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Research Paper

On General Degree-Eccentricity Index for Trees with Fixed Diameter and Number of Pendant Vertices

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Abstract

The general degree-eccentricity index of a graph G is defined by, $DEI_{a,b}(G) = \sum_{v \in V(G)} d_G^a(v) ecc_G^b(v)$ for $a, b \in \mathbb{R}$, where $V(G)$ is the vertex set of G, $ecc_G(v)$ is the eccentricity of a vertex v and $d_G(v)$ is the degree of v in G.
In this paper, we generalize results on the general eccentric connectivity index for trees. We present upper and lower bounds on the general degree-eccentricity index for trees of given order and diameter and trees of given order and number of pendant vertices. The upper bounds hold for $a > 1$ and $b \in \mathbb{R} \setminus \{0\}$ and the lower bounds hold for $0 < a < 1$ and $b \in \mathbb{R} \setminus \{0\}$. We include the case $a = 1$ and $b \in \{-1, 1\}$ in those theorems for which the proof of that case is not complicated. We present all the extremal graphs, which means that our bounds are best possible.

1 Introduction

A topological index is a numerical value that associates a chemical structure with various physical properties, chemical reactivity, or biological activity of molecules.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Order of G is defined as the number of vertices in G. Degree of a vertex $v \in V(G)$, denoted by $d_G(v)$ is the number of vertices adjacent to the vertex v. A Pendant vertex is a vertex of degree one, and a pendant edge is an edge incident with a pendant vertex. A branching vertex is a vertex of degree at least three. A pendant path is a path in which one of the end vertices is a pendant vertex and the other is branching, and all the internal vertices (if exist) have degree two. For $U \subset V(G)$ and $F \subset E(G)$, the graph obtained from G by deleting all the vertices in U (resp. edges in F is denoted by $G - U$ (res. $G - F$). On the other hand, for a pair of non-adjacent vertices u and v, in G, the graph obtained from G by adding an edge uv is denoted by $G + uv$.

For $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$ is the length of a shortest (u, v)-path in G. The eccentricity of a vertex v in G, denoted by $ec_G(v)$ is the maximum

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distance from v to any other vertex. The diameter of a graph G is the length of the shortest path between the most distanced vertices. It measures the extent of a graph and the topological length between two vertices. A tree is a connected and acyclic graph. We denote, the path and star graphs of order n by P_n and S_n, respectively. For a connected graph G and $a, b \in \mathbb{R}$, the general degree-eccentricity index of G is defined as

$$DEI_{a,b}(G) = \sum_{v \in V(G)} d_G^a(v) ec_G^b(v).$$

Several well-studied eccentricity-based topological indices are special cases of this general index. For instance, $DEI_{a,1}(G)$ is the general eccentric connectivity index, $DEI_{1,1}(G)$ is the classical eccentric connectivity index, $DEI_{1,-1}(G)$ is the connective eccentricity index, $DEI_{0,1}(G)$ is the total eccentricity index and $DEI_{0,2}(G)$ is the first Zagreb eccentricity index of G.

The mathematical properties of eccentricity-based topological indices have been extensively studied due to their wide range of applications. In [1–3], the authors studied the eccentric connectivity index for trees with a given order, order and diameter, and order and number of pendant vertices. In [4, 5], the authors studied the connective eccentricity index for graphs of given order and clique number, and order and matching number respectively.

Zagreb eccentricity indices have been investigated extensively. In [6–8], the authors studied Zagreb eccentricity indices for trees and general graphs, with prescribed domination number or bipartition. In [9], sharp upper and lower bounds on the general eccentric connectivity index of trees with prescribed order and diameter/number of pendant vertices were given. In [10], the authors introduced the general degree-eccentricity index of a graph, they also determined the general degree-eccentricity index for connected graphs of a given order in combination with given vertex connectivity, edge connectivity, number of pendant vertices, number of bridges or matching number. In [11], the same authors studied the general degree-eccentricity index for trees of a given order, and trees of a given order in combination with a given matching number, independence number, domination number or bipartition.

In this paper, motivated by the works in [10, 11], we continue to study the general degree-eccentricity index. We generalize results on the general eccentric connectivity index for trees which were presented in [9].

The main contribution of this paper is the characterization of trees with a given order and diameter, as well as trees with a given order and number of pendant vertices, that have the maximum and minimum general degree-eccentricity index. We study the upper bounds for $a > 1$ and $b \in \mathbb{R} \setminus \{0\}$ and the lower bounds for $0 < a < 1$ and $b \in \mathbb{R} \setminus \{0\}$. In some of the results, we include the case $a = 1$ and $b \in \{-1, 1\}$. We also show that all our bounds are sharp by presenting extremal graphs. Lemma 1.1 was proved in [9], and it plays an important role in the proof of our main results.

Lemma 1.1. *Let $1 \leq x < y$ and $c > 0$. For $a > 1$ or $a < 0$, we have*

$$(x + c)^a - x^a < (y + c)^a - y^a.$$

If $0 < a < 1$, then

$$(x + c)^a - x^a > (y + c)^a - y^a.$$

2 Trees of given order and diameter

In this section, we characterize trees of prescribed order n and diameter d having the maximum and the minimum $DEI_{a,b}$ index.

Let $\mathfrak{T}_{n,d}$ be the set of all n-vertex trees with diameter d. Clearly, $\mathfrak{T}_{n,2} = \{S_n\}$ and $\mathfrak{T}_{n,n-1} = \{P_n\}$. So in what follows, we consider $\mathfrak{T}_{n,d}$ for $3 \leq d \leq n-2$.

For a positive integer d, let $V_{n,d}$ be a tree obtained from a path $P_{d+1} : x_0x_1\dots x_d$ by attaching $n-d-1$ pendant vertices to vertex $x_{\lfloor \frac{d}{2} \rfloor}$. We prove that $V_{n,d}$ is the unique tree having the maximum $DEI_{a,b}$ index for $a \geq 1$ and $b < 0$, and the minimum $DEI_{a,b}$ index for $0 < a \leq 1$ and $b > 0$, in the class $\mathfrak{T}_{n,d}$.

Theorem 2.1. *Let $T \in \mathfrak{T}_{n,d}$. Then for $a \geq 1$ and $b < 0$, we have*

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-1} r^b + (n-d+1)^a \left(\frac{d}{2}\right)^b + (n-d-1)\left(\frac{d+2}{2}\right)^b + 2d^b,$$

if d is even,

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+3}{2}}^{d-1} r^b + ((n-d+1)^a + 2^a)\left(\frac{d+1}{2}\right)^b + (n-d-1)\left(\frac{d+3}{2}\right)^b + 2d^b,$$

if d is odd, with equalities if and only if T is $V_{n,d}$.

Proof. Let T_{max} be a tree with maximum $DEI_{a,b}$ index in the class $\mathfrak{T}_{n,d}$. We show that T_{max} is $V_{n,d}$.

Assume, to the contrary, that there exists a tree $G'_{n,d}$ of order n and diameter d with $DEI_{a,b}(G'_{n,d}) > DEI_{a,b}(V_{n,d})$ for some n and d. Let $G'_{n^*,d}$ be a tree having the minimum possible number of vertices n^* satisfying $DEI_{a,b}(G'_{n^*,d}) > DEI_{a,b}(V_{n^*,d})$.

Let $P : x_0x_1x_2\dots x_d$ be a diametral path in $G'_{n^*,d}$. Since $G'_{n^*,d}$ is not a path graph, it has a pendant vertex, say v other than x_0 and x_d. Let u be the neighbor of v. Let $V(G'_{n^*-1,d}) = V(G'_{n^*,d}) \setminus \{v\}$ and $E(G'_{n^*-1,d}) = E(G'_{n^*,d}) \setminus \{uv\}$. Then $G'_{n^*-1,d} \in \mathfrak{T}_{n-1,d}$.

Let $d_{G'_{n^*,d}}(u) = r$. Then we have $2 \leq r \leq n^* - d + 1$, $d_{G'_{n^*-1,d}}(u) = r-1$, $d_{G'_{n^*,d}}(v) = 1$ and $d_{G'_{n^*-1,d}}(y) = d_{G'_{n^*,d}}(y)$ for all $y \in V(G'_{n^*,d}) \setminus \{u,v\}$. Moreover, we have $ec_{G'_{n^*,d}}(v) \geq \lceil \frac{d}{2} \rceil + 1$, $ec_{G'_{n^*-1,d}}(u) = ec_{G'_{n^*,d}}(u) \geq \lceil \frac{d}{2} \rceil$ and $ec_{G'_{n^*-1,d}}(y) = ec_{G'_{n^*,d}}(y)$ for $y \in V(G'_{n^*,d}) \setminus \{u,v\}$. This implies that $ec^b_{G'_{n^*,d}}(v) \leq \left(\lceil \frac{d}{2} \rceil + 1\right)^b$, $ec^b_{G'_{n^*-1,d}}(u) = ec^b_{G'_{n^*,d}}(u) \leq \lceil \frac{d}{2} \rceil^b$ for $b < 0$. Thus

$$\begin{aligned} DEI_{a,b}(G'_{n^*,d}) - DEI_{a,b}(G'_{n^*-1,d}) &= d^a_{G'_{n^*,d}}(v) ec^b_{G'_{n^*,d}}(v) \\ &\quad + d^a_{G'_{n^*,d}}(u) ec^b_{G'_{n^*,d}}(u) - d^a_{G'_{n^*-1,d}}(u) ec^b_{G'_{n^*-1,d}}(u) \\ &= ec^b_{G'_{n^*,d}}(v) + r^a ec^b_{G'_{n^*,d}}(u) - (r-1)^a ec^b_{G'_{n^*,d}}(u) \\ &\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1\right)^b + \left\lceil \frac{d}{2} \right\rceil^b (r^a - (r-1)^a). \end{aligned}$$

Since $G'_{n^*,d}$ is a tree having the minimum possible number of vertices n^* such that $DEI_{a,b}(G'_{n^*,d}) > DEI_{a,b}(V_{n^*,d})$, we obtain $DEI_{a,b}(G'_{n^*-1,d}) \leq DEI_{a,b}(V_{n^*-1,d})$ which implies that

$$\begin{aligned} DEI_{a,b}(G'_{n^*,d}) - DEI_{a,b}(G'_{n^*-1,d}) &> DEI_{a,b}(V_{n^*,d}) - DEI_{a,b}(V_{n^*-1,d}) \\ &= \left(\left\lceil \frac{d}{2} \right\rceil + 1\right)^b + \left\lceil \frac{d}{2} \right\rceil^b ((n^* - d + 1)^a - (n^* - d)^a). \end{aligned}$$

This implies that,

$$\left(\left\lceil\frac{d}{2}\right\rceil+1\right)^b+\left\lceil\frac{d}{2}\right\rceil^b(r^a-(r-1)^a)>\left(\left\lceil\frac{d}{2}\right\rceil+1\right)^b+\left\lceil\frac{d}{2}\right\rceil^b((n^*-d+1)^a-(n^*-d)^a),$$

and hence

$$r^a-(r-1)^a>(n^*-d+1)^a-(n^*-d)^a,$$

which is a contradiction, since $2\le r\le n^*-d+1$ and $a\ge 1$ (see Lemma 1.1). Thus T_{max} is $V_{n,d}$. By simple calculation, we have

$$DEI_{a,b}(V_{n,d})=2^{a+1}\sum_{r=\frac{d+2}{2}}^{d-1}r^b+(n-d+1)^a\left(\frac{d}{2}\right)^b+(n-d-1)\left(\frac{d+2}{2}\right)^b+2d^b,$$

for even d, and

$$DEI_{a,b}(V_{n,d})=2^{a+1}\sum_{r=\frac{d+3}{2}}^{d-1}r^b+((n-d+1)^a+2^a)\left(\frac{d+1}{2}\right)^b+(n-d-1)\left(\frac{d+3}{2}\right)^b+2d^b,$$

for odd d. ■

Theorem 2.2. *Let $T\in\mathfrak{T}_{n,d}$. Then for $0<a\le 1$ and $b>0$, we have*

$$DEI_{a,b}(T)\ge 2^{a+1}\sum_{r=\frac{d+2}{2}}^{d-1}r^b+(n-d+1)^a\left(\frac{d}{2}\right)^b+(n-d-1)\left(\frac{d+2}{2}\right)^b+2d^b,$$

if d is even,

$$DEI_{a,b}(T)\ge 2^{a+1}\sum_{r=\frac{d+3}{2}}^{d-1}r^b+((n-d+1)^a+2^a)\left(\frac{d+1}{2}\right)^b+(n-d-1)\left(\frac{d+3}{2}\right)^b+2d^b,$$

if d is odd, with equalities if and only if T is $V_{n,d}$.

Proof. We present those parts of the proof of Theorem 2.2 which are different from the proof of Theorem 2.1. Let T_{min} be a tree with minimum $DEI_{a,b}$ index in the class $\mathfrak{T}_{n,d}$. We show that T_{min} is $V_{n,d}$. Assume to the contrary that there exists a tree $G'_{n,d}$ of order n and diameter d with $DEI_{a,b}(G'_{n,d})>DEI_{a,b}(V_{n,d})$ for some n and d. Let $G'_{n^*,d}$ be a tree having the minimum possible number of vertices n^* satisfying $DEI_{a,b}(G'_{n^*,d})<DEI_{a,b}(V_{n^*,d})$. Since $ec_{G'_{n^*,d}}(v)\ge\lceil\frac{d}{2}\rceil+1$ and $ec_{G'_{n^*-1,d}}(u)=ec_{G'_{n^*,d}}(u)\ge\lceil\frac{d}{2}\rceil$, we have $ec^b_{G'_{n^*,d}}(v)\ge\left(\lceil\frac{d}{2}\rceil+1\right)^b$, $ec^b_{G'_{n^*-1,d}}(u)=ec^b_{G'_{n^*,d}}(u)\ge\lceil\frac{d}{2}\rceil^b$ for $b>0$. Thus

$$\begin{aligned}DEI_{a,b}(G'_{n^*,d})-DEI_{a,b}(G'_{n^*-1,d}) &= d^a_{G'_{n^*,d}}(v)ec^b_{G'_{n^*,d}}(v)+d^a_{G'_{n^*,d}}(u)ec^b_{G'_{n^*,d}}(u)\\ &\quad -d^a_{G'_{n^*-1,d}}(u)ec^b_{G'_{n^*-1,d}}(u)\\ &= ec^b_{G'_{n^*,d}}(v)+r^a ec^b_{G'_{n^*,d}}(u)-(r-1)^a ec^b_{G'_{n^*,d}}(u)\\ &\ge \left(\left\lceil\frac{d}{2}\right\rceil+1\right)^b+\left\lceil\frac{d}{2}\right\rceil^b(r^a-(r-1)^a).\end{aligned}$$

Since $G'_{n^*,d}$ is a tree with the smallest possible order n^* such that $DEI_{a,b}(G'_{n^*,d}) < DEI_{a,b}(V_{n^*,d})$, we obtain $DEI_{a,b}(G'_{n^*-1,d}) \geq DEI_{a,b}(V_{n^*-1,d})$ which implies that

$$\begin{aligned} DEI_{a,b}(G'_{n^*,d}) - DEI_{a,b}(G'_{n^*-1,d}) &< DEI_{a,b}(V_{n^*,d}) - DEI_{a,b}(V_{n^*-1,d}) \\ &= \left(\left\lceil \frac{d}{2} \right\rceil + 1\right)^b + \left\lceil \frac{d}{2} \right\rceil^b ((n^* - d + 1)^a - (n^* - d)^a). \end{aligned}$$

This implies that,

$$\left(\left\lceil \frac{d}{2} \right\rceil + 1\right)^b + \left\lceil \frac{d}{2} \right\rceil^b (r^a - (r-1)^a) < \left(\left\lceil \frac{d}{2} \right\rceil + 1\right)^b + \left\lceil \frac{d}{2} \right\rceil^b ((n^* - d + 1)^a - (n^* - d)^a),$$

and hence

$$r^a - (r-1)^a < (n^* - d + 1)^a - (n^* - d)^a,$$

which is a contradiction, since $2 \leq r \leq n^* - d + 1$ and $0 < a \leq 1$ (see Lemma 1.1). Thus T_{min} is $V_{n,d}$. ■

For a positive integer d, the tree obtained by attaching $n - d - 1$ vertices to vertex x_1 from a path $P_{d+1} : x_0 x_1 x_2 \ldots x_d$ is denoted by $B_{n,d}$. We prove that $B_{n,d}$ is the unique tree having the maximum and the minimum $DEI_{a,b}$ index in the class $\mathfrak{T}_{n,d}$ for $a > 1$, $b > 0$, and $0 < a < 1$, $b < 0$ respectively.

Theorem 2.3. *Let $T \in \mathfrak{T}_{n,d}$. Then for $a > 1$ and $b > 0$, we have*

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b + 2^a \left(\frac{d}{2}\right)^b,$$

if d is even,

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+1}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b,$$

if d is odd, with equalities if and only if T is $B_{n,d}$.

Proof. Let T_{max} be a tree with maximum $DEI_{a,b}$ index in the class $\mathfrak{T}_{n,d}$. We show by contradiction that T_{max} is $B_{n,d}$.

Suppose that T_{max} is not $B_{n,d}$. Let $P : x_0 x_1 x_2 \ldots x_d$ be a path whose length is the diameter of T_{max}. Without any loss of generality, we assume that $d_{T_{max}}(x_1) \geq d_{T_{max}}(x_{d-1})$.

Since T_{max} is not a path graph, there is a non-pendant vertex $v \notin P$ whose all neighbors except for one are pendant vertices or a non pendant vertex $v \in P \setminus \{x_1\}$ whose all $r \geq 1$ neighbors which are not in P are pendant vertices. We denote those pendant vertices by $v_1, v_2, \ldots, v_r$, where $r \geq 1$. It follows that $d_{T_{max}}(v) = r + \epsilon$ where $\epsilon = 1$ or 2.

Let T' be a tree defined by, $T' = T_{max} - \{vv_1, vv_2, \ldots, vv_r\} + \{x_1v_1, x_1v_2, \ldots, x_1v_r\}$. Then $T' \in \mathfrak{T}_{n,d}$. We have $d_{T_{max}}(x_1) = s \geq 2$. Then $d_{T'}(x_1) = s + r$ and $d_{T'}(v) = \epsilon$. We obtain $ec_{T_{max}}(x_1) = ec_{T'}(x_1) = d - 1$, $ec_{T_{max}}(v) = ec_{T'}(v) \leq d - 1$, $ec_{T_{max}}(v_i) \leq d$ and $ec_{T'}(v_i) = d$, for $i = 1, 2, \ldots, r$. This implies that, $ec^b_{T_{max}}(v) = ec^b_{T'}(v) \leq (d-1)^b$, $ec^b_{T_{max}}(v_i) \leq ec^b_{T'}(v_i) = d^b$ for $b > 0$ and for $i = 1, 2, \ldots, r$.

For all $y \in V(T_{max}) \setminus \{x_1, v\}$, $d_{T_{max}}(y) = d_{T'}(y)$ and for $y \in V(T_{max}) \setminus \{v_1, v_2, \ldots, v_r\}$, $ec_{T_{max}}(y) = ec_{T'}(y)$ and hence $ec^b_{T_{max}}(y) = ec^b_{T'}(y)$. Thus

$$\begin{aligned} DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &= \sum_{i=1}^{r} d^a_{T'}(v_i)(ec^b_{T_{max}}(v_i) - ec^b_{T'}(v_i)) \\ &\quad + d^a_{T_{max}}(x_1)ec^b_{T_{max}}(x_1) + d^a_{T'}(x_1)ec^b_{T'}(x_1) \\ &\quad + d^a_{T_{max}}(v)ec^b_{T_{max}}(v) - \ d^a_{T'}(v)ec^b_{T'}(v) \\ &= r[ec^b_{T_{max}}(v_1) - d^b] + (d-1)^b[s^a - (s+r)^a] \\ &\quad + ec^b_{T'}(v)[(r+\epsilon)^a - \epsilon^a] \\ &\leq r[ec^b_{T_{max}}(v_1) - d^b] + (d-1)^b[s^a - (s+r)^a] \\ &\quad + (d-1)^b[(r+\epsilon)^a - \epsilon^a] \\ &\leq (d-1)^b[s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a]. \qquad (1) \end{aligned}$$

If $s > 2$ and $a > 1$, then by Lemma 1.1,

$$s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a < 0,$$

which implies that $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$.
If $s = 2$ and $a > 1$, then by Lemma 1.1,

$$s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a \leq 0,$$

with equality if and only if $\epsilon = 2$, which means that $v \in P$. Then the equality in (1) hold only if $ec_{T'}(v) = d-1$, that is $v = x_{d-1}$. But by our assumption $d_{T_{max}}(x_{d-1}) = 2$, that is there is no pendant vertex outside the path adjacent to x_{d-1}. Therefore, we get $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, which is a contradiction. Thus T_{max} is $B_{n,d}$. We have

$$DEI_{a,b}(B_{n,d}) = 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b + 2^a \left(\frac{d}{2}\right)^b,$$

for even d, and

$$DEI_{a,b}(B_{n,d}) = 2^{a+1} \sum_{r=\frac{d+1}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b,$$

for odd d. ■

Theorem 2.4. *Let $T \in \mathfrak{T}_{n,d}$. Then for $0 < a < 1$ and $b < 0$, we have*

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b + 2^a \left(\frac{d}{2}\right)^b,$$

if d is even,

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{d+1}{2}}^{d-2} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b,$$

if d is odd, with equalities if and only if T is $B_{n,d}$.

Proof. We present those parts of the proof of Theorem 2.4 which are different from the proof of Theorem 2.3.

Let T_{min} be a tree with minimum $DEI_{a,b}$ index in the class $\mathfrak{T}_{n,d}$. We show that T_{min} is $B_{n,d}$. Assume to the contrary that T_{min} is not $B_{n,d}$.

Since $ec_{T_{min}}(v) = ec_{T'}(v) \leq d-1$, and $ec_{T_{min}}(v_i) \leq ec_{T'}(v_i) = d$, we obtain $ec^b_{T_{min}}(v) = ec^b_{T'}(v) \geq (d-1)^b$, and $ec^b_{T_{min}}(v_i) \geq ec^b_{T'}(v_i) = d^b$ for $b < 0$ and for $i = 1, 2, \ldots, r$. Hence

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq r[ec^b_{T_{min}}(v_1) - d^b] \\ &\quad +(d-1)^b[s^a - (s+r)^a] + (d-1)^b[(r+\epsilon)^a - \epsilon^a] \\ &\geq (d-1)^b[s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a]. \end{aligned} \tag{2}$$

If $s > 2$ and $0 < a < 1$, then by Lemma 1.1,

$$s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a > 0,$$

which implies that $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is a contradiction.
If $s = 2$ and $0 < a < 1$, then by Lemma 1.1,

$$(s^a - (s+r)^a + (r+\epsilon)^a - \epsilon^a \geq 0,$$

with equality if and only if $\epsilon = 2$, which means that $v \in P$. Then the equality in (2) hold only if $ec_{T'}(v) = d-1$, that is $v = x_{d-1}$. But by our assumption $d_{T_{min}}(x_{d-1}) = 2$, that is there is no pendant vertex outside the path adjacent to x_{d-1}. Therefore we get $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is a contradiction. So T_{min} is $B_{n,d}$.

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3 Trees of given order and number of pendant vertices

In this section, we characterize trees of prescribed order n and number of pendant vertices p having the maximum and the minimum $DEI_{a,b}$ index.

Let $\mathbb{T}_{n,p}$ be the set of all n-vertex trees with p pendant vertices. Clearly, $\mathbb{T}_{n,2} = \{P_n\}$ and $\mathbb{T}_{n,n-1} = \{S_n\}$. So in what follows, we consider $\mathbb{T}_{n,p}$ for $3 \leq p \leq n-2$.

Let $X_{n,p}$ be a tree that consists of p paths attached to one vertex, such that the lengths of any two paths differ by at most one. We prove that $X_{n,p}$ is the only tree in the class $\mathbb{T}_{n,p}$ that has the maximum $DEI_{a,b}$ index for $a > 1$ and $b < 0$, and the minimum $DEI_{a,b}$ index for $0 < a < 1$ and $b > 0$.

Theorem 3.1. *Let $T \in \mathbb{T}_{n,p}$ and $k = \lfloor \frac{n-1}{p} \rfloor$. Then for $a > 1$ and $b < 0$, we have*

$$DEI_{a,b}(T) \leq p2^a \sum_{r=k+1}^{2k-1} r^b + k^b p^a + p(2k)^b,$$

if $n \equiv 1$ (mod p),

$$DEI_{a,b}(T) \leq p2^a \sum_{r=k+2}^{2k} r^b + (p^a + 2^a)(k+1)^b + p(2k+1)^b,$$

if $n \equiv 2$ (mod p), and

$$DEI_{a,b}(T) \leq p2^a \sum_{r=k+2}^{2k} r^b + ((2k+2)^b + 2^a(2k+1)^b)(n-kp-1) + (k+1)^b p^a + (kp-n+p+1),$$

otherwise, with equalities if and only if T is $X_{n,p}$.

Proof. Let T_{max} be a tree with maximum $DEI_{a,b}$ index in the class $\mathbb{T}_{n,p}$. We show by contradiction that T_{max} is $X_{n,p}$.

Let $P : x_0x_1x_2 \ldots x_d$ be a diametral path in T_{max} and v be a central vertex of T_{max} with a maximum degree. Clearly, $v = x_{\lfloor \frac{d}{2} \rfloor}$ or $x_{\lceil \frac{d}{2} \rceil}$ and hence $ec_{T_{max}}(v) = \lceil \frac{d}{2} \rceil$. Let $d_{T_{max}}(v) = s \geq 2$.

Claim 3.2. *$d_{T_{max}}(u) \leq 2$, for all $u \in V(T_{max}) \setminus \{v\}$.*

Suppose to the contrary that there exists a vertex $u \in V(T_{max}) \setminus \{v\}$ such that $d_{T_{max}}(u) = r \geq 3$. Let $u_1, u_2, \ldots, u_{r-2}$ be the neighbors of u not in P and not on the (u, v)-path.
Let $T' = T_{max} - \{uu_1, uu_2, \ldots, uu_{r-2}\} + \{vu_1, vu_2, \ldots, vu_{r-2}\}$. So, $T' \in \mathbb{T}_{n,p}$. We have $d_{T'}(v) = s + r - 2$, $d_{T'}(u) = 2$ and $d_{T_{max}}(y) = d_{T'}(y)$ for all $y \in V(T') \setminus \{v, u\}$.
For each $i = 1, 2, \ldots, r - 2$, there is no path in T_{max} with the first vertex u and the second vertex u_i which is longer than $\lceil \frac{d}{2} \rceil$. Since $ec_{T_{max}}(y) = d_{T_{max}}(y, x_0)$ or $ec_{T_{max}}(y) = d_{T_{max}}(y, x_d)$ for all $y \in V(T_{max})$, we have $ec_{T'}(y) \leq ec_{T_{max}}(y)$ and hence $ec^b_{T'}(y) \geq ec^b_{T_{max}}(y)$ for $b < 0$. Then

$$\begin{aligned} DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq d^a_{T_{max}}(v)ec^b_{T_{max}}(v) - d^a_{T'}(v)ec^b_{T'}(v) \\ &\quad + d^a_{T_{max}}(u)ec^b_{T_{max}}(u) - d^a_{T'}(u)ec^b_{T'}(u) \\ &\leq ec^b_{T_{max}}(v)[d^a_{T_{max}}(v) - d^a_{T'}(v)] + ec^b_{T_{max}}(u)[d^a_{T_{max}}(u) - d^a_{T'}(u)] \\ &= ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(u)[r^a - 2^a] \\ &\leq ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(v)[r^a - 2^a] \\ &= ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a + r^a - 2^a]. \end{aligned}$$

If $s \geq 3$ and $a > 1$, then by Lemma 1.1,

$$s^a - (s + r - 2)^a + r^a - 2^a < 0,$$

which gives $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, which is a contradiction.

Let $s = 2$. In this case, u is not a central vertex as v is a central vertex with maximum degree and $d_{T_{max}}(v) = 2$, so $ec_{T_{max}}(u) > ec_{T_{max}}(v)$. This implies that $ec^b_{T_{max}}(u) < ec^b_{T_{max}}(v)$ for $b < 0$. Hence

$$\begin{aligned} DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(u)[r^a - 2^a] \\ &< ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(v)[r^a - 2^a] \\ &= ec^b_{T_{max}}(v)[s^a - (s + r - 2)^a + r^a - 2^a] \\ &= ec^b_{T_{max}}(v)[2^a - r^a + r^a - 2^a] \\ &= 0, \end{aligned}$$

which gives $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, which is a contradiction.

Therefore, T_{max} has only one vertex of degree at least three, which is v. This implies that T_{max} consists of p pendant paths attached to v. Since v is a central vertex of T_{max}, there are two longest pendant paths attached to v whose lengths differ by at most 1. Let those two paths be $Q : vv_1v_2 \ldots v_t$ and $Q' : vv'_1v'_2 \ldots v'_{t-\epsilon}$, where $\epsilon = 0$ or 1.

Claim 3.3. *Any two pendant paths attached to v have lengths that differ by at most one.*

We prove the claim by contradiction. Suppose that there are two pendant paths, whose lengths differ by at least 2. It means that there is a path $vv''_1v''_2\dots v''_q$ in T_{max}, such that $1 \leq q \leq t-2$ and $d_{T_{max}}(v''_q) = 1$.

Let $T' = T_{max} - \{v''_{q-1}v''_q, v_{t-2}v_{t-1}, v_{t-1}v_t\} + \{v_{t-1}v''_{q-1}, v_{t-1}v''_q, v_{t-2}v_t\}$. Then $T_{max} \in \mathbb{T}_{n,p}$. Note that if $q = 1$, then $v''_{q-1} = v$. We have $d_{T_{max}}(v''_q) = d_{T'}(v''_q) = d_{T_{max}}(v_t) = d_{T'}(v_t) = 1$ and $d_{T_{max}}(v_{t-1}) = d_{T'}(v_{t-1}) = 2$. Moreover, $d_{T_{max}}(y) = d_{T'}(y)$ for all $y \in V(T_{max})$. We also know that $ec_{T_{max}}(v''_q) = q+t$, $ec_{T'}(v''_q) = (q+1)+t-\epsilon$, $ec_{T_{max}}(v_t) = 2t-\epsilon$, $ec_{T'}(v_t) = ec_{T_{max}}(v_{t-1}) = 2t-\epsilon-1$, and $ec_{T'}(v_{t-1}) = q+(t-\epsilon)$. Since $ec_{T'}(y) \leq ec_{T_{max}}(y)$, we have $ec^b_{T'}(y) \geq ec^b_{T_{max}}(y)$ for $b < 0$ and for all $y \in V(T_{max}) \setminus \{v''_q\}$. Then

$$\begin{aligned}
DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq d^a_{T_{max}}(v''_q)ec^b_{T_{max}}(v''_q) - d^a_{T'}(v''_q)ec^b_{T'}(v''_q) \\
&\quad + d^a_{T_{max}}(v_t)ec^b_{T_{max}}(v_t) - d^a_{T'}(v_t)ec^b_{T'}(v_t) \\
&\quad + d^a_{T_{max}}(v_{t-1})ec^b_{T_{max}}(v_{t-1}) - d^a_{T'}(v_{t-1})ec^b_{T'}(v_{t-1}) \\
&= (q+t)^b - (q+t-\epsilon+1)^b + (2t-\epsilon)^b - (2t-\epsilon-1)^b \\
&\quad + [(2t-\epsilon-1)^b - (q+t-\epsilon)^b]2^a.
\end{aligned}$$

If $\epsilon = 0$, then

$$\begin{aligned}
DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq (q+t)^b - (q+t+1)^b + (2t)^b - (2t-1)^b \\
&\quad + [(2t-1)^b - (q+t)^b]2^a \\
&= [(q+t)^b - (2t-1)^b][1-2^a] + (2t)^b - (q+t+1)^b \\
&< [(q+t)^b - (2t-1)^b][1-2^a] \\
&< 0.
\end{aligned}$$

This implies that, $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, which is a contradiction.

If $\epsilon = 1$, then

$$\begin{aligned}
DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq (2t-1)^b - (2t-2)^b + [(2t-2)^b - (q+t-1)^b]2^a \\
&< 0.
\end{aligned}$$

This implies that, $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, which contradicts our assumption. Thus T_{max} is $X_{n,p}$. By simple calculation, we have

$$DEI_{a,b}(X_{n,p}) = p2^a \sum_{r=k+1}^{2k-1} r^b + k^b p^a + p(2k)^b,$$

if $n \equiv 1 \pmod{p}$,

$$DEI_{a,b}(X_{n,p}) = p2^a \sum_{r=k+2}^{2k} r^b + (p^a + 2^a)(k+1)^b + p(2k+1)^b,$$

if $n \equiv 2 \pmod{p}$, and

$$DEI_{a,b}(X_{n,p}) = p2^a \sum_{r=k+2}^{2k} r^b + ((2k+2)^b + 2^a(2k+1)^b)(n-kp-1) + (k+1)^b p^a + (kp-n+p+1),$$

otherwise. ■

Theorem 3.4. *Let $T \in \mathbb{T}_{n,p}$ and $k = \lfloor \frac{n-1}{p} \rfloor$. Then for $0 < a < 1$ and $b > 0$ we have*

$$DEI_{a,b}(T) \geq p2^a \sum_{r=k+1}^{2k-1} r^b + k^b p^a + p(2k)^b,$$

if $n \equiv 1 \pmod p$,

$$DEI_{a,b}(T) \geq p2^a \sum_{r=k+2}^{2k} r^b + (p^a + 2^a)(k+1)^b + p(2k+1)^b,$$

if $n \equiv 2 \pmod p$,

$$DEI_{a,b}(T) \leq p2^a \sum_{r=k+2}^{2k} r^b + ((2k+2)^b + 2^a(2k+1)^b)(n-kp-1) + (k+1)^b p^a + (kp-n+p+1),$$

otherwise, with equalities if and only if T is $X_{n,p}$.

Proof. We present those parts of the proof of Theorem 3.4 which are different from the proof of Theorem 3.1. Let T_{min} be a tree with minimum $DEI_{a,b}$ index in the class $\mathbb{T}_{n,p}$. We show by contradiction that T_{min} is $X_{n,p}$.

Claim 3.5. *$d_{T_{min}}(u) \leq 2$, for all $u \in V(T_{min}) \setminus \{v\}$.*

Since $ec_{T'}(y) \leq ec_{T_{min}}(y)$, we obtain $ec^b_{T'}(y) \leq ec^b_{T_{min}}(y)$ for $b > 0$ and for all $y \in V(T_{min})$. It follows that

$$\begin{aligned}
DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq d^a_{T_{min}}(v)ec^b_{T_{min}}(v) - d^a_{T'}(v)ec^b_{T'}(v) \\
&\quad + d^a_{T_{min}}(u)ec^b_{T_{min}}(u)\ d^a_{T'}(u)ec^b_{T'}(u) \\
&\geq ec^b_{T_{min}}(v)[d^a_{T_{min}}(v) - d^a_{T'}(v)] + ec^b_{T_{min}}(u)[d^a_{T_{min}}(u) - d^a_{T'}(u)] \\
&= ec^b_{T_{min}}(v)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(u)[r^a - 2^a] \\
&\geq ec^b_{T_{min}}(v)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(v)[r^a - 2^a] \\
&= ec^b_{T_{min}}(v)[s^a - (s+r-2)^a + r^a - 2^a].
\end{aligned}$$

If $s \geq 3$ and $0 < a < 1$, then by Lemma 1.1,

$$s^a - (s+r-2)^a + r^a - 2^a > 0,$$

which gives $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is a contradiction.

Let $s = 2$. In this case, u is not a central vertex as v is a central vertex with maximum degree and $d_{T_{min}}(v) = 2$, so $ec_{T_{min}}(u) > ec_{T_{min}}(v)$. This implies that, $ec^b_{T_{min}}(u) > ec^b_{T_{min}}(v)$ for $b > 0$. Hence

$$\begin{aligned}
DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq ec^b_{T_{min}}(v)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(u)[r^a - 2^a] \\
&> ec^b_{T_{min}}(v)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(v)[r^a - 2^a] \\
&= ec^b_{T_{min}}(v)[s^a - (s+r-2)^a + r^a - 2^a] \\
&= ec^b_{T_{min}}(v)[2^a - r^a + r^a - 2^a] \\
&= 0,
\end{aligned}$$

which gives $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is a contradiction. Therefore T_{min} has only one vertex of degree at least three, which is v. This implies that T_{min} consists of p pendant paths attached to v.

Claim 3.6. *Any two pendant paths attached to v have lengths that differ by at most one.*

Since $ec_{T'}(y) \leq ec_{T_{min}}(y)$, we obtain $ec^b_{T'}(y) \leq ec^b_{T_{min}}(y)$ for $b > 0$ and for all $y \in V(T_{min}) \setminus \{v''_q\}$. Then

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq d^a_{T_{min}}(v''_q)ec^b_{T_{min}}(v''_q) - d^a_{T'}(v''_q)ec^b_{T'}(v''_q) \\ &\quad + d^a_{T_{min}}(v_t)ec^b_{T_{min}}(v_t) - d^a_{T'}(v_t)ec^b_{T'}(v_t) \\ &\quad + d^a_{T_{min}}(v_{t-1})ec^b_{T_{min}}(v_{t-1}) - d^a_{T'}(v_{t-1})ec^b_{T'}(v_{t-1}) \\ &= (q+t)^b - (q+t-\epsilon+1)^b + (2t-\epsilon)^b - (2t-\epsilon-1)^b \\ &\quad + [(2t-\epsilon-1)^b - (q+t-\epsilon)^b]2^a. \end{aligned}$$

If $\epsilon = 0$, then

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq (q+t)^b - (q+t+1)^b + (2t)^b - (2t-1)^b \\ &\quad + [(2t-1)^b - (q+t)^b]2^a \\ &= [(q+t)^b - (2t-1)^b][1-2^a] + (2t)^b - (q+t+1)^b \\ &> [(q+t)^b - (2t-1)^b][1-2^a] \\ &> 0. \end{aligned}$$

This implies that, $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which contradicts our assumption.
If $\epsilon = 1$, then

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq (2t-1)^b - (2t-2)^b + [(2t-2)^b - (q+t-1)^b]2^a \\ &> 0. \end{aligned}$$

This implies that, $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is in contradiction with our assumption. Thus T_{min} is $X_{n,p}$. ■

For each $3 \leq p \leq n-2$, let $Y_{n,p}$ be a tree obtained from a path $Q_{n-p+2} : x_0x_1x_2 \dots x_{n-p+1}$ by attaching $p-2$ pendant vertices to vertex x_1.

We show that $DEI_{a,b}(T) \leq DEI_{a,b}(Y_{n,p})$ for $a > 1$ and $b > 0$, and $DEI_{a,b}(T) \geq DEI_{a,b}(Y_{n,p})$ for $0 < a < 1$ and $b < 0$ for every tree in the class $\mathbb{T}_{n,p}$.

Theorem 3.7. *Let $T \in \mathbb{T}_{n,p}$. Then for $a > 1$ and $b > 0$, we have*

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{n-p+2}{2}}^{n-p-1} r^b + (p^a + 2^a)(n-p)^b + p(n-p+1)^b,$$

if $n-p$ is even,

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{n-p+3}{2}}^{n-p-1} r^b + 2^a \left(\frac{n-p+1}{2}\right)^b + (p^a + 2^a)(n-p)^b + p(n-p+1)^b,$$

if $n-p$ is odd, with equalities if and only if T is $Y_{n,p}$.

Proof. Let T_{max} be a tree with maximum $DEI_{a,b}$ index in the class $\mathbb{T}_{n,p}$. We show that T_{max} is $Y_{n,p}$.

Assume to the contrary that T_{max} is not $Y_{n,p}$. Let $P : x_0x_1x_2 \dots x_d$ be a longest path in T_{max}. Without any loss of generality, we assume that $d_{T_{max}}(x_1) \geq d_{T_{max}}(x_{d-1})$. Since T_{max} is not $Y_{n,p}$, we have $d_{T_{max}}(v) = r \geq 3$ for some $v \in \{x_2, x_3, \ldots, x_{d-1}\}$. Let $v_1, v_2, \ldots, v_{r-2}$ be the neighbors of v outside P.

Let $T' = T_{max} - \{vv_1, vv_2, \ldots, vv_{r-2}\} + \{x_1v_1, x_1v_2, \ldots, x_1v_{r-2}\}$. We have $d_{T_{max}}(x_1) = s \geq 2$. Then $d_{T'}(x_1) = s + r - 2$, $d_{T'}(v) = 2$ and $d_{T_{max}}(y) = d_{T'}(y)$ for all $y \in V(T_{max}) \setminus \{x_1, v\}$. We have $ec_{T_{max}}(y) \leq ec_{T'}(y)$ for all $y \in V(T_{max})$. This implies that, $ec^b_{T_{max}}(y) \leq ec^b_{T'}(y)$ for $b > 0$ and for all $y \in V(T_{max})$. Thus

$$\begin{aligned}
DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq ec^b_{T_{max}}(x_1)d^a_{T_{max}}(x_1) - ec^b_{T'}(x_1)d^a_{T'}(x_1) \\
&\quad + ec^b_{T_{max}}(v)d^a_{T_{max}}(v) - ec^b_{T'}(v)d^a_{T'}(v) \\
&\leq ec^b_{T_{max}}(x_1)[d^a_{T_{max}}(x_1) - d^a_{T'}(x_1)] + ec^b_{T_{max}}(v)[d^a_{T_{max}}(v) - d^a_{T'}(v)] \\
&= ec^b_{T_{max}}(x_1)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(v)[r^a - 2^a] \\
&\leq ec^b_{T_{max}}(x_1)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(x_1)[r^a - 2^a] \\
&= ec^b_{T_{max}}(x_1)[s^a - (s + r - 2)^a + r^a - 2^a].
\end{aligned}$$

If $s \geq 3$ and $a > 1$, then by Lemma 1.1,

$$s^a - (s + r - 2)^a + r^a - 2^a < 0,$$

which gives $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, it contradicts our assumption.

Suppose $s = 2$. Since $ec_{T_{max}}(v) \leq d - 2 < ec_{T_{max}}(x_1)$, we have $ec^b_{T_{max}}(v) \leq (d - 2)^b < ec^b_{T_{max}}(x_1)$ for $b > 0$. In this case,

$$\begin{aligned}
DEI_{a,b}(T_{max}) - DEI_{a,b}(T') &\leq ec^b_{T_{max}}(x_1)[s^a - (s + r - 2)^a] + ec^b_{T_{max}}(v)[r^a - 2^a] \\
&= ec^b_{T_{max}}(x_1)[2^a - r^a] + ec^b_{T_{max}}(v)[r^a - 2^a] \\
&= [ec^b_{T_{max}}(v) - ec^b_{T_{max}}(x_1)][r^a - 2^a] \\
&< 0.
\end{aligned}$$

Hence, $DEI_{a,b}(T_{max}) < DEI_{a,b}(T')$, it contradicts our assumption. Thus T_{max} is $Y_{n,p}$. We have

$$DEI_{a,b}(Y_{n,p}) = 2^{a+1} \sum_{r=\frac{n-p+2}{2}}^{n-p-1} r^b + (p^a + 2^a)(n - p)^b + p(n - p + 1)^b,$$

for even $n - p$, and

$$DEI_{a,b}(Y_{n,p}) = 2^{a+1} \sum_{r=\frac{n-p+3}{2}}^{n-p-1} r^b + 2^a \left(\frac{n - p + 1}{2}\right)^b + (p^a + 2^a)(n - p)^b + p(n - p + 1)^b,$$

for odd $n - p$. ■

Theorem 3.8. *Let $T \in \mathbb{T}_{n,p}$. Then for $0 < a < 1$ and $b < 0$, we have*

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{n-p+2}{2}}^{n-p-1} r^b + (p^a + 2^a)(n - p)^b + p(n - p + 1)^b,$$

if $n - p$ is even,

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{n-p+3}{2}}^{n-p-1} r^b + 2^a \left(\frac{n-p+1}{2}\right)^b + (p^a+2^a)(n-p)^b + p(n-p+1)^b,$$

if $n-p$ is odd, with equalities if and only if T is $Y_{n,p}$.

Proof. We present those parts of the proof of Theorem 3.8 which are different from the proof of Theorem 3.7.

Let T_{min} be a tree with minimum $DEI_{a,b}$ index in the class $\mathbb{T}_{n,p}$. We prove by contradiction that T_{min} is $Y_{n,p}$.

Assume that T_{min} is not $Y_{n,p}$. Since $ec_{T_{min}}(y) \leq ec_{T'}(y)$, we have $ec^b_{T_{min}}(y) \geq ec^b_{T'}(y)$ for $b<0$ and for all $y \in V(T_{min})$.

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq ec^b_{T_{min}}(x_1)d^a_{T_{min}}(x_1) - ec^b_{T'}(x_1)d^a_{T'}(x_1) \\ &\quad + ec^b_{T_{min}}(v)d^a_{T_{min}}(v) - ec^b_{T'}(v)d^a_{T'}(v) \\ &\geq ec^b_{T_{min}}(x_1)[d^a_{T_{min}}(x_1) - d^a_{T'}(x_1)] \\ &\quad + ec^b_{T_{min}}(v)[d^a_{T_{min}}(v) - d^a_{T'}(v)] \\ &= ec^b_{T_{min}}(x_1)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(v)[r^a - 2^a] \\ &\geq ec^b_{T_{min}}(x_1)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(x_1)[r^a - 2^a] \\ &= ec^b_{T_{min}}(x_1)[s^a - (s+r-2)^a + r^a - 2^a]. \end{aligned}$$

If $s \geq 3$ and $0<a<1$, then by Lemma 1.1,

$$s^a - (s+r-2)^a + r^a - 2^a > 0,$$

which gives $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$.

Since $ec_{T_{min}}(v) \leq d-2 < ec_{T_{min}}(x_1)$, we have $ec^b_{T_{min}}(v) \geq (d-2)^b > ec^b_{T_{min}}(x_1)$ for $b<0$. In this case,

$$\begin{aligned} DEI_{a,b}(T_{min}) - DEI_{a,b}(T') &\geq ec^b_{T_{min}}(x_1)[s^a - (s+r-2)^a] + ec^b_{T_{min}}(v)[r^a - 2^a] \\ &= ec^b_{T_{min}}(x_1)[2^a - r^a] + ec^b_{T_{min}}(v)[r^a - 2^a] \\ &= [ec^b_{T_{min}}(v) - ec^b_{T_{min}}(x_1)][r^a - 2^a] \\ &> 0. \end{aligned}$$

Hence, $DEI_{a,b}(T_{min}) > DEI_{a,b}(T')$, which is a contradiction. Thus T_{min} is $Y_{n,p}$. ■

4 Conclusion and open problems

In this paper, we studied the $DEI_{a,b}$ index for trees of fixed order and diameter, and trees of fixed order and the number of pendant vertices. In Section 2, we obtained upper bounds on $DEI_{a,b}(T)$ for $a>1, b \in \mathbb{R}\setminus\{0\}$ and lower bounds on $DEI_{a,b}(T)$ for $0<a<1, b \in \mathbb{R}\setminus\{0\}$, for trees T of fixed order and diameter. In Section 3, we obtained upper bounds on $DEI_{a,b}(T)$ for $a>1, b \in \mathbb{R}\setminus\{0\}$ and lower bounds on $DEI_{a,b}(T)$ for $0<a<1, b \in \mathbb{R}\setminus\{0\}$, for trees T of fixed order and the number of pendent vertices. In some of the results, we included the case $a=1$ and $b \in \{-1,1\}$. We also showed that all our bounds are sharp by presenting the corresponding extremal graphs. We state open problems on the $DEI_{a,b}$ index of trees for future research.

Problem 4.1. *Find trees having the maximum and minimum $DEI_{a,b}$ index among trees of a fixed order, where $a < 0$ or $a \geq 1$ and $b \geq 0$, and $a \leq 1$ or $b < 0$.*

Problem 4.2. *Find a tree T having the minimum $DEI_{a,b}$ index among trees of fixed order and diameter/trees of fixed order and number of pendant vertices, where $a < 0$ or $a \geq 1$ and $b \in \mathbb{R}$.*

Problem 4.3. *Find a tree T having the maximum $DEI_{a,b}$ index among trees of fixed order and diameter/trees of fixed order and number of pendant vertices, where $a \leq 1$ and $b \in \mathbb{R}$.*

Conflicts of Interest. The author declare that he has no conflicts of interest regarding the publication of this article.

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