Laplacian Energy of a Fuzzy Graph

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ABSTRACT. Let $\tilde{G}$ be a fuzzy graph with n vertices and m edges. If $\{\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n\}$ is the Laplacian spectrum of $\tilde{G}$ then the Laplacian energy of $\tilde{G}$ has been recently defined as $LE(\tilde{G}) = |\,\tilde{\mu}_i - \frac{2\sum_{1\le i<j\le n}\mu(u_i,u_j)}{n}|$. In this paper, some bounds on Laplacian energy of fuzzy graphs are given.

Keywords: Fuzzy graph, fuzzy Laplacian matrix, Laplacian spectrum, Laplacian energy of fuzzy graph.

1. INTRODUCTION

Many kinds of matrices are associated with a graph. The spectrum of one such matrix, adjacency matrix is called the spectrum of the graph. The properties of the spectrum of a graph are related to the properties of the graph. The area of graph theory that deals with this is called spectral graph theory. The spectrum of a graph first appeared in a paper by Collatz and Sinogowitz in 1957 [3]. At present, it is widely studied owing to its applications in physics, chemistry, computer science and other branches of mathematics. In chemistry, it has applications in the theory of unsaturated conjugated molecular hydrocarbons called Hückel molecular orbital theory. Graph spectrum appears in problems in statistical physics and in combinatorial optimization problems in mathematics. Spectrum of a graph also plays an important role in pattern recognition, modelling virus propagation in computer networks and in securing personal data in databases. A concept related to the spectrum of a graph is that of energy. The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. The Laplacian spectrum of the graph G, consisting of the numbers $\{\mu_1, \ldots, \mu_n\}$, is the spectrum of its Laplacian matrix L of G. The

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Received : December 22, 2013; Accepted: February 12, 2014.

Laplacian energy of a graph G is equal to the sum of distances of the Laplacian eigenvalues of G and the average degree $d(G)$ of G. The Laplacian energy $LE(G)$ and the ordinary energy $E(G)$ were found [6] to have a number of analogous properties, but also some noteworthy differences between them have been recognized [6]. Rosenfeld in 1975 considered fuzzy relations on fuzzy sets [11] and developed the theory of fuzzy graphs, and then some basic fuzzy graph theoretic concepts and applications have been indicated, many authors found deeper results and fuzzy analogues of many other graph theoretic concepts. The energy of a fuzzy graph and some bounds on energy of fuzzy graphs are studied in [1,4].

In this paper we introduce the concept of Laplacian energy of fuzzy graphs. Section 2 consists of preliminaries and definition of Laplacian energy of a fuzzy graph and in Section 3, we present some results on Laplacian energy of a fuzzy graph.

2. PRELIMINARIES

Let G be a simple graph possessing n vertices and m edges. Energy of a simple graph $G = (V, E)$ with adjacency matrix A is defined as the sum of absolute values of eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of A [2]. This quantity introduced by Ivan Gutman, has noteworthy chemical applications. For details see [7,8]. It is denoted by $E(G)$ as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The Laplacian energy $LE(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ of G and the average degree $d(G)$ of G. This concept has been investigated in [6,12] by

$$LE(G) = \sum_{i=1}^{n} |\gamma_i|,$$

where;

$$\begin{cases} \gamma_i = \mu_i - \dfrac{2m}{n}, \\ \displaystyle\sum_{i=1}^{n} \mu_i = 2m, \\ \displaystyle\sum_{i=1}^{n} {\mu_i}^2 = 2m + \sum_{i=1}^{n} d^2(v_i). \end{cases}$$

The analogous relations for the Laplacian eigenvalues read as given by

$$\begin{cases} \sum_{i=1}^{n} \gamma_i = 0, \\ \sum_{i=1}^{n} {\gamma_i}^2 = 2M, \end{cases}$$

where;

$$M = m + \frac{1}{2}\sum_{i=1}^{n}(d(v_i) - \frac{2m}{n})^2.$$

with $d(v_i)$ denoting the degree of the *i*–th vertex of *G*. From [6,12], we have the following bounds for $LE(G)$ as

$$\begin{cases} LE(G) \leq \sqrt{2Mn}, \\ LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2M - (\frac{2m}{n})^2]}, \\ 2\sqrt{M} \leq LE(G) \leq \sqrt{2M}, \end{cases}$$

Definition 2. 1. (See [10]) A fuzzy graph with V as the underlying set is a pair of functions $\tilde{G} = (\sigma, \mu)$ where $\sigma: V \to [0,1]$ is a fuzzy subset and $\mu: V \times V \to [0,1]$ is a symmetric fuzzy relation on the fuzzy subset σ where for all $u, v \in V$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. The underlying crisp graph of $\tilde{G} = (\sigma, \mu)$ is denoted by $G^* = (V, E)$ where $E \subseteq V \times V$.

A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix $M = [m_{ij}]$, *where* $m_{ij} = \mu(u_i, u_j)$ [9].

Throughout this paper, we suppose $\tilde{G}$ is undirect, without loops and $\sigma(u) = 1$, for each $u \in V$.

Definition 2. 2. (See [10]) Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with *n* vertices and *m* edges. The adjacency matrix of $\tilde{G} = (\sigma, \mu)$ is a square matrix of order *n* whose $\tilde{A}(\tilde{G})=[\tilde{a}_{ij}]$, where $\tilde{a}_{ij} = \mu(u_i, u_j)$ entry is as the strength of relation between the vertices u_i and u_j.

Example 2. 3. Suppose $\tilde{G}_1$ is a fuzzy graph depicted in Figure 1. Adjacency matrix of the fuzzy graph $\tilde{G}_1$ is

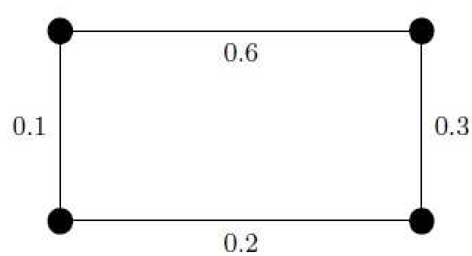

Figure 1. A Fuzzy Graph $\tilde{G}_1$.

$$\tilde{A} = \begin{bmatrix} 0 & 0.6 & 0 & 0.1 \\ 0.6 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0.2 \\ 0.1 & 0 & 0.2 & 0 \end{bmatrix}.$$

Definition 2. 4. (See [9]) Let u be a vertex of the fuzzy graph $\tilde{G} = (\sigma, \mu)$. The degree of u is defined as

$$d_{\tilde{G}}(u) = \sum_{uv \in E(\tilde{G})} \mu(uv).$$

Definition 2. 5. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph on $G^* = (V, E)$. If $d_{\tilde{G}}(v) = k$ for all $v \in V$, (i.e. if each vertex has same degree k), then $\tilde{G}$ is said to be a regular fuzzy graph of degree k or a k–regular fuzzy graph [5].

Definition 2. 6. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with n vertices and m edges. The degree matrix of $\tilde{G} = (\sigma, \mu)$ is a square matrix of order n whose $D(\tilde{G})=[\tilde{d}_{ij}]$ and we have:

$$\tilde{d}_{ij} := \begin{cases} d_{\tilde{G}}(v_i), & if\ i = j \\ 0, & O.W \end{cases}$$

Theorem 2.7. (See [1,4]) Suppose $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n\}$ are the eigenvalues of the fuzzy adjacency matrix $\tilde{A}(\tilde{G})$ with $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_n$; then

a) $\sum_{i=1}^{n} \tilde{\lambda}_i = 0$,
b) $\sum_{i=1}^{n} \tilde{\lambda}_i^{\,2} = 2\sum_{1 \leq i < j \leq n} m_{ij}^{\,2}$.

Proof: a) The diagonal is zero since there are no loops. Since $\tilde{A}$ is a symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero.

b) Since $(\tilde{a}_{i,i})^2$ is the degree vertex u_i and it's equal to the total degree of each vertex in $\tilde{G}$. Using the trace properties of matrix, we have

$$\sum_{i=1}^{n} d(u_i) = tr(\tilde{A}_{i,i}^{\,2}) = \sum_{i=1}^{n} \tilde{\lambda}_i^{\,2},$$

where,

$$\begin{aligned} tr\left(\tilde{A}_{i,i}^{\,2}\right) = {} & (0 + \mu^2(u_1, u_2) + \cdots + \mu^2(u_1, u_n)) \\ & + (\mu^2(u_2, u_1) + 0 + \cdots + \mu^2(u_2, u_n)) \\ & \vdots \\ & + (\mu^2(u_n, u_1) + \mu^2(u_n, u_2) + \cdots + 0) = 2\sum_{1 \leq i < j \leq n} m_{ij}^{\,2}, \end{aligned} \tag{2.1}$$

proving the result. □

Example 2. 8. Let $\tilde{G}_2$ be a fuzzy graph with $|V| = n$ vertices. The adjacency matrix of $\tilde{G}_2$ in Figure 2 is,

$$\widetilde{A} = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0 & 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.3 & 0 \end{bmatrix}.$$

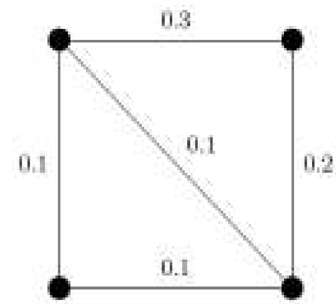

Figure 2. Fuzzy Graph $\tilde{G}_2$.

The spectrum of $\tilde{G}_2$ is {–0.34, –0.1, 0, 0.44}. We also have

$$\begin{cases} \sum_{i=1}^{4} \tilde{\lambda}_i = -0.34 + \ -0.1 + 0 + \ 0.44 = 0, \\ \sum_{i=1}^{n} \tilde{\lambda}_i{}^2 = 0.1156 + 0.01 + 0 + 0.1936 = 2 \sum_{1 \le i < j \le n} m^2{}_{ij} = 0.32. \end{cases}$$

Definition 2. 9. Let $A(\tilde{G})$ be a adjacency matrix and $D(\tilde{G})$ = $[d_{ij}]$ be a degree matrix of $\tilde{G}$=(σ, μ). The matrix $L(\widetilde{G})$=$D(\widetilde{G})-A(\widetilde{G})$ is defined as fuzzy Laplacian matrix of $\widetilde{G}$.

Example 2. 10. The Laplacian matrix of the fuzzy graph $\tilde{G}_2$ is

$$\widetilde{L} = \begin{bmatrix} 0.2 & -0.1 & 0 & -0.1 \\ -0.1 & 0.4 & -0.2 & -0.1 \\ 0 & -0.2 & 0.5 & -0.3 \\ -0.1 & -0.1 & -0.3 & 0.5 \end{bmatrix}.$$

Definition 2. 11. Let $L(\tilde{G})$ be a fuzzy Laplacian matrix of $\tilde{G}=(\sigma,\mu)$. The Laplacian polynomial of G is the characteristic polynomial of its Laplacian matrix, $\Phi(\tilde{G},\tilde{\mu}) = det(\tilde{\mu}I_n - L(\tilde{G}))$. The roots of $\Phi(\tilde{G},\tilde{\mu})$ is the fuzzy Laplacian eigenvalues of $\tilde{G}$.

Theorem 2. 12. Let $\tilde{G} = (\sigma,\mu)$ be a fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ is the Laplacian eigenvalues of the fuzzy graph $\tilde{G} = (\sigma,\mu)$, then

a) $\sum_{i=1}^{n} \tilde{\mu}_i = 2\sum_{1\leq i<j\leq n} m_{ij}$,

b) $\sum_{i=1}^{n} \tilde{\mu}_i^{\,2} = 2\sum_{1\leq i<j\leq n} m^2{}_{ij} + \sum_{i=1}^{n} d^2{}_{\tilde{G}}(u_i)$.

Proof. a) Since $\tilde{L}$ is a symmetric matrix and these Laplacian eigenvalues are nonnegative such that

$$tr(\tilde{L}) = \sum_{i=1}^{n} d_{\tilde{G}}(u_i) = 2\sum_{1\leq i<j\leq n} m_{ij}. \tag{2.2}$$

b) According to definition of Laplacian matrix, we have:

$$\begin{bmatrix} d_{\tilde{G}}(u_1) & \cdots & -\mu(u_1,u_n) \\ \vdots & \ddots & \vdots \\ -\mu(u_n,u_1) & \cdots & d_{\tilde{G}}(u_n) \end{bmatrix}.$$

Then we obtain,

$$tr\left(\tilde{L}_{i,i}{}^2\right) = \sum_{i=1}^{n} \tilde{\mu}_i^{\,2}, \tag{2.3}$$

where

$$\begin{aligned} tr\left(\tilde{L}_{i,i}{}^2\right) &= (d^2{}_{\tilde{G}}(u_1) + \mu^2(u_1,u_2) + \cdots + \mu^2(u_1,u_n)) \\ &\quad + (\mu^2(u_2,u_1) + d^2{}_{\tilde{G}}(u_2) + \cdots + \mu^2(u_2,u_n)) \\ &\quad \vdots \\ &\quad + (\mu^2(u_n,u_1) + \mu^2(u_n,u_2) + \cdots + d^2{}_{\tilde{G}}(u_n)) \\ &= 2\sum_{1\leq i<j\leq n} m^2{}_{ij} + \sum_{i=1}^{n} d^2{}_{\tilde{G}}(u_i), \end{aligned} \tag{2.4}$$

proving the result. □

Corollary 2. 13. Let $\tilde{G} = (\sigma,\mu)$ be a fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ is the Laplacian eigenvalues of fuzzy graph $\tilde{G} = (\sigma,\mu)$, where $\tilde{\gamma}_i = \tilde{\mu}_i - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}}$ then we obtain

a) $\sum_{i=1}^{n} \tilde{\gamma}_i = 0$,

b) $\sum_{i=1}^{n} \tilde{\gamma}_i^{\,2} = 2M$,

where $M = \sum_{1\leq i<j\leq n} m^2{}_{ij} + \frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2$.

Definition 2. 14. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with |V| = n vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ be the Laplacian eigenvalues of $\tilde{G} = (\sigma, \mu)$. The Laplacian energy of fuzzy graph $\tilde{G} = (\sigma, \mu)$ is defined as

$$LE(\tilde{G}) = |\ \tilde{\mu}_i - \frac{2\sum_{1\leq i<j\leq n}\mu(u_i,u_j)}{n}|.$$

Example 2. 15. For the graph depicted in Figure 3, the Laplacian spectrum of $\tilde{G}$ is

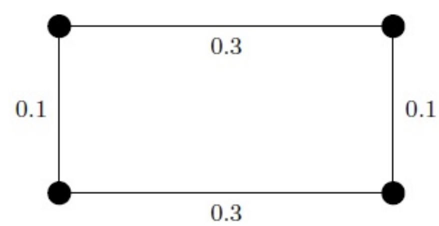

Figure 3. Fuzzy Graph $\tilde{G}$**.**

$\{\tilde{\mu}_1 = 0.8, \tilde{\mu}_2 = 0.6, \tilde{\mu}_3 = 0.2, \tilde{\mu}_4 = 0\}$. We also have

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3. RESULTS

Theorem 3. 1. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ is the Laplacian eigenvalues of fuzzy graph $\tilde{G} = (\sigma, \mu)$; then

$$LE(\tilde{G}) \leq \sqrt{2\sum_{1\leq i<j\leq n} {m^2}_{ij} + \sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2)n}\ .$$

Proof. Apply Cauchy–Schwarz inequality to (1,…,1) and $(|\tilde{\gamma}_1|, |\tilde{\gamma}_2|, \ldots, |\tilde{\gamma}_n|)$, we get

$$|\sum_{i=1}^{n}\tilde{\gamma}_i|^2 \leq n\sum_{i=1}^{n}|\tilde{\gamma}_i|^2,$$

where

$$LE(\tilde{G}) \leq \sqrt{n\ \sum_{i=1}^{n}|\tilde{\gamma}_i|^2} = \sqrt{2\ M\ n}\ .$$

Since $M = \sum_{1\leq i<j\leq n} {m^2}_{ij} + \frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{G}}(u_i)\ -\ \frac{2\ \sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2$, we have

$$LE(\tilde{G}) \leq \sqrt{2\sum_{1\leq i<j\leq n} {m^2}_{ij} + \sum_{i=1}^{n}(d_{\tilde{G}}(u_i)\ -\ \frac{2\ \sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2)n}\ .$$

Hence the result. □

Theorem 3. 2. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ be the Laplacian eigenvalues of $\tilde{G} = (\sigma, \mu)$. Then

$$LE(\tilde{G}) \geq 2\sqrt{\sum_{1 \leq i < j \leq n} m^2{}_{ij} + \frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2}\,.$$

Proof. From Definition 2.14, we have:

$$(LE(\tilde{G}))^2 = (\sum_{i=1}^{n} |\tilde{\gamma}_i|)^2 = \sum_{i=1}^{n} |\tilde{\gamma}_i|^2 + 2\sum_{1\leq i<j\leq n} |\tilde{\gamma}_i||\tilde{\gamma}_j| \geq 4M,$$

we get

$$LE(\tilde{G}) \geq 2\sqrt{\sum_{1 \leq i < j \leq n} m^2{}_{ij} + \frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2}\,.$$

Hence the result. □

Theorem 3. 3. Let $\tilde{G} = (\sigma, \mu)$ be a fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ be the Laplacian eigenvalues of fuzzy graph $\tilde{G} = (\sigma, \mu)$; then

$$LE(\tilde{G}) \leq \tilde{\gamma}_1 + \sqrt{(n-1)(2\sum_{1\leq i<j\leq n} m^2{}_{ij} + \sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2 - \tilde{\gamma}_1{}^2)}\,.$$

Proof. Apply Cauchy–Schwarz inequality to (1,…,1) and $(|\tilde{\gamma}_2|, |\tilde{\gamma}_3|, \ldots, |\tilde{\gamma}_n|)$, we get

$$(|\sum_{i=2}^{n} \tilde{\gamma}_i|)^2 \leq (|\tilde{\gamma}_2|^2 + \cdots + |\tilde{\gamma}_n|^2)(n-1),$$

$$LE(\tilde{G}) - \tilde{\gamma}_1 \leq \sqrt{(\mathrm{n}-1)(2\mathrm{M} - \tilde{\gamma}_1{}^2)}\,,$$

Since $M = \sum_{1\leq i<j\leq n} m^2{}_{ij} + \frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2$, thus

$$LE(\tilde{G}) \leq \tilde{\gamma}_1 + \sqrt{(n-1)(2\sum_{1\leq i<j\leq n} m^2{}_{ij} + \sum_{i=1}^{n}(d_{\tilde{G}}(u_i) - \frac{2\sum_{1\leq i<j\leq n} m_{ij}}{\boldsymbol{n}})^2 - \tilde{\gamma}_1{}^2)}\,. \tag{3.1}$$

This completes the proof. □

Corollary 3. 4. Let $\tilde{G} = (\sigma, \mu)$ be a *k*-regular fuzzy graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_n$ be the Laplacian eigenvalues of fuzzy graph $\tilde{G} = (\sigma, \mu)$; then

$$LE(\tilde{G}) \leq \tilde{\gamma}_1 + \sqrt{(n-1)(2\sum_{1\leq i<j\leq n} m^2{}_{ij} - \tilde{\gamma}_1{}^2)}\,.$$

Proof. Since $\tilde{G}$ is a k–regular fuzzy graph and

$$k = d_{\tilde{G}}(u_i) = \frac{2\ \sum_{1\le i<j\le n} m_{ij}}{\boldsymbol{n}}\,, \tag{3.2}$$

Substituting (3.2) in (3.1) we have

$$LE(\tilde{G}) \le\ \tilde{\gamma}_1 + \sqrt{(n-1)(\ 2\ \sum_{1\le i<j\le n} m^2{}_{ij}\ \ -\ \tilde{\gamma}_1{}^2)}\,. \tag{3.3}$$

This proves our result. □

4. Conclusion

The Laplacian matrix and energy for a fuzzy graph are defined. Some results on Laplacian energy bounds extended to fuzzy graphs. Further study on energy and the Laplacian spectra of fuzzy graphs may reveal more analogous results of these kinds and will be discussed in the forthcoming papers.

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