Then

$$\mathrm{ABC}_{GG}(G)=\sum_{i=1}^{r}\sum_{e_i\in E_i}|E_i|\times\sqrt{\frac{n-n_{uv}-2}{n_u n_v}}. \qquad (6)$$

Proof. Let $E_1, \ldots, E_r$ be the orbits of G. For two edges $e = uv$ and $f = ab$ in the same orbit of G, we can prove that $\{n_u, n_v\}=\{n_a, n_b\}$. This completes the proof.

Theorem 5. Consider the fullerene graph C_{12n}, where n is even and $n \geq 12$. Then

$$\begin{aligned}\mathrm{ABC}_{GG}(C_{12n}) &= 12\times\sqrt{\frac{34}{18\times18}}+12\times\sqrt{\frac{12n-18}{15\times(12n-31)}}+24\times\sqrt{\frac{12n-6}{24\times(12n-28)}}+\\ &12\times\sqrt{\frac{12n-2}{18\times(12n-18)}}+24\times\sqrt{\frac{12n-4}{32\times(12n-34)}}+12\times\sqrt{\frac{12n-2}{30\times(12n-30)}}+\\ &24\times\sqrt{\frac{12n-4}{40\times(12n-42)}}+12\times\sqrt{\frac{12n-2}{42\times(12n-42)}}+24\times\sqrt{\frac{12n-3}{50\times(12n-51)}}+\\ &12\times\sqrt{\frac{12n-2}{54\times(12n-54)}}+24\times\sqrt{\frac{12n-3}{60\times(12n-61)}}+12\times\sqrt{\frac{12n-2}{6n\times6n}}\\ &12\sum_{i=0}^{(n/2)-6}\sqrt{\frac{12n-2}{(12n-66-2i)\times(66+2i)}}+24\sum_{i=0}^{(n/2)-6}\sqrt{\frac{12n-2}{(12n-66-(2i+1))\times(66+(2i+1))}}.\end{aligned}$$

Proof. For every edge $e=uv$, the value of $n(u)$, $n(v)$ of the edge $e = uv$ is determined as reported in Table 1, see [8]. If $n \geq 12$, then by using Table 1, for every edge $e=uv$, one can determine the value of $\sqrt{(n-n_{uv}-2)/n_u n_v}$ and applying the Eq. (6) yields the proof.

Edges	**C_{24}**			**C_{48}**			**C_{72}**			**C_{96}**			**C_{120}**		
e_1	10	10	4	16	16	16	18	18	36	18	18	60	18	18	84
e_2	11	9	4	15	21	12	15	41	16	15	65	16	15	89	16
e_3	8	8	8	18	24	6	23	44	5	24	68	4	24	92	4
e_4	-	-	-	18	30	0	18	54	0	18	78	0	18	102	0
e_5	-	-	-	22	22	4	29	40	3	32	62	2	32	86	2
e_6	-	-	-	-	-	-	30	42	0	30	66	0	30	90	0
e_7	-	-	-	-	-	-	34	34	4	39	54	3	40	78	2
e_8	-	-	-	-	-	-	-	-	-	42	54	0	42	78	0
e_9	-	-	-	-	-	-	-	-	-	47	47	2	50	69	1
e_{10}	-	-	-	-	-	-	-	-	-	-	-	-	54	66	0
e_{11}	-	-	-	-	-	-	-	-	-	-	-	-	59	59	2

Table 2. Some exceptional cases of the fullerene graph C_{12n} (n is even).

For $n \leq 12$, the exceptional cases can be computed by data of Table 2 as follows:

$$ABC_{GG}(C_{24}) = 12\times\sqrt{\frac{18}{100}} + 12\times\sqrt{\frac{18}{99}} + 12\times\sqrt{\frac{14}{64}} = 12(\sqrt{\frac{18}{100}} + \sqrt{\frac{18}{99}} + \sqrt{\frac{14}{64}}).$$

$$ABC_{GG}(C_{48}) = 12\times\sqrt{\frac{30}{256}} + 12\times\sqrt{\frac{34}{315}} + 24\times\sqrt{\frac{40}{432}} + 12\times\sqrt{\frac{46}{540}} + 12\times\sqrt{\frac{42}{484}}$$

$$= 12\times(\sqrt{\frac{30}{256}} + \sqrt{\frac{34}{315}} + 2\sqrt{\frac{40}{432}} + \sqrt{\frac{46}{540}} + \sqrt{\frac{42}{484}}).$$

$$ABC_{GG}(C_{72}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{54}{615}} + 24\times\sqrt{\frac{65}{1012}} + 12\times\sqrt{\frac{70}{972}} + 24\times\sqrt{\frac{67}{1160}}$$

$$+ 12\times\sqrt{\frac{70}{1260}} + 12\times\sqrt{\frac{66}{1156}}$$

$$= 12\times(\sqrt{\frac{34}{324}} + \sqrt{\frac{54}{615}} + 2\sqrt{\frac{65}{1012}} + \sqrt{\frac{70}{972}} + 2\sqrt{\frac{67}{1160}} + \sqrt{\frac{70}{1260}} + \sqrt{\frac{66}{1156}}).$$

$$ABC_{GG}(C_{96}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{78}{975}} + 24\times\sqrt{\frac{90}{1632}} + 12\times\sqrt{\frac{94}{1404}} + 24\times\sqrt{\frac{92}{1984}}$$

$$+ 12\times\sqrt{\frac{94}{1980}} + 24\times\sqrt{\frac{91}{2106}} + 12\times\sqrt{\frac{94}{2268}} + 12\times\sqrt{\frac{92}{2209}}$$

$$= 12\times(\sqrt{\frac{34}{324}} + \sqrt{\frac{78}{975}} + 2\sqrt{\frac{90}{1632}} + \sqrt{\frac{94}{1404}} + 2\sqrt{\frac{92}{1984}} + \sqrt{\frac{94}{1980}}$$

$$+ 2\sqrt{\frac{91}{2106}} + \sqrt{\frac{94}{2268}} + \sqrt{\frac{92}{2209}}).$$

$$ABC_{GG}(C_{120}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{102}{1335}} + 24\times\sqrt{\frac{114}{2208}} + 12\times\sqrt{\frac{118}{1836}}$$

$$+ 24\times\sqrt{\frac{116}{2752}} + 12\times\sqrt{\frac{118}{2700}} + 24\times\sqrt{\frac{116}{3120}} + 12\times\sqrt{\frac{118}{3276}} + 24\times\sqrt{\frac{117}{3450}}$$

$$+ 12\times\sqrt{\frac{118}{3564}} + 12\times\sqrt{\frac{116}{3481}} = 12\times(\sqrt{\frac{34}{324}} + \sqrt{\frac{102}{1335}} + 2\sqrt{\frac{114}{2208}} + \sqrt{\frac{118}{1836}}$$

$$+ 2\sqrt{\frac{116}{2752}} + \sqrt{\frac{118}{2700}} + 2\sqrt{\frac{116}{3120}} + \sqrt{\frac{118}{3276}} + 2\sqrt{\frac{117}{3450}} + \sqrt{\frac{118}{3564}} + \sqrt{\frac{116}{3481}}).$$

Theorem 6. The fullerene graph C_{12n} (n is odd) has exactly $n + 1$ orbits under the action of automorphism group on the set of edges.

Proof. Suppose n is an odd number. The graph C_{12n} has $n + 1$ levels such that the first and the last levels have 6 edges and the other levels have exactly 12 edges. There are 6 edges between i-th and i+1-th levels that we call them as i-th set of vertical edges. By rotation $\rho = 60°$ around the central hexagon one can see that the edges of the first level and the last level are in the same orbit. Also, a reflection α which maps the central hexagon to the last one maps the edges of the first level to the edges of the last level. This means that the edges of the first and the last levels are in the same orbit under the action of automorphism group on the set of edges.

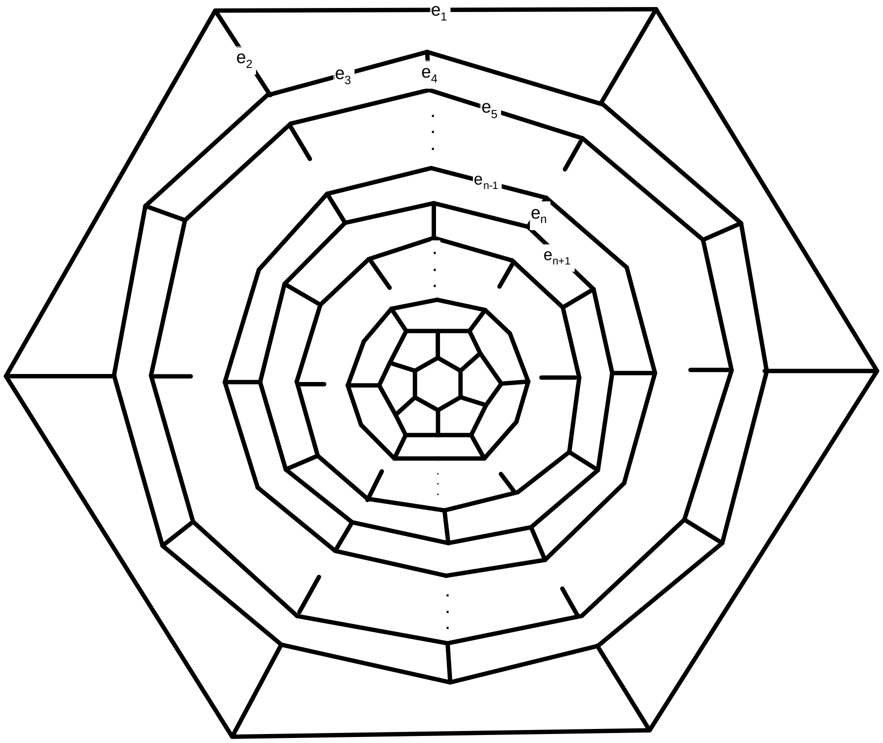

Figure 1. The fullerene graph C_{12n} (n is even).

One can prove that i-th and $(n+1-i)$-th $(1 \le i \le (n-1)/2)$ set of vertical edges are in the same orbit under the action of the group $<\rho, \alpha>$. Note that in the case that $i = (n+1)/2$, we have an orbit of size 6 and the other orbits are of size 12. Also, by a similar argument, one can see that the edges of the j-th and $(n+2-j)$-th $(1 \le j \le (n+1)/2)$ levels are in the same orbit. In other words, C_{12n} has $1+(n-1)/2$ orbits of size 12, $(n-1)/2$ orbits of size 24 and one orbit of size 6. This means that the number of orbits of C_{12n} on the set of edges is $n + 1$.

By a similar argument, we can compute the values n_u, n_v, n_{uv} for arbitrary edge $e = uv$ in the case that n is even. The resulted values can be found in [8].

Theorem 7. The Graovac-Ghorbani index of the fullerene graph C_{12n}, n is odd and $n \ge 11$, can be computed as follows:

$$\begin{aligned}
\mathrm{ABC}_{GG}(C_{12n}) &= 12\times\sqrt{\frac{34}{18\times 18}}+12\times\sqrt{\frac{12n-18}{15\times(12n-31)}}+24\times\sqrt{\frac{12n-6}{24\times(12n-28)}}+\\
&12\times\sqrt{\frac{12n-2}{18\times(12n-18)}}+24\times\sqrt{\frac{12n-4}{32\times(12n-34)}}+12\times\sqrt{\frac{12n-2}{30\times(12n-30)}}+\\
&24\times\sqrt{\frac{12n-4}{40\times(12n-42)}}+12\times\sqrt{\frac{12n-2}{42\times(12n-42)}}+24\times\sqrt{\frac{12n-3}{50\times(12n-51)}}+\\
&12\times\sqrt{\frac{12n-2}{54\times(12n-54)}}+24\times\sqrt{\frac{12n-3}{60\times(12n-61)}}+6\times\sqrt{\frac{12n-2}{6n\times 6n}}\\
&12\sum_{i=0}^{(n-13)/2}\sqrt{\frac{12n-2}{(12n-66-2i)\times(66+2i)}}+24\sum_{i=0}^{(n-13)/2}\sqrt{\frac{12n-2}{(12n-66-(2i+1))\times(66+(2i+1))}}.
\end{aligned}$$

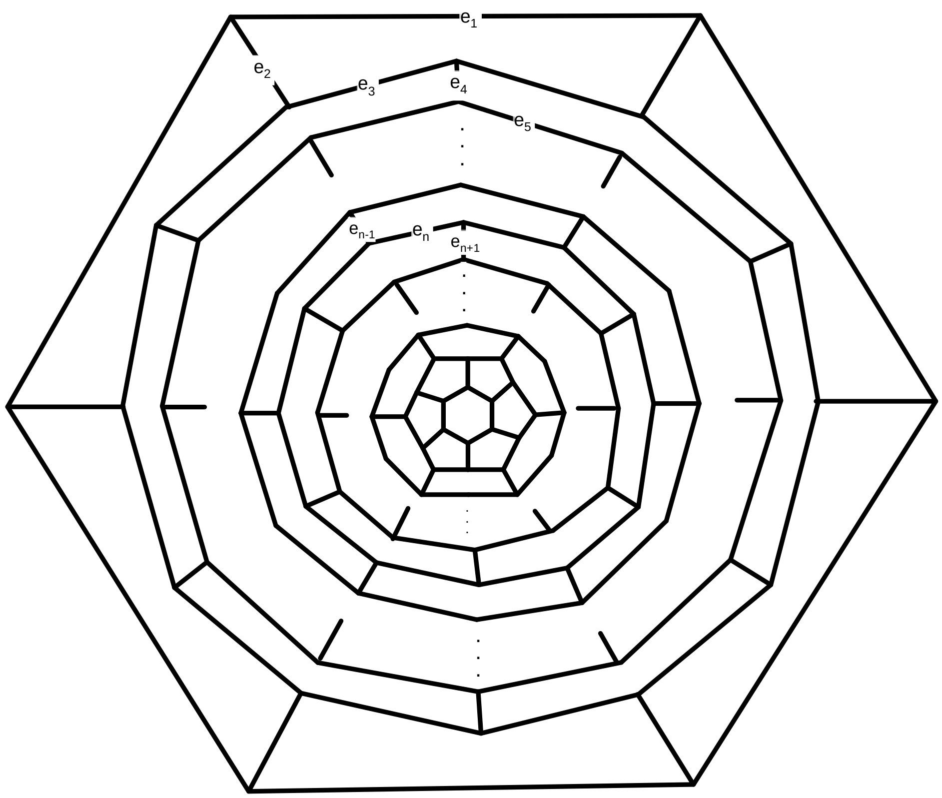

Figure 2. The fullerene graph C_{12n} (n is odd).

Proof. If $n \geq 11$, then by using Table 3, one can determine the value of $\sqrt{(n-n_{uv}-2)/n_u n_v}$ for every edge $e=uv$. Now applying Eq. (6) yields the proof.

Edges	$\boldsymbol{n_u, n_v, n_{uv}}$	***Number***
e_1	$18,18,12n-36$	12
e_2	$15,12n-31,16$	12
e_3	$24,12n-28,4$	24
e_4	$18,12n-18,0$	12
e_5	$32,12n-34,2$	24
e_6	$30,12n-30,0$	12
e_7	$40,12n-42,2$	24
e_8	$42,12n-42,0$	12
e_9	$50,12n-51,1$	24
e_{10}	$54,12n-54,0$	12
e_{11}	$60,12n-61,1$	24
e_{12}	$66+0\times6,12n-(66+0\times6),0$	12
e_{13}	$66+1\times6,12n-(66+1\times6),0$	24
e_{14}	$66+2\times6,12n-(66+2\times6),0$	12
e_{15}	$66+3\times6,12n-(66+3\times6),0$	24
⋮	⋮	⋮
e_{n-1}	$66+(n-13)\times6,12n-(66+(n-13)\times6),0$	12
e_n	$66+(n-12)\times6,12n-(66+(n-12)\times6),0$	24
e_{n+1}	$66+(n-11)\times6,12n-(66+(n-11)\times6),0$	6

Table 3. Computing n_u,n_v and n_{uv} for edges in the different orbits of fullerene graph C_{12n}, (n is odd and $n \geq 11$).

For $n \leq 9$, the exceptional cases can be computed by data of Table 4 as follows:

$$ABC_{GG}(C_{36}) = 12 \times \sqrt{\frac{26}{196}} + 12 \times \sqrt{\frac{26}{196}} + 24 \times \sqrt{\frac{28}{224}} + 6 \times \sqrt{\frac{34}{324}}$$
$$= 6 \times (2\sqrt{\frac{26}{196}} + 2\sqrt{\frac{26}{196}} + 4\sqrt{\frac{28}{224}} + \sqrt{\frac{34}{324}}).$$

$$ABC_{GG}(C_{60}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{43}{450}} + 24\times\sqrt{\frac{53}{714}} + 12\times\sqrt{\frac{58}{756}} + 24\times\sqrt{\frac{54}{780}} + 6\times\sqrt{\frac{58}{900}}$$
$$= 6\times(2\sqrt{\frac{34}{324}} + 2\sqrt{\frac{43}{450}} + 4\sqrt{\frac{53}{714}} + 2\sqrt{\frac{58}{756}} + 4\sqrt{\frac{54}{780}} + \sqrt{\frac{58}{900}}).$$

$$ABC_{GG}(C_{84}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{66}{795}} + 24\times\sqrt{\frac{78}{1344}} + 12\times\sqrt{\frac{82}{1188}} + 24\times\sqrt{\frac{79}{1550}} + 12\times\sqrt{\frac{82}{1620}} + 24\times\sqrt{\frac{79}{1628}} + 6\times\sqrt{\frac{82}{1764}}$$
$$= 6\times(2\sqrt{\frac{34}{324}} + 2\sqrt{\frac{66}{795}} + 4\sqrt{\frac{78}{1344}} + 2\sqrt{\frac{82}{1188}} + 4\sqrt{\frac{79}{1550}} + 2\sqrt{\frac{82}{1620}} + 4\sqrt{\frac{79}{1628}} + \sqrt{\frac{82}{1764}}).$$

$$ABC_{GG}(C_{108}) = 12\times\sqrt{\frac{34}{324}} + 12\times\sqrt{\frac{90}{1155}} + 24\times\sqrt{\frac{102}{1920}} + 12\times\sqrt{\frac{106}{1620}} + 24\times\sqrt{\frac{104}{2368}} + 12\times\sqrt{\frac{106}{2340}} + 24\times\sqrt{\frac{104}{2640}} + 12\times\sqrt{\frac{106}{2772}} + 24\times\sqrt{\frac{104}{2793}} + 6\times\sqrt{\frac{106}{2916}}$$
$$= 6\times(2\sqrt{\frac{34}{324}} + 2\sqrt{\frac{90}{1155}} + 4\sqrt{\frac{102}{1920}} + 2\sqrt{\frac{106}{1620}} + 4\sqrt{\frac{104}{2368}} + 2\sqrt{\frac{106}{2340}} + 4\sqrt{\frac{104}{2640}} + 2\sqrt{\frac{106}{2772}} + 4\sqrt{\frac{104}{2793}} + \sqrt{\frac{106}{2916}}).$$

Edges	**C_{36}**			**C_{60}**			**C_{84}**			**C_{108}**		
e_1	14	14	8	18	18	24	18	18	48	18	18	72
e_2	14	14	8	15	30	15	15	53	16	15	77	16
e_3	14	16	6	21	34	5	24	56	4	24	80	4
e_4	18	18	0	18	42	0	18	66	0	18	90	0
e_5	-	-	-	26	30	4	31	50	3	32	74	2
e_6	-	-	-	30	30	0	30	54	0	30	78	0
e_7	-	-	-	-	-	-	37	44	3	40	66	2
e_8	-	-	-	-	-	-	42	42	0	42	66	0
e_9	-	-	-	-	-	-	-	-	-	49	57	2
e_{10}	-	-	-	-	-	-	-	-	-	54	54	0

Table 4. Some exceptional cases of the fullerene graph C_{12n} (n is odd).

REFERENCES

1. D. Dimitrov, B. Ikica and R. Škrekovski, Remarks on the Graovac-Ghorbani index of bipartite graphs, arXiv:1609.01406v1.
2. E. Estrada, L. Torres, L. Rodríguez and I. Gutman, An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, *Indian J. Chem*. **37A** (1998) 849−855.
3. E. Estrada, Atom-bond connectivity and the energetic of branched alkanes, *Chem. Phys. Lett*. **463** (2008) 422−425.
4. B. Furtula, I. Gutman and K. C. Das, On atom-bond molecule structure descriptors, *J. Serb. Chem. Soc*. **80** (2015) 1−7.
5. B. Furtula and I. Gutman, in: I. Gutman, B. Furtula (Eds.) *Novel Molecular Structure Descriptors − Theory and Applications*, Vol. I, University of Kragujevac, Kragujevac, 2010, pp. 137−172.
6. B. Furtula, Atom-bond connectivity index versus Graovac-Ghorbani analog, *MATCH Commun. Math. Comput. Chem*. **75** (2016) 233−242.
7. M. Ghorbani, Enumeration of heterofullerenes: A survey, *MATCH Commun. Math. Comput. Chem*. **68** (2012) 381−414.
8. M. Ghorbani and Sh. Rahmani, Study of Mostar index of fullerene graphs, *Iranian J. Math. Sci. Inf*., in press.
9. A. Graovac and M. Ghorbani, A new version of the atom-bond connectivity index, *Acta Chim. Slov*. **57** (2010) 609−612.
10. I. Gutman and A. A. Dobrynin, The Szeged index − a success story, *Graph Theory Notes New York* **34** (1998) 37−44.
11. M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, Vertex and edge PI indices of Cartesian product graphs, *Discrete Appl. Math.* **156** (2008) 1780−1789.
12. H. W. Kroto, R. J. Heath, S. C. O'Brien, R. F. Curl and R. E. Smalley, C_{60}: Buckminsterfullerene, *Nature* **318** (1985) 162−163.
13. H. W. Kroto, J. E. Fichier and D. E. Cox, *The Fulerene*, Pergamon Press, New York, 1993.
14. R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, 2000.
15. N. Trinajstić, *Chemical Graph Theory*, 2nd revised edn, CRC Press, Boca Raton, FL, 1992.
16. N. Trinajstić and I. Gutman, Mathematical Chemistry, *Croat. Chem. Acta*. **75** (2002) 329 – 356.
17. H. Wiener, Structural determination of the paraffin boiling points, *J. Am. Chem. Soc*. **69** (1947) 17−20.