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Further Results on Betweenness Centrality of Graphs

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ABSTRACT

Betweenness centrality is a distance-based invariant of graphs. In this paper, we use link and lexicographic products to compute betweenness centrality of some important classes of graphs. Finally, we pose some open problems related to this topic.

1 INTRODUCTION

All graphs in this paper are finite and simple. A graph G is an ordered pair (V_G, E_G) consisting of a set V_G of vertices and a set E_G, disjoint from V_G, of edges, together with an incidence function f_G that associates with each edge of G an unordered pair of (not necessarily G distinct) vertices of G. A path in a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another. The distance $d_G(u, v)$ between the vertices u and v of a graph G is equal to the length of a shortest path that connects u and v.

The betweenness centrality, B_G, was first introduced by Bavelas [3] as the number of times a node acts as a bridge along the shortest path between two other nodes. In other words, for a vertex $v \in V_G$, $B_G(v) = \sum_{s\neq v\neq t\in V_G} \frac{\sigma_G^v(s,t)}{\sigma_G(s,t)}$, where $\sigma_G(s,t)$

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is total number of shortest paths from node s to node t and $\sigma_G^v(s,t)$ is the number of those paths that pass through v [7].

This invariant has important role in Psychology to study on mental illnesses. We encourage readers to see [6, 8, 9, 12 – 17] for the role of betweenness centrality in analysis of social networks, computer networks, and many other types of network data models.

The lexicographic product $G[H]$ of graphs G and H, studied first by Felix Hausdor in 1914, is the graph with vertex set $V_G \times V_H$ and (g_1, h_1) is adjacent with (g_2, h_2) whenever (g_1 is adjacent to g_2) or ($g_1 = g_2$ and h_1 is adjacent to h_2). We encourage the reader to consult the book Handbook of Product Graphs, written by Hammack, Imrich and Klavžar, for more information on results on this product.

Suppose G and H are graphs with disjoint vertex sets, $x \in V_G$ and $y \in V_H$. A link of G and H by vertices y and z is a graph operation defined as the graph $(G \sim H)(x; y)$ obtained by joining x and y by an edge in the union of these graphs, see [2, 5]. Let $V_G = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = [a_{ij}]$ is an $n \times n$ matrix for which $a_{ij}=1$ if $v_iv_j \in E_G$ and $a_{ij}=0$ otherwise [10].

The degree of a vertex v in G is denoted by $deg_G(v)$. We use $N_G[v]$ to denote the ball of radius one centered at the vertex v in G. Also, we use the notations P_n, C_n and K_n to denote the path, cycle, complete graph with n vertices, respectively. Our other notations are standard and taken mainly from the standard books of graph theory such as [4].

2. BETWEENNESS CENTRALITY UNDER LEXICOGRAPHIC AND LINK PRODUCTS

In this section, we compute the betweenness centrality of link and lexicographic products from the betweenness centrality of their initial factors.

Theorem 2.1. Let (g, h) be a vertex of $G[H]$. Then

$$B_{G[H]}((g,h)) = |V_H|B_G(g) + \frac{1}{|V_H|}\left(\binom{|V_H|}{2} - |E_H| - \sum_{1\le i<j\le |V_H|} I(a_{ij}^{(2)})\right)\sum_{gg'\in E_G}\frac{1}{deg_G(g')}$$
$$+ \sum_{g'\in N_G[g],\, d_H(h',h")=2}\frac{1}{|V_H|deg_G(g') + \sigma_H(h',h")}$$

where $a_{ij}^{(2)}$ is ij-th entry of $A^2(G)$ and $I(x) = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{otherwise} \end{cases}$.

Proof. Let (g, h), (g_1, h_1) and (g_2, h_2) be three different vertices of $G[H]$. Thus, there are four cases in which $\sigma_{G[H]}^{(g,h)}((g_1,h_1),(g_2,h_2)) \neq 0$, as follows:

1. $g_1 = g_2 = g$ and $d_H(h_1, h_2) = 2$. Then $\sigma_{G[H]}((g_1, h_1), (g_2, h_2)) = |V_H|deg_G(g) + \sigma_H(h_1, h_2)$ and $\sigma_{G[H]}^{(g,h)}\big((g_1, h_1), (g_2, h_2)\big) = 1$. Set
$$B_1 = \sum_{h_1,h_2 \in V_H, d_H(h_1,h_2)=2} \frac{1}{|V_H|deg_G(g)+\sigma_H(h_1,h_2)}.$$

2. $g_1 = g_2$, $gg_1 \in E_G$ and $d_H(h_1, h_2) = 2$. Then $\sigma_{G[H]}((g_1, h_1), (g_2, h_2)) = |V_H|deg_G(g_1) + \sigma_H(h_1, h_2)$ and $\sigma_{G[H]}^{(g,h)}\big((g_1, h_1), (g_2, h_2)\big) = 1$. Set
$$B_2 = \sum_{h_1,h_2 \in V_H, d_H(h_1,h_2)=2, gg' \in E_G} \frac{1}{|V_H|deg_G(g')+\sigma_H(h_1,h_2)}.$$

3. $g_1 = g_2$, $gg_1 \in E_G$ and $d_H(h_1, h_2) > 2$. Then $\sigma_{G[H]}((g_1, h_1), (g_2, h_2)) = |V_H|deg_G(g_1)$ and $\sigma_{G[H]}^{(g,h)}\big((g_1, h_1), (g_2, h_2)\big) = 1$. Set
$$B_3 = \sum_{h_1,h_2 \in V_H, d_H(h_1,h_2)>2, gg' \in E_G} \frac{1}{|V_H|deg_G(g')}$$
and so $B_3 = \frac{1}{|V_H|}\left(\binom{|V_H|}{2} - |E_H| - \sum_{1 \le i < j \le |V_H|} I(a_{ij}^{(2)})\right)\sum_{gg' \in E_G} \frac{1}{deg_G(g')}$.

4. $g_1 \neq g \neq g_2$ and $d_G(g_1, g_2) \geq 2$. Then
$\sigma_{G[H]}((g_1, h_1), (g_2, h_2)) = \sigma_G(g_1, g_2)|V_H|^{d_G(g_1,g_2)-1}$,
$\sigma_{G[H]}^{(g,h)}((g_1, h_1), (g_2, h_2)) = \sigma_G^g(g_1, g_2)\,|V_H|^{d_G(g_1,g_2)-2}$.
Set $B_4 = \sum_{\{h_1,h_2\} \subseteq V_H, d_G(g',g'') \geq 2} \frac{\sigma_G^g(g',g'')|V_H|^{d_G(g',g'')-2}}{\sigma_G(g',g'')|V_H|^{d_G(g',g'')-1}}$ and so $B_4 = |V_H|B_G(g)$.

Therefore, by summation of B_1, B_2, B_3 and B_4 , the result can be proved. □

Corollary 2.1. If (g, h) is a vertex of $G[C_n]$ and $n > 4$, then
$$B_{G[C_n]}\big((g,h)\big) = nB_G(g) + \frac{n-5}{2}\sum_{gg' \in E_G} \frac{1}{deg_G(g')} + n\sum_{g' \in N_G[g]} \frac{1}{ndeg_G(g')+1}.$$
Also, if G is a k-regular graph, we have
$$B_{G[C_n]}\big((g,h)\big) = nB_G(g) + \frac{n(k+1)}{nk+1} + \frac{n-5}{2}.$$

Corollary 2.2. Let (g, h) be a vertex of $G[C_4]$, then
$$B_{G[C_4]}\big((g,h)\big) = 4B_G(g) + 2\sum_{g' \in N_G[g]} \frac{1}{4\deg(g')+2}.$$
Moreover, if G is a k-regular graph, then
$$B_{G[C_4]}\big((g,h)\big) = 4B_G(g) + \frac{k+1}{2k+1}.$$

Corollary 2.3. If (g, h) is a vertex of $G[C_3]$, then $B_{G[C_3]}\big((g,h)\big) = 3B_G(g)$.

Theorem 2.2. Let G and H be graphs with disjoint vertex sets, $x \in V_G$ and $y \in V_H$. Then

$$B_{(G\sim H)(x;y)}(u) = \begin{cases} B_G(u) + |V_H| \sum\limits_{t\in V_G} \frac{\sigma_G^u(t,x)}{\sigma_G(t,x)} & if\ u \in V_G \\ B_H(u) + |V_G| \sum\limits_{t\in V_H} \frac{\sigma_H^u(t,y)}{\sigma_H(t,y)} & if\ u \in V_H \end{cases}.$$

Proof. Supose *u*, *s* and *t* are three different vertices of $(G \sim H)(x;y)$. There are two cases as follow:

1. $u \in V_G$. In this case, if $s,t \in V_G$, then

$$\sigma_{(G\sim H)(x;y)}(s,t) = \sigma_G(s,t) \quad \text{and } \sigma^u_{(G\sim H)(x;y)}(s,t) = \sigma^u_G(s,t)$$

and if $s \in V_G$ and $t \in V_H$, then

$$\sigma_{(G\sim H)(x;y)}(s,t) = \sigma_G(s,x)\sigma_H(y,t) \quad \text{and } \sigma^u_{(G\sim H)(x;y)}(s,t) = \sigma^u_G(s,x)\sigma_H(y,t).$$

Note that if $s,t \in V_H$, then $\sigma^u_{(G\sim H)(x;y)}(s,t) = 0$. Therefore,

$$B_{(G\sim H)(x;y)}(u) = B_G(u) + |V_H| \sum_{s\in V_G} \frac{\sigma_G^u(t,x)}{\sigma_G(t,x)}.$$

2. $u \in V_H$. Using a similar argument applied in the first case, we have

$$B_{(G\sim H)(x;y)}(u) = B_H(u) + |V_G| \sum_{t\in V_H} \frac{\sigma_H^u(t,y)}{\sigma_H(t,y)},$$

which completes our proof. □

3. APPLICATIONS

In this section, we apply our results to compute the betweenness centrality of some well-known graphs.

Example 3.1. Consider the Catlin graph $C_5[C_3]$ shown in Figure 1. Then $\sum_{1\le i\le j\le 3} I\left(a_{ij}^{(2)}\right) = 0$. On the other hand, by [20], $B_{C_n}(v) = \begin{cases} \frac{1}{8}(n-2)^2 & 2|n \\ \frac{1}{8}(n-1)(n-3) & 2\nmid n \end{cases}$. Therefore, by Corollary 1.1, we have $B_{C_5[C_3]}((g,h)) = 3$.

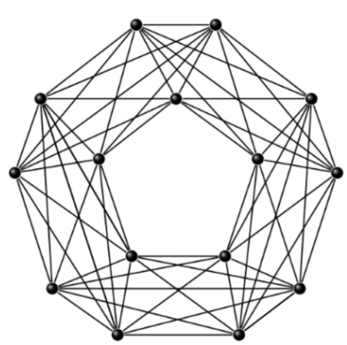

Figure 1. The Catlin graph.

Example 3.2. Let G be the closed fence graph shown in Figure 2. It is clear that the lexicographic product of C_n and P_2 is isomorphic to G. Then, by Theorem 1, we have

$$B_G(v) = B_{C_n[P_2]}(v) = \begin{cases} \frac{1}{4}(n-2)^2 & 2|n \\ \frac{1}{4}(n-1)(n-3) & 2 \nmid n \end{cases}.$$

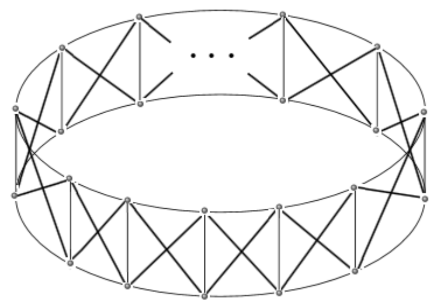

Figure 2. Closed fence graph.

Example 3.3. Let G be the open fence graph depicted in Figure 3. It is not difficult to check that $G \cong P_n[P_2]$ and $B_{P_n}(v_1) = (i-1)(n-i)$. Then, by Theorem 1, we have

$$B_G\left((g_i, h_j)\right) = B_{P_n[P_2]}\left((g_i, h_j)\right) = 2(n-i)(i-1).$$

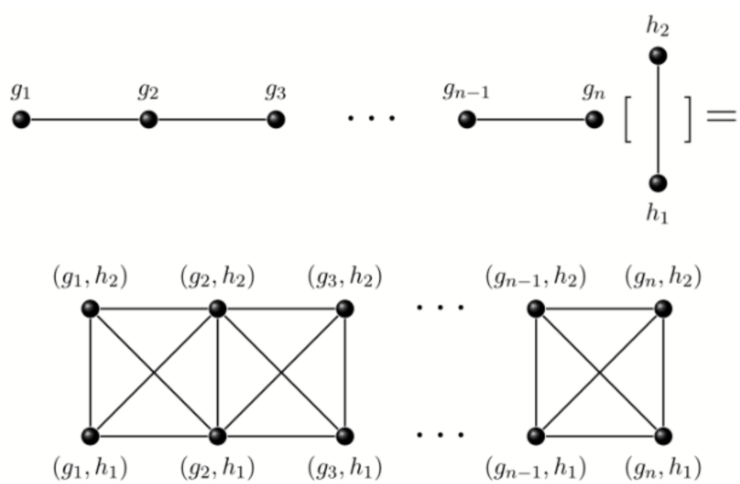

Figure 3. Open fence graph.

The Wiener index, W, is equal to the sum of the lengths of the shortest paths between all pairs of vertices. Kumar and Balakrishnan [11] gave the following relation between the Wiener index and the betweenness centrality index for a graph G:

$$W(G) = \sum_{v \in V_G} B_G(v) + \binom{|V_G|}{2}.$$

Thus, we can use betweenness centrality instade of Wiener index. Therefore, if $B(v) = B(u)$ for each $u,v \in V_G$, then $B_G(v) = \frac{W(G)-\binom{|V_G|}{2}}{|V_G|}$. For example, Since $B_{C_n}(v) = B_{C_n}(u)$ for each $u,v \in V_{C_n}$, then $B_{C_n}(v) = \frac{W(C_n)-\binom{n}{2}}{n}$.

Example 3.4. Consider the dendrimer D_1 shown in Figure 4. As one can see in this figure, $D_1 = (G \sim H)(x; y)$. On the other hand, if u is the vertex of G shown in Figure 4, it is not difficult to check that $B_G(u) = 2$ and $\sum_{s \in V_G} \frac{\sigma_G^u(t,x)}{\sigma_G(t,x)} = 0$. Therefore, by Theorem 2, we have $B_{D_1}(u) = B_{(G\sim H)(x;y)}(u) = 2$. Also, by the previous argument,

$$W(D_1)=\sum_{u \in V_{D_1}} B_{D_1}(u) + \binom{|V_{D_1}|}{2}.$$

Using a similar argument, $B_{D_n}(u) = 2$, where u is the vertex of D_n shown in Figure 4.

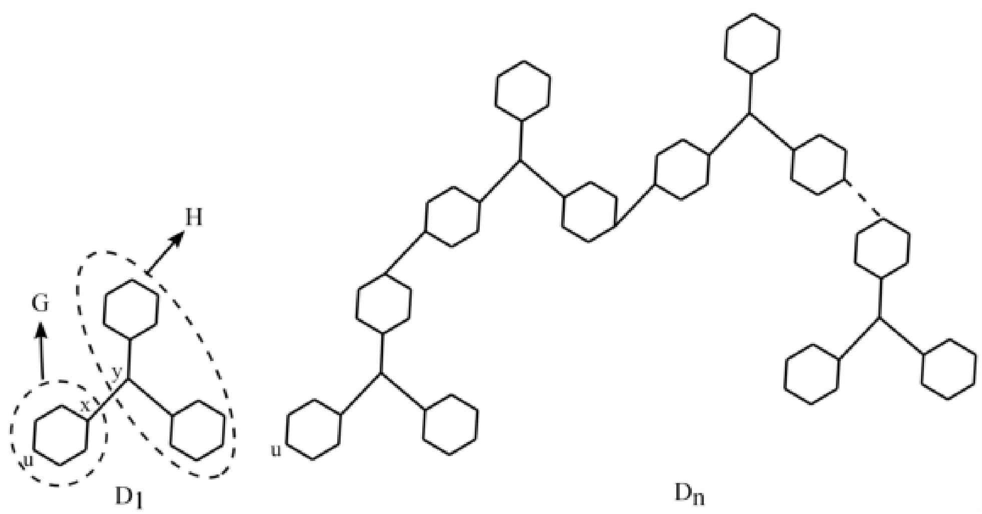

Figure 4. Dendrimers D_1 and D_n.

Example 3.5. A k-almost tree is a graph in which each biconnected component is obtained by adding at most k edges to a tree. Akutsu and Nagamochi [1] studied these graphs as an example of chemical graphs.

Consider graph G, graph H and the almost tree Γ shown in Figure 5. As one can see, $\Gamma = (G\sim H)(x;y)$. Then, by Theorem 2 and this fact that $B_G(u)=\frac{1}{2}$, we have

$$B_\Gamma(u)=B_{(G\sim H)(x;y)}(u)=\frac{1}{2}.$$

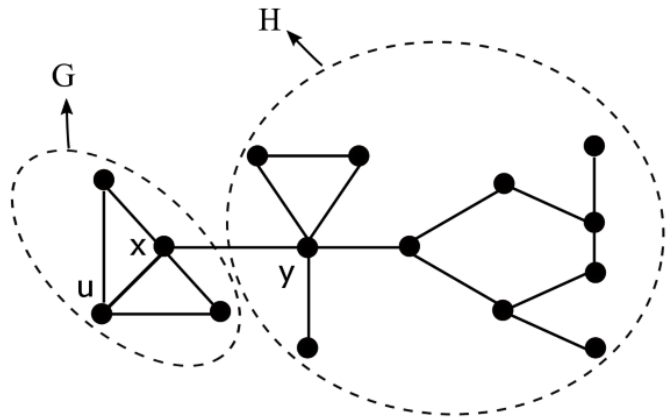

Figure 5. The almost tree Γ.

Example 3.6. For handcuffs graph $C_n \sim C_m$, we have

$$B_{(C_n \sim C_m)(x;y)}(u) = \begin{cases} \frac{1}{8}(n-2)^2 + m \sum\limits_{t \in V_{C_n}} \frac{\sigma^u_{C_n}(t,x)}{\sigma_{C_n}(t,x)} & if\ u \in V_{C_n}\ \&\ 2|n \\ \frac{1}{8}(n-1)(n-3) + m \sum\limits_{t \in V_{C_n}} \frac{\sigma^u_{C_n}(t,y)}{\sigma_{C_n}(t,x)} & if\ u \in V_{C_n}\ \&\ 2 \nmid n \\ \frac{1}{8}(m-2)^2 + n \sum\limits_{t \in V_{C_m}} \frac{\sigma^u_{C_m}(t,x)}{\sigma_{C_m}(t,x)} & if\ u \in V_{C_m}\ \&\ 2|m \\ \frac{1}{8}(m-1)(m-3) + n \sum\limits_{t \in V_{C_m}} \frac{\sigma^u_{C_m}(t,y)}{\sigma_{C_m}(t,x)} & if\ u \in V_{C_m}\ \&\ 2 \nmid m \end{cases}$$

4. OPEN PROBLEMS

In this section, we pose two open problems to develop the topic of betweenness centrality on other graph operations. The tensor product $G \otimes H$ of graphs G and H is the graph with vertex set $V_G \times V_H$ and (g_1, h_1) is adjacent with (g_2, h_2) whenever (g_1 is adjacent to g_2) and (h_1 is adjacent to h_2), see [10, 18] for details. The strong product $G \oplus H$ of graphs G and H is the graph with vertex set $V_G \times V_H$ and (g_1, h_1) is adjacent with (g_2, h_2) whenever (g_1 is adjacent to g_2 and $h_1 = h_2$) or (h_1 is adjacent to h_2 and $g_1 = g_2$) or (g_1 is adjacent to g_2 and h_1 is adjacent to h_2), see [10, 19].

We end this paper by the following two open questions:

1. Let G and H be two graphs and (g, h) be a vertex of $G \otimes H$. What is the value of $B_{G \otimes H}((g,h))$?

2. Let G and H be two graphs and (g, h) be a vertex of $G \oplus H$. What is the value of $B_{G\oplus H}((g,h))$?

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