Iranian J. Math. Chem. 9 (3) September (2018) 187 – 199

Iranian Journal of Mathematical Chemistry

Journal homepage:ijmc.kashanu.ac.ir

A New Family of High-Order Difference Schemes for the Solution of Second Order Boundary Value Problems

MORTEZA BISHEH–NIASAR, ABBAS SAADATMANDI• AND MOSTAFA AKRAMI–ARANI

Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, Iran

ARTICLE INFO

Article History:
Received 9 August 2017
Accepted 1 February 2018
Published online 30 August 2018
Academic Editor: Hassan Yousefi-Azari

Keywords:
Boundary value problems
Finite difference methods
Bratu's problem
Troesch's problem
High accuracy

ABSTRACT

Many problems in chemistry, nanotechnology, biology, natural science, chemical physics and engineering are modeled by two point boundary value problems. In general, analytical solution of these problems does not exist. In this paper, we propose a new class of high-order accurate methods for solving special second order nonlinear two point boundary value problems. Local truncation errors of these methods are discussed. To illustrate the potential of the new methods, we apply them for solving some well-known problems, including Troesch's problem, Bratu's problem and certain singularly perturbed problem. Bratu's and Troech's problems, may be used to model some chemical reaction-diffusion and heat transfer processes. We also compare the results of this work with some existing results in the literature and show that the new methods are efficient and applicable.

1. INTRODUCTION

The study through boundary value problem is an interesting in recent years. This interest can be attributed due to its wide range of application in scientific research. In general, nonlinear boundary value problems do not always have solutions which we can obtain using analytical methods. Therefore, techniques for rapidly computing approximate solutions of boundary value problem are very importance.

In this paper, we introduce two fast and accurate numerical schemes for the solution of second-order nonlinear differential equations of the form

$$y'' = f(x, y), \qquad a < x < b, \tag{1}$$

•Corresponding Author (Email: saadatmandi@kashanu.ac.ir)
DOI: 10.22052/ijmc.2018.94933.1306

subject to the boundary conditions:

$$y(a) = \alpha, \qquad y(b) = \beta, \tag{2}$$

where a, b, α and β are the given constants. The existence and uniqueness of the solutions to problem (1)−(2) are discussed in [1]. The literature on the numerical approximation of solutions of boundary value problems is large and still growing rapidly. Among the most recent works concerned with numerical methods, we can consider direct implicit block method [2], Chebyshev finite difference method [3], sinc collocation method [4, 5], compact finite difference method [6], non-standard finite difference method [7, 8] and rational finite difference method [9, 10]. Also, Ramos [11] presented a non-standard explicit algorithm for initial-value problems.

In this paper a new class of novel non-classical difference methods is proposed for the solution of problem (1)−(2). Our methods are based on the idea behind in [10, 11]. Two point boundary value problems (1)−(2) covers many interesting problems. Three of these important problems, which we consider in this paper, are as follows:

1.1 TROESCH'S PROBLEM

Troesch's problem is defined by

$$\begin{cases} y'' - \mu \sinh(\mu y(x)) = 0, \quad 0 \le x \le 1, \\ y(0) = 0, \quad y(1) = 1, \end{cases} \tag{3}$$

where μ is a positive constant. This problem arises in an investigation of the confinement of a plasma column under radiation pressure [12]. Also, this problem comes from the theory of gas porous electrodes [13]. Moreover, as pointed out in [14], Troesch's problems may be used to model some chemical reaction-diffusion and heat transfer processes.

The known closed-form solution of this problem in terms of the Jacobi elliptic function is (see [15])

$$y(x) = \frac{2}{\mu} \sinh^{-1}\left\{\frac{y'(0)}{2}\, \mathrm{sc}\left(\mu x \,\middle|\, 1 - \frac{1}{4} y'(0)^2\right)\right\}.$$

Here $y'(0) = 2\sqrt{1-m}$, and the constant m satisfies the transcendental equation

$$\frac{\sinh\left(\frac{\mu}{2}\right)}{\sqrt{1-m}} = \mathrm{sc}(\mu|m),$$

where, $\mathrm{sc}(\mu|m)$ is the Jacobi elliptic function. As is said in [16], this problem is inherently unstable and difficult, especially when the sensitivity parameter μ is large. Therefore, Troesch's problem has become a widely used test problem, and has been studied extensively. In the last decade, variational spline method [14], discontinuous Galerkin finite element method [17], variational iteration method [18], shooting method [19], B-spline collocation method [20], Christov collocation method [21], sinc-Galerkin method [22], nonstandard finite difference method [7], finite difference method [23] and homotopy analysis method [24] are used to solve this problem.

1.2 BRATU'S PROBLEM

The classical Bratu's problem is given as:

$$\begin{cases} y'' + \lambda \exp(y) = 0, & 0 \le x \le 1, \\ y(0) = y(1) = 0, \end{cases} \tag{4}$$

where λ is a constant. For $\lambda > 0$, the analytical solution to this problem reads [24, 25, 26, 27],

$$y(x) = -2\ln\left[\frac{\cosh\left(\left(x - \frac{1}{2}\right)\theta/2\right)}{\cosh(\theta/4)}\right], \tag{5}$$

where θ satisfies $\theta = \sqrt{2\lambda}\cosh(\theta/4)$. It is well known that, this problem has zero, one, or two solutions when $\lambda > \lambda c$, $\lambda = \lambda c$ and $\lambda < \lambda c$, respectively. Here λc, called the critical value, is given by $\lambda c = 3.513830719$ [24, 25].

The Bratu model appears in a large variety of applications such as the model of thermal reaction process, questions in geometry and relativity about the Chandrasekhar model, radiative heat transfer, nanotechnology and the fuel ignition model of the thermal combustion theory (for example, we refer the reader to see [24, 25, 26, 27, 28, 29, 30], and the references therein). Various numerical methods such as homotopy analysis method [24], Adomian decomposition method [25, 28], sinc-Galerkin method [26], B-spline method [27], pseudospectral method [29] and finite difference method [29] have been applied to this problem. Also, recently, Temimi and Ben-Romdhane [30] proposed an iterative finite difference method to solve the Bratu's problem.

1.3 SINGULARLY PERTURBED PROBLEM

We consider a class of singularly perturbed boundary value problems given in [6, 31, 32] as

$$\begin{cases} -\epsilon y''(x) + p(x)y(x) = q(x), & 0 \le x \le 1, p(x) > 0, \\ y(0) = \alpha, \quad y(1) = \beta, \end{cases} \tag{6}$$

where α, β are given constants and $\epsilon \in (0, \epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter. Further, $p(x)$ and $q(x)$ are assumed to be sufficiently continuously differentiable functions.

This type of problem occurs in many fields of science and engineering (see [6, 31, 32]). As pointed out in [32], usual numerical treatment of singular-perturbation problems gives major computational difficulties. This problem, has been studied by several researchers. Gelu et al. [6] used sixth-order compact finite difference method and Rashidinia et al. [31] employed quantic spline method. Khan et al. [32] solved this problem by sixth-order method based on sextic splines. Also, we refer the interested readers to [33, 34, 35, 36, 37].
The organization of the rest of this paper is as follows. In Section 2, the methods are described and also local truncation errors are discussed. In section 3, the numerical results of applying the methods of this paper on three test problems are presented. Finally a conclusion is drawn in Section 4.

2. THE PROPOSED METHODS

To approximate the solution of problem (1)–(2), first of all, the domain $[a,b]$ is divided into N equal subintervals of fixed mesh length $h = (b-a)/N$. The grid points are given by $x_i = a + ih, i = 0,\dots,N$, in which N is a positive integer. For convenience let $y^{(k)}(x_i) = y_i^{(k)}$, and $f^{(k)}\big(x_i, y(x_i)\big) = f_i^{(k)}, k = 0,1,2,\cdots$. Now, following the ideas in [11, 10], we suggest the following difference equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\frac{h^2}{1+g(h)}} = f_i, \tag{7}$$

equivalently,

$$(y_{i+1} - 2y_i + y_{i-1})\big(1 + g(h)\big) = h^2 f_i, \tag{8}$$

where $g(h) \neq -1$ is a sufficiently differentiable unknown function that has to be determined. Expanding $g(h)$ in Taylor's expansion about $h = 0$ and also expanding y_{i+1} and y_{i-1} on the left side of Eq. (8) in the neighborhood of x_i by Taylor's expansion, we obtain

$$\left(h^2 y_i'' + \frac{h^4}{12} y_i^{(4)} + \frac{h^6}{360} y_i^{(6)} + \cdots\right)\left(1 + g(0) + hg'(0) + \frac{h^2}{2} g''(0) + \cdots\right) = h^2 f_i. \tag{9}$$

Now, we rewrite Eq. (9) as follows

$$h^2\left[y_i''\left(1+g(0)\right)-f_i\right]+h^3[y_i''g'(0)]+h^4\left[\frac{y_i''\left(g''(0)\right)}{2}+\frac{y_i^{(4)}}{12}\left(1+g(0)\right)\right]$$
$$+h^5\left[\frac{y_i''g^{(3)}(0)}{6}+\frac{y_i^{(4)}g'(0)}{12}\right]$$
$$+h^6\left[\frac{y_i''g^{(4)}(0)}{24}+\frac{y_i^{(4)}g''(0)}{24}+\frac{y_i^{(6)}\left(1+g(0)\right)}{360}\right]+O(h^7)$$
$$=0. \tag{10}$$

In order to obtain a fourth-order scheme, the coefficients of h^2, h^3 and h^4 in Eq.(10) must be zero. So, we have

$$g(0)=0, \qquad g'(0)=0, \qquad g''(0)=-\frac{1}{6}\frac{y_i^{(4)}}{y_i''}. \tag{11}$$

By substituting the above values in the Taylor series of $g(h)$ we obtain

$$g(h)=-\frac{h^2}{12}\frac{y_i^{(4)}}{y_i''}+O(h^3). \tag{12}$$

From Eqs.(8) and (12) we get

$$(y_{i+1}-2y_i+y_{i-1})\left(1-\frac{h^2}{12}\frac{y_i^{(4)}}{y_i''}\right)-h^2f_i=0. \tag{13}$$

Therefore, using Eq. (13) and having in mind the problem (1)−(2), we obtain the numerical method given by

$$\textit{Scheme 1:}\begin{cases}(y_{i+1}-2y_i+y_{i-1})\left(1-\frac{h^2}{12}\frac{f_i^{(2)}}{f_i}\right)=h^2f_i, \quad i=1,2,\cdots,N-1,\\ y_0=\alpha,\ y_N=\beta.\end{cases} \tag{14}$$

Similarly, in order to obtain a sixth-order scheme, the coefficients of h^2, h^3, h^4, h^5 and h^6 in Eq.(10) must be zero. So, we obtain

$$\begin{aligned}&g(0)=g'(0)=g^{(3)}(0)=0,\ g''(0)=-\frac{1}{6}\frac{y_i^{(4)}}{y_i''},\\ &g^{(4)}(0)=-\frac{y_i^{(4)}g''(0)}{y_i''}+y_i^{(6)}\frac{1+g(0)}{15y_i''}.\end{aligned} \tag{15}$$

Therefore,

$$g(h) = -\frac{h^2}{12}\frac{y_i^{(4)}}{y_i''} + \frac{h^4}{y_i''}\left(\frac{1}{144}\frac{\left(y_i^{(4)}\right)^2}{y_i''} - \frac{y_i^{(6)}}{360}\right) + O(h^5). \tag{16}$$

Employing Eqs. (1), (2), (16) and (8), we obtain the numerical method given by

$$\textit{Scheme 2}: \begin{cases} (y_{i+1} - 2y_i + y_{i-1})\left(1 - \frac{h^2}{12}\frac{f_i^{(2)}}{f_i} + \frac{h^4}{f_i}\left(\frac{\left(f_i^{(2)}\right)^2}{144 f_i} - \frac{f_i^{(4)}}{360}\right)\right) = h^2 f_i, \\ i = 1,2,\cdots,N-1, \\ y_0 = \alpha,\ y_N = \beta\,. \end{cases} \tag{17}$$

2.1 LOCAL TRUNCATION ERROR

It follows from the construction of the methods in Eqs. (14) and (17) that the new *Scheme 1* and *Scheme 2* are at least of fourth-order and sixth-order respectively. In fact, for *Scheme 1*, let us define

$$LTE_i^1 = (y(x_i + h) - 2y(x_i) + y(x_i - h))\left(1 - \frac{h^2}{12}\frac{f^{(2)}(x_i, y(x_i))}{f(x_i, y(x_i))}\right) - h^2 f(x_i, y(x_i)). \tag{18}$$

After expanding each term on the right side of Eq. (18) in Taylor series about x_i and collecting terms in h we get

$$LTE_i^1 = \left(-\frac{1}{144}\frac{\left(y^{(4)}(x_i)\right)^2}{y''(x_i)} + \frac{1}{360}y^{(6)}(x_i)\right)h^6 + O(h^8). \tag{19}$$

Similarly, for *Scheme 2*, we have

$$LTE_i^2 = \left(\frac{1}{1728}\frac{\left(y^{(4)}(x_i)\right)^3}{\left(y''(x_i)\right)^2} - \frac{1}{2160}\frac{y^{(4)}(x_i)y^{(6)}(x_i)}{y''(x_i)} + \frac{1}{20160}y^{(8)}(x_i)\right)h^8 + O(h^{10}). \tag{20}$$

3. NUMERICAL RESULTS

In this section, to validate the application of the presented methods to problem (1)−(2), we consider three test problems. We have computed the numerical results by Maple programming.

Example 1. (*Troesch's problem*) In this example we will consider Troesch's problem given in Eq. (3) for different values of the parameter μ. We solved this problem, by applying the techniques described in Section 2. Taking $\mu = 0.5$ and $\mu = 1$, in Tables 1 and 2 we compare our results with the exact solutions given in [7]. Also, in Table 3 the numerical solution obtained by *Scheme 1* and *Scheme 2* for $\mu = 5$ is compared with the numerical approximation of the exact solutions given by a Fortran code [20] and the numerical solution obtained by B-spline collocation method [20]. From Tables 1–3 we see that *Scheme1* and *Scheme 2* yields a reasonable numerical solution for $\mu = 0.5, 1$ and 5. As said in [20, 23], the stiffness ratio near $x = 1$ increases as μ increases. For this reason, most common numerical methods fail to provide enough accurate solutions for large values of μ. In Table 4 the numerical solution obtained by the *Scheme 2* with $N = 300$, for $\mu = 10, 30$, is compared with the results obtained in [20] by the adaptive collocation method over a non-uniform mesh using $N = 330$ and those obtained in [23] by finite difference method (FDM) for mesh size $N = 2000$. It can be seen from Table 4 that the results obtained using Scheme 2 have a good agreement with the results obtained in [20, 23].

Table 1: Results for Troesch's problem ($\mu = 0.5$).

x	Exact	*Scheme1*		*Scheme2*	
		$N = 10$	$N = 20$	$N = 10$	$N = 20$
0.1	0.0959443493	5.0(-10)	1.0(-10)	8.0(-10)	1.0(-10)
0.2	0.1921287477	1.0(-9)	1.0(-10)	1.4(-9)	1.0(-10)
0.3	0.2887944009	1.3(-9)	1.0(-10)	2.0(-9)	0
0.4	0.3861848464	1.7(-9)	1.0(-10)	1.0(-10)	0
0.5	0.4845471647	1.8(-9)	1.0(-10)	2.7(-9)	0
0.6	0.5841332484	1.9(-9)	1.0(-10)	2.8(-9)	0
0.7	0.6852011483	1.8(-9)	1.0(-10)	2.7(-9)	1.0(-10)
0.8	0.7880165227	1.5(-9)	1.0(-10)	2.3(-9)	1.0(-10)
0.9	0.8928542161	9.0(-9)	0	1.3(-9)	0

Example 2. (*Bratu's problem*) As the second example, we consider Bratu's problem given in Eq. (4) for different values of the parameter λ. Taking $\lambda = 1, 2$, Tables 5 and 6, show the numerical solution obtained by our methods with $N = 200$ compared to the exact solution given by Eq. (5), as well as to the values computed by iterative finite difference (IFD) method with $N = 1000$ given in [30] and B-spline method given in [27]. Moreover, for the critical value $\lambda = 3.51$, in Table 7 the numerical solution obtained by the present methods with $N = 300$, is compared with the B-spline method [27] and IFD method [30]. As pointed by [30], many existing numerical methods for Bratu's problem fail to compute the solution for $\lambda = 3.51$. From Tables 5–7, we see that the present methods are in excellent agreement with the exact values and the IFD method. Also, the present methods are clearly reliable if compared with the B-spline method.

Table 2: Results for Troesch's problem ($\mu = 1$).

x	Exact	*Scheme 1*		*Scheme 2*	
		$N = 10$	$N = 20$	$N = 10$	$N = 20$
0.1	0.0846612565	2.6(-8)	1.7(-9)	4.3(-8)	2.7(-9)
0.2	0.1701713582	5.2(-8)	3.3(-9)	8.4(-8)	5.4(-9)
0.3	0.2573939080	7.6(-8)	4.7(-9)	1.2(-7)	7.8(-9)
0.4	0.3472228551	9.7(-8)	6.1(-9)	1.5(-7)	1.0(-8)
0.5	0.4405998351	1.1(-7)	7.0(-9)	1.8(-7)	1.1(-8)
0.6	0.5385343980	1.2(-7)	7.6(-9)	2.0(-7)	1.2(-8)
0.7	0.6421286091	1.2(-7)	7.5(-9)	2.0(-7)	1.2(-8)
0.8	0.7526080939	1.0(-7)	6.5(-9)	1.7(-7)	1.1(-8)
0.9	0.8713625196	6.9(-8)	4.1(-9)	1.1(-7)	7.3(-9)

Example 3. Consider the following singularly perturbed problem [6, 31]:

$$\begin{cases} -\epsilon y'' + y = x, \quad 0 \le x \le 1, \\ y(0) = 1, \quad y(1) = 1 + \exp\left(\frac{1}{\sqrt{\epsilon}}\right). \end{cases} \tag{21}$$

The exact solution of this problem is

$$y(x) = x + \exp\left(-\frac{x}{\sqrt{\epsilon}}\right). \tag{22}$$

This problem is solved in [6] by sixth-order compact finite difference method. Also, in [31] the authors used quintic spline method to solve this problem. For the purpose of comparison in Table 8, we compare maximum absolute errors of our methods, for different values of ϵ and N, together with the maximum absolute errors given in [6, 31].

Furthermore, we have calculated the computational orders of our methods (denoted by C-order) with the following formula:

$$\frac{\log(E_N) - \log(E_{2N})}{\log(2)},$$

where E_N and E_{2N} are maximum absolute errors obtained using N and $2N$ mesh intervals, respectively. The results are summarized in Tables 9 and 10. From Tables 9 and 10, we see that the computational and theoretical orders of Scheme 1 and Scheme 2 are very close to each other, i.e the order of *Scheme 1* and *Scheme 2* are $O(h^4)$ and $O(h^6)$, respectively.

4. CONCLUSION

In this paper, a new family of schemes for numerically solving two point boundary value problems is presented. We showed that, the order of *Scheme 1* and *Scheme 2* are $O(h^4)$ and $O(h^6)$, respectively. These schemes are used for solving Troesch's problem, Bratu's problem and certain singularly perturbed problem. According to the numerical results, *Scheme 1* and *Scheme 2* can handle these kind of problems effectively and the comparison show that the proposed methods are in good agreement with the existing results in the literature. Also numerical results confirm the theoretical results of the proposed techniques.

Table 3: Comparison of numerical solutions for Troesch's problem ($\mu = 5$).

x	Fortran code [20]	*Scheme 1* $N = 20$	*Scheme 2* $N = 20$	B-spline [20]
0.2	0.01075342	0.01071950	0.01070406	0.01002027
0.4	0.03320051	0.03309592	0.03304801	0.03099793
0.6	0.25821664	0.25735421	0.25695699	0.24170496
0.8	0.45506034	0.45335039	0.45258050	0.42461830

Table 4: Comparison of numerical solutions for Troesch's problem ($\mu = 10, 30$).

	$\mu = 10$			$\mu = 30$	
	Scheme 2	B-spline[20]	FDM	*Scheme 2*	FDM[23]
x	$N = 300$	$N = 330$	$N = 2000$	$N = 300$	$N = 2000$
0	0	0	0	0	0
0.1	4.204824(−5)	4.207335(−5)	4.211194(−5)	3.614375(−13)	2.500056(−13)
0.2	1.297676(−4)	1.298517(−4)	1.299642(−4)	7.277661(−12)	5.033929(−12)
0.3	3.584358(−4)	3.586905(−4)	3.589786(−4)	1.461766(−10)	1.011094(−10)
0.4	9.764246(−4)	9.771828(−4)	9.779034(−4)	2.936036(−9)	2.030831(−9)
0.5	2.655001(−3)	2.657239(−3)	2.659022(−3)	5.897186(−8)	4.079021(−8)
0.6	7.218002(−3)	7.224571(−3)	7.228934(−3)	1.184481(−6)	8.192908(−7)
0.7	1.963429(−2)	1.965351(−2)	1.966406(−2)	2.379094(−5)	1.645584(−5)
0.8	5.364813(−2)	5.370517(−2)	5.373034(−2)	4.778560(−4)	3.305241(−4)
0.9	1.518614(−1)	1.520568(−1)	1.521140(−1)	9.614584(−3)	6.644214(−3)
0.95	2.757046(−1)	2.761735(−1)		4.460814(−2)	3.026175(−2)
0.97	3.713175(−1)	3.721473(−1)		8.991531(−2)	5.753674(−2)
0.98	4.468330(−1)	4.481030(−1)		1.441330(−1)	8.223035(−2)
0.99	5.714501(−1)	5.739404(−1)		5.218877(−1)	1.269861(−1)
1	1	1	1	1	1

Table 5: Comparison of numerical solutions for Bratu's problem ($\lambda = 1$).

x	Exact	*Scheme 1*	*Scheme 2*	B-spline[27]	IDF[30]
0.1	0.049846791245	0.049846791245	0.049846791245	0.0498438103	0.049846791445
0.2	0.089189934629	0.089189934628	0.089189934629	0.0891844690	0.089189934988
0.3	0.117609095768	0.117609095767	0.117609095768	0.1176017599	0.117609096243
0.4	0.134790253884	0.134790253883	0.134790253884	0.1347817559	0.134790254431
0.5	0.140539214400	0.140539214399	0.140539214400	0.1405303221	0.140539214971
0.6	0.134790253884	0.134790253883	0.134790253884	0.1347817559	0.134790254430
0.7	0.117609095768	0.117609095767	0.117609095768	0.1176017599	0.117609096243
0.8	0.089189934629	0.089189934628	0.089189934629	0.0891844690	0.089189934988
0.9	0.049846791245	0.049846791245	0.049846791245	0.0498438103	0.049846791444

Table 6: Comparison of numerical solutions for Bratu's problem ($\lambda = 2$).

x	Exact	*Scheme 1*	*Scheme 2*	B-spline[27]	IDF[30]
0.1	0.114410743268	0.114410743264	0.114410743265	0.1143935651	0.114410743957
0.2	0.206419116488	0.206419116481	0.206419116483	0.2063865190	0.206419117764
0.3	0.273879311826	0.273879311817	0.273879311820	0.2738344125	0.273879313548
0.4	0.315089364226	0.315089364215	0.315089364220	0.3150365062	0.315089366227
0.5	0.328952421341	0.328952421330	0.328952421335	0.3288968072	0.328952423437
0.6	0.315089364226	0.315089364215	0.315089364220	0.3150365062	0.315089366228
0.7	0.273879311826	0.273879311817	0.273879311820	0.2738344125	0.273879313550
0.8	0.206419116488	0.206419116481	0.206419116483	0.2063865190	0.206419117767
0.9	0.114410743268	0.114410743264	0.114410743265	0.1143935651	0.114410743961

Table 7: Comparison of numerical solutions for Bratu's problem ($\lambda = 3.51$).

x	Exact	*Scheme 1*	*Scheme 2*	B-spline[27]	IDF[30]
0.1	0.364335803565	0.364335803086	0.364335802967	0.357388461	0.364335803565
0.2	0.677869705682	0.677869704751	0.677869704528	0.664283874	0.677869705683
0.3	0.922214197098	0.922214195783	0.922214195480	0.902930838	0.922214197097
0.4	1.078634240752	1.078634239178	1.078634238825	1.055419782	1.078634240752
0.5	1.132617978282	1.132617976616	1.132617976246	1.107989815	1.132617978283
0.6	1.078634240752	1.078634239178	1.078634238825	1.055419782	1.078634240752
0.7	0.922214197097	0.922214195783	0.922214195480	0.902930838	0.922214197097
0.8	0.677869705682	0.677869704751	0.677869704528	0.664283874	0.677869705683
0.9	0.364335803565	0.364335803086	0.364335802967	0.357388461	0.364335803565

Table 8: Comparison of maximum absolute errors for Example 3.

ϵ	N = 16	N = 32	N = 64
Scheme 1			
1/16	2.96(−6)	1.85(−7)	1.15(−8)
1/32	1.19(−5)	7.45(−7)	4.67(−8)
1/64	4.74(−5)	2.98(−6)	1.87(−7)
1/128	1.78(−4)	1.19(−5)	7.46(−7)
Scheme 2			
1/16	7.34(−9)	1.14(−10)	1.79(−12)
1/32	5.90(−8)	9.25(−10)	1.45(−11)
1/64	4.71(−7)	7.41(−9)	1.16(−10)
1/128	3.54(−6)	5.90(−8)	9.25(−10)
Method of [6]			
1/16	8.03(−9)	1.26(−10)	1.97(−12)
1/32	6.41(−8)	1.01(−9)	1.59(−11)
1/64	5.06(−7)	8.10(−9)	1.27(−10)
1/128	3.72(−6)	6.42(−8)	1.01(−9)
Method of [31]			
1/16	2.96(−6)	1.85(−7)	1.15(−8)
1/32	1.18(−5)	7.54(−7)	4.67(−8)
1/64	4.74(−5)	2.96(−6)	1.86(−7)
1/128	1.78(−4)	1.18(−5)	7.46(−7)

Table 9: Errors and computational orders obtained by *Scheme 1*, for Example 3.

	$\epsilon = 1/16$		$\epsilon = 1/32$		$\epsilon = 1/64$	
N	E_N	C-order	E_N	C-order	E_N	C-order
16	2.96(−6)	--	1.19(−5)	--	4.74(−5)	--
32	1.85(−7)	3.9999	7.45(−7)	3.9975	2.98(−6)	3.9915
64	1.15(−8)	4.0078	4.67(−8)	3.9957	1.87(−7)	3.9942
128	7.26(−10)	3.9855	2.92(−9)	3.9993	1.16(−8)	4.0108

Table 10: Errors and computational orders obtained by *Scheme 2*, for Example 3.

	$\epsilon = 1/16$		$\epsilon = 1/32$		$\epsilon = 1/64$	
N	E_N	C-order	E_N	C-order	E_N	C-order
16	7.34(−9)	--	5.90(−8)	--	4.71(−7)	--
32	1.14(−10)	6.0086	9.25(−10)	5.9951	7.41(−9)	5.9901
64	1.79(−12)	5.9929	1.45(−11)	5.9953	1.16(−10)	5.9972
128	2.80(−14)	5.9983	2.26(−13)	6.0035	1.81(−12)	6.0019

ACKNOWLEDGEMENT. The authors would like to thank the three referees for their comments and suggestions that have improved the paper.

REFERENCES

1. H. B. Keller, *Numerical Methods for Two−Points Boundary−Value Problems*, Dover, New York, 1992.
2. Z. A. Majid, N. A. Azmi, M. Suleiman, Solving second order ordinary differential equation ns using two point four step direct implicit block method, *Eur. J. Sci. Res.* **31** (2009) 29−36.
3. A. Saadatmandi, M. R. Azizi, Chebyshev finite difference method for a two-point boundary value problems with applications to chemical reactor theory, *Iranian J. Math. Chem*. **3** (2012) 1−7.
4. M. Dehghan, A. Saadatmandi, The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method, *Math. Comput. Modelling* **46** (2007) 1434−1441.
5. S Yeganeh, Y. Ordokhani, A. Saadatmandi, A sinc-collocation method for second-order boundary value problems of nonlinear integro-differential equation, *J. Inform. Comput. Sci.* **7** (2012) 151−160.
6. F. W. Gelu, G. F. Duressa, T. A. Bullo, Sixth-order compact finite difference method for singularly perturbed 1D reaction diffusion problems, *J. Taibah Univ. Sci.* **11** (2017) 302−308.

7. U. Erdugan, T. Ozis, A smart nonstandard finite difference scheme for second order nonlinear boundary value probles, *J. Comput. Phys.* **230** (2011) 6464−6474.
8. R. E. Mickens, *Advances in the Applications of Nonstandard Finite Difference Schemes*, Wiley−Interscience, Singapore, 2005.
9. P. K. Pandey, Rational finite difference approximation of high order accuracy for nonlinear two point boundary value problems, *Sains Malays* **43** (2014) 1105−1108.
10. P. K. Pandey, A non-classical finite difference method for solving two point boundary value problems, *Pac. J. Sci. Technol.* **14** (2013) 147−152.
11. H. Ramos, A non-standard explicit integration scheme for initial-value problems, *Appl. Math. Comput.* **189** (2007) 710−718.
12. E. S. Weibel, On the confinement of a plasma by magnetostatic fields, *Phys. Fluids* **2** (1959) 52−56.
13. D. Gidaspow, B. S. Baker, A model for discharge of storage batteries, *J. Electrochem. Soc.* **120** (1973) 1005−1010.
14. A. Kouibia, M. Pasadas, Z. Belhaj, A. Hananel, The variational spline method for solving Troesch’s problem, *J. Math. Chem.* **53** (2015) 868−879.
15. S. M. Roberts, J. S. Shipman, On the closed form solution of Troesch’s problem, *J. Comput. Phys*. **21** (1976) 291−304.
16. J. P. Chiou, T. Y. Na, On the solution of Troesch’s nonlinear two-point boundary value problem using an initial value method, *J. Comput. Phys.* **19** (1975) 311−316.
17. H. Temimi, A discontinuous Galerkin finite element method for solving the Troesch’s problem, *Appl. Math. Comput.* **219** (2012) 521–529.
18. S. H. Chang, A variational iteration method for solving Troesch’s problem, *J. Comput. Appl. Math.* **234** (2010) 3043−3047.
19. S. H. Chang, Numerical solution of Troesch’s problem by simple shooting method, *Appl. Math. Comput.* **216** (2010) 3303−3306.
20. S. A. Khuri, A. Sayfy, Troesch’s problem: A B-spline collocation approach, *Math. Comput. Modelling* **54** (2011) 1907−1918.
21. A. Saadatmandi, T. Abdolahi-Niasar, Numerical solution of Troesch's problem using Christov rational functions, *Comput. Methods Differ. Equ.* **3** (2015) 247−257.
22. M. Zarebnia, M. Sajjadian, The sinc-Galerkin method for solving Troesch’s problem, *Math. Comput. Modelling* **56** (2012) 218−228.

23. H. Temimi, M. Ben-Romdhane, A. R. Ansari, G. I. Shishkin, Finite difference numerical solution of Troesch's problem on a piecewise uniform Shishkin mesh, *Calcolo* **54** (2017) 225−242.
24. H. N. Hassan, M. A. El-Tawilb, An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method, *Math. Methods Appl. Sci.* **34** (2011) 977−989.
25. A. M. Wazwaz, A reliable study for extensions of the Bratu problem with boundary conditions, *Math. Methods Appl. Sci.* **35** (2012) 845−856.
26. J. Rashidinia, K. Maleknejad, N. Taheri, Sinc-Galerkin method for numerical solution of the Bratu's problems, *Numer. Algorithms* **62** (2013) 1−11.
27. H. Caglar, N. Caglar, M. Ozer, A. Valarıstos, A. N. Anagnostopoulos, B-spline method for solving Bratu's problem, *Int. J. Comput. Math.* **87** (2010) 1885−1891.
28. E. Deeba, S. A. Khuri, S. Xie, An algorithm for solving boundary value problems, *J. Comput. Phys.* **159** (2000) 125−138.
29. J. Karkowski, Numerical experiments with the Bratu equation in one, two and three dimensions, *Comput. Appl. Math.* **32** (2013) 231−244.
30. H. Temimi, M. Ben-Romdhane, An iterative finite difference method for solving Bratu's problem, *J. Comput. Appl. Math.* **292** (2016) 76−82.
31. J. Rashidinia, R. Mohammadi, S. H. Moatamedoshariati, Quintic spline methods for the solution of Singularly perturbed boundary-value problems, *Int. J. Comput. Methods Eng. Sci. Mech.* **11** (2010) 247−257.
32. A. Khan, I. Khan, T. Aziz, Sextic spline solution of a singularly perturbed boundary value problems, *Appl. Math. Comput.* 181 (2006) 432−439.
33. A. Saadatmandi, Z. Akbari, Transformed Hermite functions on a finite interval and their applications to a class of singular boundary value problems, *Comput. Appl. Math.* **36** (2017) 1085−1098.
34. A. Saadatmandi, N. Nafar, S. P. Toufighi, Numerical study on the reaction cum diffusion process in a spherical biocatalyst, *Iranian J. Math. Chem.* **5** (2014) 47−61.
35. T. Caraballo, M. Herrera-Cobos, P. Marín-Rubio, An iterative method for non-autonomous nonlocal reaction-diffusion equations, *Appl. Math. Nonlinear Sci.* **2** (2017) 73−82.
36. F. Balibrea, On problems of Topological Dynamics in non-autonomous discrete systems, *Appl. Math. Nonlinear Sci.* **1** (2016) 391−404.
37. E Babolian, A Eftekhari, A Saadatmandi, A sinc-Galerkin approximate solution of the reaction–diffusion process in an immobilized biocatalyst pellet, *MATCH Commun. Math. Comput. Chem.* **71** (2014) 681−697.