Electro–Spunorganic Nanofibers Elaboration Process Investigations using BPs Operational Matrices

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ABSTRACT In this paper operational matrix of Bernstein Polynomials (BPs) is used to solve Bratu equation. This nonlinear equation appears in the particular elecotrospun nanofibers fabrication process framework. Elecotrospun organic nanofibers have been used for a large variety of filtration applications such as in non-wovens and filtration industries. By using operational matrix of integration and multiplication the investigated equations are turned into set of algebraic equations. Numerical solutions show both accuracy and simplicity of the suggested approach.

KEYWORDS Bratu equation • Elecotrospun nanofibers • Bernstein polynomials • Operational matrix.

1. INTRODUCTION

Electrospinning has been recognized as one of the most convenient, direct and economical methods for the fabrication of polymer nanofibers. Various polymers have been successfully electrospun into ultrafine fibers in recent years mostly in solvent solution and some in melt form. Electrospinning is a process for elaborating nanofibers with diameters about $20nm$ by forcing a fluidified polymer through a spinneret by an electric field. The elements required for electrospinning include a polymer source, a high-voltage supply (HV), and a collector (as shown in Fig. 1) [4]. Through several different collection methods, this process yields nonwoven, nanoporous materials. The basis of electrospinning is derived from a large change in electric potential. Many electrospinning device were designed in vibration-electrospinning [14, 9], magneto-electrospinning [18], bubble-electrospinning [12, 10].

In this paper, a mathematical model of the electrospinning process has been associated to Bratu equation through thermo-electro-hydrodynamics balance equations. This model is considered in terms of fluid velocity at the level of the outer edge of the syringue. It has been showed that the problem can be expressed through second-order nonlinear ordinary differential Bratue quation:

$$u''(x) + \lambda e^{u(x)} = 0, \;\; 0 < x < 1, \qquad \lambda \text{ is constant} \tag{1}$$

with initial conditions $u(0) = b_0 = 0$ and $u'(0) = b_1 = 0$ will be investigated.

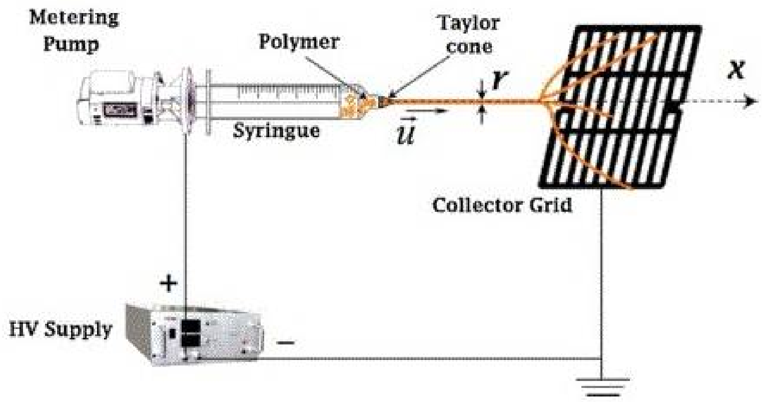

Figure 1. Electrospinning process setup.

Colantoni and Boubaker established a model which is the monodimensional Bratu equation as following [4]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \lambda e^u = 0, \\ with\!: \lambda = \dfrac{18\, E^2(I - r^2 kE)^2}{\rho^2 r^4}. \end{cases}$$

where ρ is material density, $r$is is the radius of the jet atxial coordinate x (Fig. 1), $I$is the electrical current intensity, $k$is a constant which depend only on temperature in the case of an in compressible and E is electric field in the axial direction.

The approximation and numerical techniques are utilized to solve this equation. Some of these methods were B–spline method [3], Chebyshev wavelets method [16], Adomian decomposition method [15], Variational iteration method [1, 8] and other method [6,7,13].

In this study, we will generalize the operational matrix for fractional integration and multiplication within Bernstein Polynomials. Bernstein polynomials (B-polynomials) have many useful properties. They play a prominent role in various areas of mathematics and have frequently been used in the solution of integral equations, differential equations and approximation theory; see e.g., [5, 17]. The core of this approach is to convert the given problem into a system of algebraic equations. This transformation is possible by expanding the unspecified function within Bernstein Polynomials. The speed of the computation increases. To evaluate the unknown coefficients which appear in this approach, we utilized the operational matrix of integral and multiple.

Now we are ready to present the organization of our wok: In Section 2, some properties of Bernstein polynomials is presented. The operational matrix is computed for integration and produc in section 3. The suggested approach is used to approximate the Bratu equation in the next section. After that we apply the proposed technique to Bratu equation in section 5. A conclusion part in Section 6 closed the manuscript.

2. Bernstein Polynomials and Their Properties

2.1 Definition of Bernstein Polynomials

The Bernstein polynomials of the mth degree on the interval $[0, 1]$ are defined as [2]:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad 0 \le x \le m. \tag{2}$$

The following Bernstein polynomials satisfy recursive definition:

$$B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x), \qquad i = 0,1,\cdots,m. \tag{3}$$

It can easily be shown that each of the Bernstein polynomials is positive and also the sum of all the Bernstein polynomials is unity for all real $x \in [0,1]$, i.e., $\sum_{i=0}^{m} B_{i,m}(x) = 1$. By using the binomial expansion of $(1-x)^{m-i}$, Bernstein polynomials can be show in terms of linear combination of the basis functions

$$\begin{aligned} B_{i,m}(x) &= \binom{m}{i} x^i (1-x)^{m-i} = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} x^k\right) \\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k}, \qquad i = 0,1,\cdots,m. \end{aligned} \tag{4}$$

We can show the Bernstein polynomials by $B_{i,m}(x) = A_{i+1} T_m(x)$, for $i = 0, 1, \cdots, m$, where

$$A_{i+1} = \left[\overbrace{0,0,\cdots,0}^{i\ times}, (-1)^0 \binom{m}{i}, (-1)^1 \binom{m}{i}\binom{m-i}{1}, \cdots, (-1)^{m-i} \binom{m}{i}\binom{m-i}{m-i}\right],$$

and

$$T_m(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^m \end{bmatrix}.$$

Now if we define $(m+1) \times (m+1)$ matrix A such that

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m+1} \end{bmatrix},$$

then we have $\phi(x) = AT_m(x)$, where $\phi(x) = [B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)]^T$ and A is an upper triangular matrix given by:

$$A = \begin{bmatrix} (1)^0\binom{m}{0} & (1)^1\binom{m}{0}\binom{m-0}{1-0} & \cdots & (1)^{m-0}\binom{m}{0}\binom{m-0}{m-0} \\ 0 & (1)^0\binom{m}{i} & \cdots & (1)^{m-i}\binom{m}{i}\binom{m-i}{m-i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (1)^0\binom{m}{m} \end{bmatrix},$$

and $|A| = \prod_{i=0}^{m} m\binom{m}{i}$, so A is an invertible matrix.

2.2 APPROXIMATION OF FUNCTION

The set of Bernstein polynomials$\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$ in Hilbert space $L^2[0,1]$ is a complete basis [11]. Therefore, any polynomial of degree $m$can be expanded in terms of linear combination of $B_{i,m}$:

$$f(x) = \sum_{i=0}^{m} c_i B_{i,m} = C^T \phi, \tag{5}$$

where $\phi^T = [B_{0,m}, B_{1,m}, \cdots, B_{m,m}]$ and $C^T = [c_0, c_1, \cdots, c_m]$. Then C^Tcan be obtained by

$$C^T \langle \phi, \phi \rangle = \langle f, \phi \rangle, \tag{6}$$

where

$$\langle f, \phi \rangle = \int_0^1 f(x)\phi(x)^T dx = [\langle f, B_{0,m} \rangle, \langle f, B_{1,m} \rangle, \cdots, \langle f, B_{m,m} \rangle], \tag{7}$$

and $\langle \phi, \phi \rangle$ is called dual matrix of ϕ which is showed by Q, and the Q is obtained as:

$$Q = \langle \phi, \phi \rangle = \int_0^1 \phi(x)\phi(x)^T dx, \tag{8}$$

and then

$$C^T = \left(\int_0^1 f(x)\phi(x)^T dx \right) Q^{-1}. \tag{9}$$

The elements of the dual matrix, Q, are easily computed by

$$\begin{aligned}
(Q)_{i+1,j+1} &= \int_0^1 B_{i,m}(x)\, B_{j,m}(x) dx \\
&= \binom{n}{i}\binom{n}{j} \int_0^1 (1-x)^{2n-(i+j)} x^{i+j}\, dx \\
&= \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}, \qquad i, j = 0, 1, \cdots, m.
\end{aligned}$$

3. OPERATIONAL MATRIX OF BERNSTEIN POLYNOMIALS

3.1 THE OPERATIONAL MATRIX OF INTEGRAL

In this section, we describe breifley operational matrix for the Riemann-Liouville integral on the basis of BPs from order m as[17]:

$$\int_0^x \phi(t) dt \simeq P\phi(x), \tag{10}$$

by substituting $\phi(x) = AT_m(x)$ in Eq. (10) we get:

$$\begin{aligned}
\int_0^x \phi(t) dt &= A \int_0^x T_m(t)\, dt = A\left[\int_0^x 1\, dt, \int_0^x t\, dt, \cdots, \int_0^x t^m\, dt\right]^T \\
&= A\left[x, \frac{x^2}{2}, \cdots, \frac{x^{m+1}}{m+1}\right]^T = AD\overline{T}_m,
\end{aligned} \tag{11}$$

where D is an $(m+1)\times(m+1)$ matrix given by

$$D=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \frac{1}{2} & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & \frac{1}{m+1}\end{bmatrix},$$

and

$$\overline{T}_m=\begin{bmatrix}x\\ x^2\\ \vdots\\ x^{1+m}\end{bmatrix}.$$

Now we approximate x^{i+1} by $m+1$ terms of the Bernstein basis:

$$x^{i+1}\simeq E_i^T\phi(x). \tag{12}$$

Therefore we have

$$\begin{aligned}E_i &= Q^{-1}\left(\int_0^1 x^{i+1}\phi_m(x)dx\right)\\ &= Q^{-1}\left[\int_0^1 x^{i+1}B_{0,m}(x)dx,\int_0^1 x^{i+1}B_{1,m}(x)dx,\cdots,\int_0^1 x^{i+1}B_{m,m}(x)dx,\right]^T\\ &= Q^{-1}\overline{E}_i.\end{aligned} \tag{13}$$

where $\overline{E}_i=\left[\overline{E}_{i,0},\overline{E}_{i,1},\cdots,\overline{E}_{i,m}\right]$ and

$$\overline{E}_{i,j}=\int_0^1 x^{i+1}B_{i,j}(x)dx=\frac{m!\Gamma(i+j+2)}{j!\Gamma(i+m+3)},\quad i,j=0,1,\cdots,m, \tag{14}$$

where E is an $(m+1)\times(m+1)$ matrix that has vector $Q^{-1}\overline{E}_i$ for ith columns. Therefore, we can write

$$P\phi(x)=AD[E_0^T\phi(x),E_1^T\phi(x),\cdots,E_m^T\phi(x)]^T=ADE^T\phi(x). \tag{15}$$

Finally, we obtain

$$\int_0^1\phi(t)\,dt\simeq P\phi(x), \tag{16}$$

where

$$P=ADE, \tag{17}$$

is called the Bernstein polynomials operational matrix of fractional integration.

3.2 B–Polynomials Operational Matrix of Product

It is always necessary to evaluate the product of $\phi(x)$ and $\phi(x)^T$, which is called the product matrix for the Bernstein polynomials basis. The operational matrices for the product $\hat{C}$ is given by

$$C^T\phi(x)\phi(x)^T\simeq\phi(x)^T\hat{C}, \tag{18}$$

where $\hat{C}$ is an $(m+1)\times(m+1)$ matrix. So we have

$$\begin{aligned}C^T\phi(x)\phi(x)^T &= C^T\phi(x)(T_m(x)^TA^T)=\left[C^T\phi(x),x\left(C^T\phi(x)\right),\cdots,x^m\left(C^T\phi(x)\right)\right]A^T\\ &= \left[\sum_{i=0}^m c_i\,B_{i,m},\sum_{i=0}^m c_i x\,B_{i,m},\cdots,\sum_{i=0}^m c_i\,x^m B_{i,m}\right]\end{aligned} \tag{19}$$

Now, we approximate all functions $x^k B_{i,m}(x)$ in terms of $\{B_{i,m}(x)\}_{i=0}^{m}$ for $i,k = 0,1,\cdots,m$. By (5), we have

$$x^m B_{i,m} \simeq e_{k,i}^T \phi_m(x) \tag{20}$$

that $e_{k,i} = \left[e_{k,i}^0, e_{k,i}^1, \cdots, e_{k,i}^m\right]^T$, then we obtain the components of the vector of $e_{k,i}$

$$\begin{aligned} e_{k,i} &= Q^{-1}\left(\int_0^1 x^k B_{i,m}(x)\phi(x)dx\right) \\ &= Q^{-1}\left[\int_0^1 x^k B_{i,m}(x)B_{0,m}(x)dx, \int_0^1 x^k B_{i,m}(x)B_{1,m}(x)dx, \cdots, \int_0^1 x^k B_{i,m}(x)B_{m,m}(x)dx\right]^T \\ &= \frac{Q^{-1}}{2m+k+1}\left[\frac{\binom{m}{0}}{\binom{2m+k}{i+k}}, \frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}}, \cdots, \frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}}\right]^T, \qquad i,k = 0,1,\cdots,m. \end{aligned} \tag{21}$$

Thus we obtain finally

$$\begin{aligned} \sum_{i=0}^{m} c_i x^k B_{i,m}(x) &= \sum_{i=0}^{m} c_i\left(\sum_{j=0}^{m} e_{k,i}^j B_{j,m}(x)\right) = \sum_{j=0}^{m} B_{j,m}(x)\left(\sum_{i=0}^{m} c_i e_{k,i}^j\right) \\ &= \phi(x)^T\left[\sum_{i=0}^{m} c_i e_{k,i}^0, \sum_{i=0}^{m} c_i e_{k,i}^1, \cdots, \sum_{i=0}^{m} c_i e_{k,i}^m\right]^T \\ &= \phi(x)^T\left[e_{k,0}, e_{k,1}, \cdots, e_{k,m}\right]C = \phi(x)^T V_{k+1} C, \end{aligned} \tag{22}$$

where $V_{k+1}(k = 0,1,\cdots,m)$ is an $(m+1)\times(m+1)$ matrix that has vectors $e_{k,i}(i = 0,1,\cdots,m)$ given, for each columns. If we choose an $(m+1)\times(m+1)$ matrix $\bar{C} = [V_1 c, V_2 c, \cdots, V_{m+1} c\ ]$, from (19) and (22) we can write:

$$C^T\phi(x)\phi(x)^T \simeq \phi(x)^T \bar{C} A^T \tag{23}$$

and therefore we obtain the operational matrix of product, $\hat{C} = A^T$.

4. SOLUTION OF BRATU EQUATION

Consider Bratu equation given in (1). We first approximate derivative by the Bernstein basis ϕ as follows:

$$u''(x) = C^T\phi(x) \tag{24}$$

where

$$C^T = [c_0, c_1, \cdots, c_m], \tag{25}$$

$$\phi^T = \left[B_{0,m}, B_{1,m}, \cdots, B_{m,m}\right], \tag{26}$$

are unknowns. Using initial conditions $u(x)$ can be represented as

$$u(x) = C^T P^2 \phi = G^T \phi, \tag{27}$$

where $C^T P^2 = G^T$ and P is the operational matrix of integration. Here we use the Taylor expansion of the strongly nonlinear term as:

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!}$$

Also using (5) and (23) we approximate constant functions 1 and nonlinear terms by the Bernstein basis as:

$$1 = d^T \phi, \tag{28}$$

$$u^2(x) = G^T \phi \phi^T G = \phi^T \hat{G} G, \tag{29}$$

$$u^3(x) = \phi^T \hat{G}^2 G, \tag{30}$$

$$u^4(x) = \phi^T \hat{G}^3 G, \tag{31}$$

Now, by substituting (27) and (28)–(31), into (1) we have

$$\phi^T C = \lambda \left(\phi^T d + \phi^T G + \tfrac{1}{2} \phi^T \hat{G} G + \tfrac{1}{3!} \phi^T \hat{G}^2 G + \tfrac{1}{4!} \phi^T \hat{G}^3 G \right) \tag{32}$$

or

$$\phi^T - \left(C - \lambda \left(\phi^T d + \phi^T G + \tfrac{1}{2} \phi^T \hat{G} G + \tfrac{1}{3!} \phi^T \hat{G}^2 G + \tfrac{1}{4!} \phi^T \hat{G}^3 G \right) \right) = 0. \tag{33}$$

Finally, we obtain the following linear system of algebraic equations:

$$\left(C - \lambda \left(\phi^T d + \phi^T G + \tfrac{1}{2} \phi^T \hat{G} G + \tfrac{1}{3!} \phi^T \hat{G}^2 G + \tfrac{1}{4!} \phi^T \hat{G}^3 G \right) \right) = 0, \tag{34}$$

that by solving this system we can obtain the vector C. Consequently determine the approximate value of $u(x)$ can be calculated from (27).

5. Illustrative Example

Below we use the presented approach in order to solve a Bratu equation.

Example. *Consider the second-order initial value problem [1,3,15]*

$$u''(x) - \lambda e^{u(x)} = 0, \qquad 0 < x < 1, \tag{35}$$

subject to the initial condition $u(0) = u'(0) = 0$. The exact solution is $u(x) = 2\ln(\cos(x))$. By applying the technique described in Section 4, in Figure 2 the exact solution together with the approximate solutions $u(x)$ show for different values of $m = 6, 8, 12$ and $\lambda = 2$. The approximate values of $u(x)$ converge to the exact solutions with increase in the number of the Bernstein basis. In Table 1, the obtained results of BPs with $m = 12$ and methods in [4] are showed.

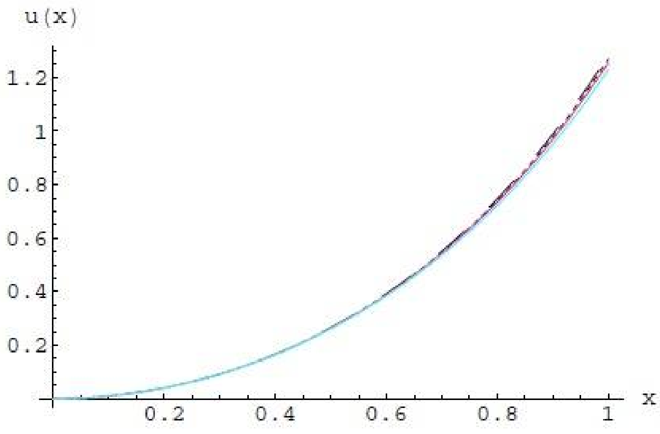

Figure 2. The exact solution: (blue line) and when $\lambda = 2$ approximation solutions for $m = 12$ (red line), $m = 8$ (dotted) and $m = 6$ (Long–dashed).

Table 1. Solution of Bratu equation.

x	Exact	BPEs	EVIM	BPs
0.03448	0.00118911	0.00118	0.00117	0.00118912
0.10345	0.010721	0.01061	0.0105	0.0107219
0.17241	0.0298737	0.02958	0.02929	0.0298804
0.24138	0.058839	0.05825	0.05766	0.0588668
0.31034	0.097897	0.09692	0.09592	0.0979798
0.37931	0.147465	0.14689	0.14632	0.147662
0.44828	0.20807	0.20599	0.20391	0.208484
0.51724	0.280393	0.27761	0.27483	0.281178
0.58621	0.365339	0.36178	0.35822	0.366712
0.65517	0.464004	0.45943	0.45485	0.466255
0.72414	0.577847	0.57211	0.56638	0.581339
0.79313	0.708731	0.70165	0.69462	0.713882
0.86207	0.858899	0.85038	0.84186	0.866119
0.93103	1.03165	1.02144	1.01122	1.04121
1	1.23125	1.21906	1.20687	1.24298

6. CONCLUSION

In this work we have performed an accurate and efficient approachbased using the Bernstein polynomials for solving the second-order initial value problems of Bratu–type. The Bernstein polynomials operational matrixes of integration and multiplication are used to reduce the problem to the solution of nonlinear algebraic equations. Illustrative example are presented to demonstrate the applicability and validity of the approach. We used Mathematica for computations.

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