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Topological Efficiency of Some Product Graphs

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ABSTRACT

The topological efficiency index of a connected graph G, denoted by $\rho(G)$ is defined as $\rho = [2W(G)/|V(G)|\underline{w}(G)]$, where $\underline{w}(G) = \min\{w_v(G) : v \in V(G)\}$ and $W(G)$ is the Wiener index of G. In this paper, we obtain the value of topological efficiency index ρ for some composite graphs such as tensor product, strong product, symmetric difference and disjunction of two connected graphs. Further, we have obtained the topological efficiency index for a double graph of a given graph.

1. INTRODUCTION

Throughout this paper, we consider only simple connected graphs. We use the notation $d_G(v)$ to denote the degree of a vertex v in a graph G. Let $d_G(u,v)$ denote the distance between two vertices u and v in G and let $w_v(G)$ denote the sum of all distances from v to all other vertices in G, that is,

$$w_v(G) = \sum_{u \in V(G)} d_G(u,v) \text{ with } \underline{w}(G) = \min\{w_v(G) : v \in V(G)\}.$$

The complete graph on n vertices is denoted by K_n.

The topological indices (also known as the molecular descriptors) have been received much attention by various authors in the past decades, and they have been found to be useful in the structure activity relationships (*SAR*), structure-property relationships (*SPR*), and pharmaceutical drug design in organic chemistry, see [4, 5, 15]. Indeed, the topological index of a graph G can be viewed as a graph invariant under the isomorphism of graphs, that is, for any topological index TI, $TI(G) = TI(H)$ if $G \cong H$.

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One of the most thoroughly studied topological indices was the Wiener index which was proposed by Wiener in 1947 [17]. This index has been shown to possess close relation with the graph distance, which is an important concept in pure graph theory. It is also well correlated with many physical and chemical properties of a variety of classes of chemical compounds. For more details, see[9, 10, 11, 12, 18]. The Wiener index of a graph G, denoted by $W(G)$, is defined as $W(G)=\sum_{u,v\in V(G)} d_G(u,v)/2$, where the summation goes over all pairs of vertices in G. One can easily observe that $W(G)=\sum_{v\in V(G)} w_v(G)/2$.

Vukičević et al. [16] proposed a new graph descriptor ρ, called topological efficiency index based on minimal vertex contribution w defined for a connected graph G as

$$\rho(G)=\frac{2W(G)}{|V(G)|\underline{w}(G)}.$$

The topological efficiency index of C_{66} fullerene graph was computed in [16]. In [3], the topological efficiency of some product graphs such as Cartesian product, join, corona product, composition and hierarchical product were given. The value of topological efficiency index of some types of fullerenes and nanocones are obtained in [7, 8]. In this paper, we obtain the topological efficiency index for some composite graphs such as tensor product, strong product, symmetric difference and disjunction of two connected graphs. Further, we have obtained topological efficiency index for a double graph.

2. BOUNDS FOR TOPOLOGICAL EFFICIENCY INDEX

In this section, we obtain the bounds for ρ of the given graph G. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *complement of* G denoted by $\overline{G}$ is a simple graph on the same set of vertices of G in which two vertices u and v are adjacent in $\overline{G}$ if and only if they are nonadjacent in G.

Theorem 2.1. *For any graph G with n vertices and m edges*

$$\frac{2((n-1)n-m)}{n\underline{w}(G)}\leq\rho(G)\leq\frac{2W(G)}{2n(n-1)-n\Delta(G)}.$$

Proof. For a vertex v in G, there are $d_G(v)$ vertices which are at distance 1 from v and the remaining $n-1-d_G(v)$ vertices are at distance at least 2. Therefore

$$w_v(G)\geq d_G(v)+2(n-1-d_G(v))=2(n-1)-d_G(v).$$

By the definition of $W(G)$,

$$W(G)=\frac{1}{2}\sum_{v\in V(G)} w_v(G)\geq\frac{1}{2}\sum_{v\in V(G)}(2(n-1)-d_G(v)=\big(n-1\big)n-m.$$

Hence

$$\rho(G) = \frac{2W(G)}{n\underline{w}(G)} \geq \frac{2((n-2)n-m)}{n\underline{w}(G)}.$$

By definition of $\underline{w}(G)$, we get

$$\underline{w}(G) = \min\{w_v(G) : v \in V(G)\}$$
$$\geq 2(n-1) - \Delta(G).$$

Hence

$$\rho(G) = \frac{2W(G)}{n\underline{w}(G)} \leq \frac{2W(G)}{2n(n-1) - n\Delta(G)}.$$ ■

By Using Theorem 2.1 for the graph $\overline{G}$, we obtain the following corollary.

Corollary 2.2. *Let G be a graph with n vertices and m edges and let $\overline{G}$ be its complement. Then*

$$\rho(\overline{G}) \geq \frac{n(n-1)+2m}{n\underline{w}(\overline{G})}.$$

3. TENSOR PRODUCT

For two graphs G_1 *and* G_2, *the tensor product* denoted by $G_1 \times G_2$, has vertex set $V(G_1) \times V(G_2)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2$ is an edge in G_1 and $h_1 h_2$ is an edge in G_2. Note that if G_1 and G_2 are connected graphs, then $G_1 \times G_2$ is connected only if at least one of them is non-bipartite, see [6]. The tensor product of graph has extensively been studied in relation to the areas such as graph colorings, graph recognition, decompositions of graphs and design theory [1, 2, 6].

Theorem 3.1. *Let G be a connected graph and let λ be the number of edges not in a triangle of G. Then*

$$\underline{w}(G \times K_r) = r(G) + \delta(G) + 2(r-1) + \lambda.$$

Proof. Let $V(G) = \{u_1, u_2, ..., u_n\}$ and $V(K_r) = \{v_1, v_2, ..., v_r\}$. The notation denotes the vertex (u_i, v_j) in $G \times K_r$. First we compute the sum of the distances between a fixed vertex x_{ij} to all other vertices in $G \times K_r$. From the structure of $G \times K_r$, we have following three cases.

Case 1. The distance between x_{ij} and x_{ik} is 2. Thus, $\sum_{k=1}^{r-1} d_{G \times K_r}(x_{ij}, x_{ik}) = 2(r-1)$.

Case 2. One can observe that, $d_{G\times K_r}(x_{ij},x_{kj})=d_G(u_i,u_k)$ if $u_iu_k\notin E(G)$. Define $E_1=\{e\in E(G): e$ is in a triangle of $G\}$ and $E_2=\{e\in E(G): e$ is not in a triangle of $G\}$. If $e=u_iu_k\in E(G)$, then

$$d_{G\times Kr}(x_{ij},x_{kj})=\begin{cases}2 & \text{if } e=u_iu_k\in E_1,\\ 3 & \text{if } e=u_iu_k\in E_2.\end{cases}$$

Hence,

$$\begin{aligned}\sum\nolimits_{i,k=1,i\neq k}^{n} d_{G\times K_r(x_{ij},x_{kj})} &= \sum\nolimits_{u_iu_k\in E(G)} d_{G\times K_r}(x_{ij},x_{kj})+\sum\nolimits_{u_iu_k\notin E(G)} d_{G\times K_r}(x_{ij},x_{kj})\\ &= \sum\nolimits_{u_iu_k\in E_1} d_{G\times K_r}(x_{ij},x_{kj})+\sum\nolimits_{u_iu_k\in E_2} d_{G\times K_r}(x_{ij},x_{kj})\\ &\quad+\sum\nolimits_{u_iu_k\notin E(G)} d_{G\times K_r}(x_{ij},x_{kj})\\ &= \sum\nolimits_{u_iu_k\in E_1} 2+\sum\nolimits_{u_iu_k\in E_2} 3+\sum\nolimits_{u_iu_k\notin E(G)} d_G(u_i,u_{kj}).\end{aligned}$$

For an edge $u_i,u_k\in E_1\cup E_2$, $d_G(u_i,u_k)=1$, we have

$$\begin{aligned}\sum\nolimits_{i,k=1,i\neq k}^{n} d_{G\times K_r(x_{ij},x_{kj})} &= \sum\nolimits_{u_iu_k\in E_1}(1+d_G(u_i,u_k))+\sum\nolimits_{u_iu_k\notin E_2}(2+d_G(u_i,u_k))\\ &\quad+\sum\nolimits_{u_iu_k\notin E(G)} d_G(u_i,u_k)\\ &= \sum\nolimits_{u_iu_k\in E_1}(1)+\sum\nolimits_{u_iu_k\in E_2}(2)+w_{u_i}(G)\\ &= \sum\nolimits_{u_iu_k\in E(G)}(1)+\sum\nolimits_{u_iu_k\in E_2}(1)+w_{u_i}(G)\\ &= d_G(u_j)+\lambda+w_{u_i}(G).\end{aligned}$$

Case 3. The distance between x_{ij} to x_{kl} is $d_G(u_i,u_k)$. Thus

$$\sum\nolimits_{i,k=1,i\neq k}^{n}\sum\nolimits_{j,l=1,j\neq l}^{r} d_{G\times K_r}(x_{ij},x_{kl})=\sum\nolimits_{i,k=1,i\neq k}^{n}(r-1)d_G(u_i,u_k)=(r-1)w_{u_i}(G).$$

Combining the above three cases, we obtain

$$\begin{aligned}w_{x_{ij}}(G\times K_r) &= 2(r-1)+w_{u_i}(G)+d_G(u_i)+\lambda+(r-1)w_{u_i}(G)\\ &= rw_{u_i}(G)+d_G(u_i)+\lambda+2(r-1).\end{aligned}$$

From the definition of $\underline{w}(G)$, we obtain

$$\underline{w}(G\times K_r)=\min\{rw_{u_i}(G)+d_G(u_i)+\lambda+2(r-1)\}=r(G)+\underline{w}(G)+\lambda+2(r-1),$$

where $\delta(G)$ is the minimum degree of the graph G and λ is the number of edges not in a triangle of G. ■

Recall from [13] that the Wiener index of the tensor product of G and K_r is given by the formula $W(G\times K_r)=r^2W(G)+(m+\lambda)r+nr(r-1)$, where n, m and λ are the numbers of vertices, edges and edges not in triangle in G, respectively. Using Theorem 3.1 and $W(G\times K_r)$, we obtain the ρ value for tensor product of G and K_r.

Theorem 3.2. *Let G be a graph with m edges. If λ is a number of edges not lie on a triangle in G, then*

$$\rho(G\times K_r)=\frac{2(rW(G)+(m+\lambda)r+n(r-1))}{n(r\underline{w}(G)+\delta(G)+2(r-1))}.$$

4. STRONG PRODUCT

The strong product of two graphs G_1 *and* G_2 denoted by $G_1\otimes G_2$, is the graph with vertex set $V(G_1)\times V(G_2)$ = $\{(u, v): u\in V(G_1),\ v\in V(G_1)\}$ and $(u, x)(v,y)$ is an edge whenever (*i*) $u=v$ and $xy\in E(G_2)$, or (*ii*) $uv\in E(G_1)$ and $x=y$, or (*iii*) $uv\in E(G_1)$ and $xy\in E(G_2)$.

Theorem 4.1. *Let G be a connected graph. Then* $\underline{w}(G\otimes K_r)=r(G)+r-1$.

Proof. Let $V(G)=\{u_1,u_2,...,u_n\}$ and $V(K_r)=\{v_1,v_2,...,v_r\}$. The notation denotes the vertex (u_i,v_j) in $W(G\otimes K_r)$.

Case 1. From the structure of $W(G\otimes K_r)$, the distance between x_{ij} and x_{il} equals 1. Thus,

$$\sum_{l=1}^{r-1} d_{G\otimes K_r}(x_{ij},x_{il})=r-1.$$

Case 2. The distance between x_{ij} and x_{kj} equals $d_G(u_i,u_k)$. Then

$$\sum_{i,k=1,i\neq k}^{n} d_{G\otimes K_r}(x_{ij},x_{kj})=\sum_{u_k\in V(G)} d_G(u_i,u_k)=w_{u_i}(G).$$

Case 3. The distance between x_{ij} to x_{kl} equals $d_G(u_i,u_k)$. Thus

$$\sum_{i,k=1,i\neq k}^{n}\sum_{j,l=1,j\neq l}^{r} d_{G\otimes K_r}(x_{ij},x_{kl})=\sum_{i,k=1,i\neq k}^{n}(r-1)d_G(u_i,u_k)=(r-1)w_{u_i}(G).$$

Combining the above cases, we obtain

$$w_{x_{ij}}(G\otimes K_r)=r-1+w_{u_i}(G)+(r-1)w_{u_i}(G)=rw_{u_i}(G)+r-1.$$

By the definition of $\underline{w}(G)$, we have

$$\underline{w}(G\otimes K_r)=\min\{r-1+rw_{u_i}(G)\}=r\underline{w}(G)+r-1.$$ ■

Recall from [14] that the Wiener index of the strong product of G and K_r is given by the formula of $W(G\otimes K_r)=r^2w(G)+n(r-1)/2$, where n is the numbers of vertices of G. Using Theorem 4.1 and $W(G\otimes K_r)$, we obtain the ρ value for strong product of G and K_r.

Theorem 4.2. *Let G be a graph on n vertices. Then*

$$\rho\ (G \otimes K_r) = \frac{2[rw(G)+n(r-1)/2]}{n[r\underline{w}(G)+r-1]}.$$

5. SYMMETRIC DIFFERENCE

For a given graph G_1 *and* G_2, their *symmetric difference* $G_1 \oplus G_2$ is the graph with the vertex set $V(G_1)\times V(G_2)$ and the edge set

$$E(G_1 \oplus G_2) = \{(u,x)(v,y) \,|\, uv \in E(G_1) \text{ or } xy \in E(G_2) \text{ but not both}\}.$$

The following lemma follows easily from the structure of $G_1 \oplus G_2$.

Lemma 5.1. *Let* G_1 *and* G_2 *be two connected graphs with* n_1 *and* n_2 *vertices respectively. Then*

(*i*) $|V(G_1 \oplus G_2)| = |V(G_1)| \times |V(G_2)|$.

(*ii*) *The distance between two vertices* (u, x) *and* (v,y) *of* $G_1 \oplus G_2$ *is given by*

$$d_{G_1\oplus G_2}((u,x),(v,y)) = \begin{cases} 1 & uv \in E(G_1) \text{ or } xy \in E(G_2) \text{ but not both}, \\ 2 & \text{otherwise.} \end{cases}$$

(*iii*) *The degree of a vertex* (u, x) *in* $G_1 \oplus G_2$ *is*

$$d_{G_1\oplus G_2}(u,x) = n_2 d_{G_1}(u) + n_1 d_{G_2}(x) - 2d_{G_1}(u) d_{G_2}(x).$$

Theorem 5.2. *Let* G_1 *and* G_2 *be two connected graphs with* n_1 *and* n_2 *vertices. Then*

(*i*) $\underline{w}(G_1 \oplus G_2) = 2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + 2\,\delta(G_1)\delta(G_2)$.

(*ii*) $W(G_1 \oplus G_2) = n_1n_2(n_1n_2-1) - (m_1n_2^2 + m_2n_1^2) + 4m_1m_2$, *where* $\delta(G_1)$ *and* $\delta(G_2)$ *are the minimum degree of* G_1 *and* G_2*, respectively.*

Proof: (*i*) For any vertex (u, x) in $G_1 \oplus G_2$, there are $d_{G_1\oplus G_2}(u,x)$ vertices which are at distance 1 from (u, x) and the remaining $|V(G_1 \oplus G_2)| - 1 - d_{G_1\oplus G_2}(u,x)$ vertices are at distance 2, therefore by Lemma 5.1, we obtain

$$\begin{aligned} w_{(u,x)}(G_1 \oplus G_2) &= (n_2(u) + n_1(x) - 2(u)(x))(1) \\ &\quad + (n_1n_2 - 1 - (n_2(u) + n_1(x) - 2(u)(x)))(2) \\ &= 2n_1n_2 - 2 - n_2(u) - n_1(x) + 2(u)(x). \end{aligned}$$

By the definition of $\underline{w}(G)$, we have

$$\begin{aligned} W(G_1 \oplus G_2) &= \min\{2n_1n_2 - 2 - n_2(u) - n_1(x) + 2(u)(x)\} \\ &= 2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + 2\,\delta(G_1)\,\delta(G_2). \end{aligned}$$

(*ii*) By case (*i*), and the definition of Wiener index, we have

$$W(G_1 \oplus G_2) = \frac{1}{2}\sum_{u\in V(G_1)}\sum_{x\in V(G_2)} w_{(u,x)}(G_1 \oplus G_2)$$
$$= n_1n_2(n_1n_2 - 1) - (m_1n_2^2 + m_2n_1^2) + 4\, m_1m_2.$$ ■

Using Theorem 5.2, we obtain the ρ value for symmetric difference of G_1 and G_2.

Theorem 5.3. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1,2$. Then*

$$\rho(G_1 \oplus G_2) = \frac{2(n_1n_2(n_1n_2\text{-}1) - (m_1n_2^2 + m_2n_1^2) + 4m_1m_2)}{n_1n_2(2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + 2\delta(G_1)\delta(G_2))}.$$

6. DISJUNCTION

The *disjunction* of the graphs G_1 *and* G_2, denoted by $G_1 \vee G_2$, has the vertex set $V(G_1)\times V(G_2)$ and edge set $E(G_1 \vee G_2)$ = $\{(u, x)(v, y) \mid uv\in E(G_1)$ (or) $xy\in E(G_2)\}$. The following lemma is easily follows from the structure of $G_1 \vee G_2$.

Lemma 6.1. *Let G_1 and G_2 be two connected graphs with n_1 and n_2 vertices respectively. Then*

(*i*) $|V(G_1 \vee G_2)| = |V(G_1)| \times |V(G_2)|$.

(*ii*) *The distance between two vertices* (u,x) *and* (v,y) *of* $G_1 \vee G_2$ *is given by*

$$d_{G_1\vee G_2}((u,x),\ (v,y)) = \begin{cases} 1 & uv \in E(G_1) \text{ or } xy \in E(G_2), \\ 2 & \text{otherwise.} \end{cases}$$

(*iii*) *The degree of a vertex* (u,x) *in* $G_1 \vee G_2$ *is*

$$d_{G_1 \vee G_2}((u,x)) = n_2d_{G_1}(u) + n_1d_{G_2}(u) - d_{G_1}(u)d_{G_2}(x).$$

Theorem 6.2. *For two graphs G_1 and G_2, we have*

(*i*) $\underline{w}(G_1 \vee G_2) = 2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + \delta(G_1)\ \delta(G_2)$.

(*ii*) $W(G_1 \vee G_2) = n_1n_2(n_1n_2 - 1) - (m_1n_2^2 + m_2n_1^2) + 2m_1m_2$.

Proof. (*i*) For any vertex (u,x) in $G_1 \vee G_2$, there are $d_{G_1\vee G_2}(u,x)$ vertices which are at distance 1 and the remaining $|V(G_1 \vee G_2)| - 1 - d_{G_1\vee G_2}(u,x)$ vertices are at distance 2. By Lemma 6.1, and the definition of $w_{(u,x)}(G_1 \vee G_2)$ we obtain

$$w_{(u,x)}(G_1 \vee G_2) = (n_2(u) + n_1(x) - (u)(x))(1) + (n_1n_2 - 1 - n_2(u) + n_1(x) - 2(u))(2)$$
$$= 2n_1n_2 - 2 - n_2(u) - n_1(x) + 2(u)(x).$$

By the definition of $\underline{w}(G)$, we have

$$\underline{w}(G_1 \vee G_2) = \min\{2n_1n_2 - 2 - n_2(u) - n_1(x) + (u)(x)\}$$
$$= 2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + \delta(G_1)\,\delta(G_2).$$

(*ii*) By case (*i*), and the definition of Wiener index, we have

$$W(G_1 \vee G_2) = \frac{1}{2}\sum_{u\in V(G_1)}\sum_{x\in V(G_2)} w(G_1 \vee G_2)$$
$$= n_1n_2(n_1n_2 - 1) - (m_1n_2^2 + m_2n_1^2) + 2m_1m_2.$$

■

Using Theorem 6.2, we have the following theorem.

Theorem 6.3. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1,2$. Then*

$$\rho(G_1 \vee G_2) = \frac{2(n_1n_2(n_1n_2 - 1) - (m_1n_2^2 + m_2n_1^2) + 2m_1m_2}{n_1n_2(2n_1n_2 - 2 - n_2\delta(G_1) - n_1\delta(G_2) + \delta(G_1)\delta(G_2))}.$$

7. DOUBLE GRAPH

Let us denote the double graph of a graph G by G^*, which is constructed from two copies of G in the following manner. Let the vertex set of G be $V(G)=\{v_1, v_2, ..., v_n\}$ and the vertices of G^* are given by two sets $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. *The double graph G^** includes the initial edge set of each copies of G, and for any edge $v_iv_j \in E(G)$, two more edges x_iy_j and x_jy_i are added.

Lemma 7.1. *Let G be a connected graphs with n vertices. Then the distance between two vertices of G^* are given as follows,*

(*i*) $|V(G^*)| = 2n$ *and* $|E(G^*)| = 4n$.

(*ii*) $d_{G^*}(x_i, x_j) = d_G(x_i, x_j), \quad i, j \in \{1, 2, ..., n\}$.

(*iii*) $d_{G^*}(x_i, y_j) = d_G(x_i, y_j), \quad i, j \in \{1, 2, ..., n\}$.

(*vi*) $d_{G^*}(x_i, y_i) = 2, \quad i \in \{1, 2, ..., n\}$.

Theorem 7.2. *Let G be a connected graph with n vertices. Then*

(*i*) $\underline{w}(G^*) = 2\underline{w}(G) + 2$,

(*ii*) $W(G^*) = 2W(G) + n$.

Proof. (*i*) Using Lemma 7.1, we compute the sum of distances between a fixed vertex x_i to all other vertices in G^*.

Case 1. From the structure of G^*, the sum of distances from a fixed vertex x_i to x_j in G is $d_{G*}(x_i, x_j) = w_{xi}(G)$.

Case 2. The sum of distances from a fixed vertex x_i to y_j in G^* is $d_{G*}(x_i, y_j) = w_{xi}(G)$.

Case 3. The sum of distances from a fixed vertex x_i to y_i in G^* is $d_{G*}(x_i, y_i) = 2$.

Combining the above cases, we have $w_{xi}(G^*) = 2w_{xi}(G) + 2$. From the definition of $\underline{w}(G)$, we obtain:

$$\underline{w}(G^*) = min\{2w_{xi}(G) + 2\} = 2\underline{w}(G) + 2.$$

(*ii*) By Case (*i*), and the definition of Wiener index, we have

$$W(G^*) = \frac{1}{2}\sum_{u \in V(G)} w(G^*) = \sum_{u \in V(G)} (2\underline{w}(G) + 2) = 2W(G) + n.$$

■

Using Theorem 7.2, we obtain the ρ value for double graph of G.

Theorem 7.3. *Let G be a connected graph with n vertices. Then*

$$\rho(G^*) = \frac{2W(G) + n}{2n(\underline{w}(G) + 1)}.$$

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