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On the General Eccentric Distance Sum of Graphs and Trees

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ABSTRACT

We obtain some sharp bounds on the general eccentric distance sum for general graphs, bipartite graphs and trees with given order and diameter 3, graphs with given order and domination number 2, and for the join of two graphs with given order and number of vertices having maximum possible degree. Extremal graphs are presented for all the bounds.

1. INTRODUCTION

We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The order of G is the number of vertices of G. The degree of a vertex u, $deg_G(u)$, is the number

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of edges incident with u. The distance $d_G(u,v)$ between two vertices u and v is the number of edges in a shortest path between u and v. The eccentricity $ecc_G(u)$ of u in G is the distance between u and a farthest vertex from u in G. The distance between any two farthest vertices in G is the diameter of G. A pendant vertex is a vertex of a graph having degree 1.

A graph whose vertices can be partitioned into two (partite) sets V_1 and V_2, such that no two vertices in the same set are adjacent is called a bipartite graph. A tree is a connected graph containing no cycles. The complete graph and the empty graph of order n are denoted by K_n and $\overline{K_n}$, respectively. For $k \geq 2$, let us denote by $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ the graph obtained from graphs $G_1, G_2, \dots, G_k$ by joining every vertex of G_{i-1} with every vertex of G_i, where $i = 2,3, \dots, k$. The graph $G_1 \oplus G_2$ is called the join of two graphs G_1 and G_2. For $U \subseteq V(G)$, an induced subgraph $G[U]$ of a graph G consists of the vertices in U and all the edges of G connecting two vertices in U.

Topological indices have been investigated due to their extensive applications, especially in chemistry. The general eccentric distance sum of a connected graph G is defined as $EDS_{a,b}(G) = \sum_{u \in V(G)} [ecc_G(u)]^a [D_G(u)]^b$, where $D_G(u) = \sum_{v \in V(G)} d_G(u,v)$ and $a, b \in \mathbb{R}$.

We believe that it is important to study general topological indices. Then, results for particular topological indices are special cases of general results. Note that $EDS_{1,1}(G) = EDS(G)$ is the classical eccentric distance sum of G, $EDS_{1,0}(G)$ is the total eccentricity index and $EDS_{2,0}(G)$ is the first Zagreb eccentricity index of G. So, those topological indices are special cases of the general eccentric distance sum.

The classical eccentric distance sum EDS belongs to the most well-known distance-based topological indices. It has been widely studied. A lower bound on EDS for trees with given order was presented in [9] and [20], a lower bound for trees with prescribed order and domination number was presented in [6] and trees were studied also in [14]. A lower bound on EDS for graphs with prescribed order and vertex connectivity was given in [10], graph operations were investigated in [3], Sierpiński graphs in [4], graphs related to groups in [1], bipartite graphs in [5] and [12], cubic transitive graphs in [19], fullerances in [7], relationships with some other indices in [2] and [8], and exact values for several basic graphs were given in [13]. Some related distance-based indices were studied for example in [15] and [18]. First results on the general eccentric distance sum were given in [16].

We present some bounds on the general eccentric distance sum for general graphs, bipartite graphs and trees with given order and diameter 3, graphs with given order and domination number 2, and for the join of two graphs with given order and number of vertices having maximum possible degree. First, let us state two lemmas. Lemma 1 was given in [16] and it is used in the proofs of Theorems 2, 3 and 4.

Lemma 1. *Let G be a connected graph with two non-adjacent vertices u and v. For $a \geq 0$ and $b > 0$, we have*

$$EDS_{a,b}(G+uv) < EDS_{a,b}(G).$$

For $a \leq 0$ and $b < 0$, we have

$$EDS_{a,b}(G+uv) > EDS_{a,b}(G).$$

The following lemma was given in [17] and it is used in the proofs of Theorems 4, 5, 6 and 9.

Lemma 2. *Let* $1 \leq x < y$ *and* $c > 0$. *Then for* $b > 1$ *and* $b < 0$,

$$(x+c)^b - x^b < (y+c)^b - y^b.$$

If $0 < b < 1$, then

$$(x+c)^b - x^b > (y+c)^b - y^b.$$

2. Results for General Graphs and Bipartite Graphs

Let us present bounds on $EDS_{a,b}(G_1 \oplus G_2)$ for the join of two graphs G_1 and G_2 with given order and number of vertices having maximum possible degree.

Theorem 1. *For* $i = 1,2$, *let* G_i *be a graph of order* n_i *with* k_i *vertices of degree* $n_i - 1$. *Let* $a, b \in \mathbb{R}$. *Then for* $b > 0$,

$$EDS_{a,b}(G_1 \oplus G_2) \geq (k_1+k_2)(n_1+n_2-1)^b + (n_1+n_2-k_1-k_2)2^a(n_1+n_2)^b,$$

and for $b < 0$,

$$EDS_{a,b}(G_1 \oplus G_2) \leq (k_1+k_2)(n_1+n_2-1)^b + (n_1+n_2-k_1-k_2)2^a(n_1+n_2)^b.$$

The equalities hold if and only if G_i contains $n_i - k_i$ vertices of degree $n_i - 2$, where $n_i - k_i$ is even; $i = 1,2$.

Proof. For $i = 1,2$, let us denote the set of vertices of degree $n_i - 1$ in $V(G_i)$ by S_i. We have $|V(G_i)| = n_i$ and $|S_i| = k_i$. Then $ecc_{G_1 \oplus G_2}(v) = 1$ and $D_{G_1 \oplus G_2}(v) = n_1 + n_2 - 1$ for $v \in S_1 \cup S_2$. For $v \in (V(G_1) \backslash S_1) \cup (V(G_2) \backslash S_2)$, we get $ecc_{G_1 \oplus G_2}(v) = 2$. For $v_1 \in V(G_1) \backslash S_1$, we have

$$\begin{aligned} D_{G_1 \oplus G_2}(v_1) &= n_2 + deg_{G_1}(v_1) + 2\left[n_1 - 1 - deg_{G_1}(v_1)\right] \\ &= n_2 + 2n_1 - 2 - deg_{G_1}(v_1) \\ &\geq n_1 + n_2, \end{aligned}$$

since $deg_{G_1}(v_1) \leq n_1 - 2$. Similarly, $D_{G_1 \oplus G_2}(v_2) \geq n_1 + n_2$ for $v_2 \in V(G_2) \backslash S_2$. Thus, for $i = 1,2$, we obtain

$$[D_{G_1 \oplus G_2}(v_i)]^b \geq (n_1+n_2)^b,$$

if $b > 0$, and

$$[D_{G_1 \oplus G_2}(v_i)]^b \leq (n_1+n_2)^b,$$

if $b < 0$. Consequently, for $b > 0$, $EDS_{a,b}(G_1 \oplus G_2) \geq k_1(n_1 + n_2 - 1)^b + k_2(n_1 + n_2 - 1)^b + (n_1 - k_1)2^a(n_1 + n_2)^b + (n_2 - k_2)2^a(n_1 + n_2)^b$,
and for $b < 0$, $EDS_{a,b}(G_1 \oplus G_2) \leq k_1(n_1 + n_2 - 1)^b + k_2(n_1 + n_2 - 1)^b + (n_1 - k_1)2^a(n_1 + n_2)^b + (n_2 - k_2)2^a(n_1 + n_2)^b$.

The equalities are achieved when $deg_{G_i}(v_i) = n_i - 2$ for every $v_i \in V(G_i)\backslash S_i$, where $i = 1,2$. Note that $n_i - k_i$ must be even, since for a graph with k_i vertices of degree $n_i - 1$ and $n_i - k_i$ vertices of degree $n_i - 2$, from Handshaking lemma, we have

$$\begin{aligned} 2|E(G_i)| &= \textstyle\sum_{v \in V(G_i)} deg_{G_i}(v) \\ &= k_i(n_i - 1) + (n_i - k_i)(n_i - 2) \\ &= n_i(n_i - 1) - (n_i - k_i). \end{aligned}$$

■

Now, we focus on graphs of diameter 3. In Theorems 2 and 3, we give bounds on $EDS_{a,b}(G)$ for general graphs G. For $a = b = 1$, the graphs of given order and diameter with the smallest $EDS_{a,b}$ were presented in [11].

Theorem 2. *Let G be a graph of order $n \geq 4$ and diameter 3. Then for $a \geq 0$ and $0 < b < 1$, we have*

$$EDS_{a,b}(G) \geq 3^a[(n + 2)^b + (2n - 2)^b] + 2^a(n - 2)n^b,$$

with equality if and only if G is $K_1 \oplus K_{n-3} \oplus K_1 \oplus K_1$.

Proof. Suppose that G' is a graph with the minimum $EDS_{a,b}$ among graphs of order n and diameter 3. Let u_0 and u_3 be any two vertices of distance 3 in G'. For $i = 0,1,2,3$, let $U_i = \{u \in V(G') : d_{G'}(u_0, u) = i\}$. Then $V(G') = U_0 \cup U_1 \cup U_2 \cup U_3$. According to Lemma 1, adding an edge will decrease $EDS_{a,b}$. Thus, $G'[U_{i-1} \cup U_i]$ is a complete graph for $i = 1,2,3$. Note that $|U_3| = 1$ (otherwise, if $|U_3| \geq 2$, we can add edges to G' to obtain G'' with $E(G'') = E(G') \cup \{uu_3 : u \in U_1\}$, and by Lemma 1, $EDS_{a,b}(G'') < EDS_{a,b}(G')$). So, G' has the form $G_p = K_1 \oplus K_{n-p-2} \oplus K_p \oplus K_1$ where $1 \leq p \leq \lfloor\frac{n}{2}\rfloor - 1$. We have

$$ecc_{G_p}(u_0) = ecc_{G_p}(u_3) = 3, \qquad D_{G_p}(u_0) = n + p + 1 \quad \text{and} \quad D_{G_p}(u_3) = 2n - p - 1.$$

For all $u \in V(G_p)\backslash\{u_0, u_3\}$, we have $ecc_{G_p}(u) = 2$ and $D_{G_p}(u) = n$. Thus

$$EDS_{a,b}(G_p) = 3^a[(n + p + 1)^b + (2n - p - 1)^b] + 2^a(n - 2)n^b = f(p).$$

Then the derivative

$$f'(p) = 3^a b[(n + p + 1)^{b-1} - (2n - p - 1)^{b-1}].$$

Since $0 < b < 1$, we have $f'(p) > 0$ for $1 \leq p < \lfloor\frac{n}{2}\rfloor - 1$ and $f'(p) = 0$ for $p = \lfloor\frac{n}{2}\rfloor - 1$. Thus, $f(p)$ is increasing for $1 \leq p \leq \lfloor\frac{n}{2}\rfloor - 1$ and $0 < b < 1$. So, $EDS_{a,b}(G_1) < EDS_{a,b}(G_p)$, where $2 \leq p \leq \lfloor\frac{n}{2}\rfloor - 1$. Hence G' is $G_1 = K_1 \oplus K_{n-3} \oplus K_1 \oplus K_1$ and

$$EDS_{a,b}(K_1 \oplus K_{n-3} \oplus K_1 \oplus K_1) = 3^a[(n+2)^b + (2n-2)^b] + 2^a(n-2)n^b. \quad \blacksquare$$

Theorem 3. *Let* G *be a graph of order* $n \geq 4$ *and diameter* 3. *Then for* $a \leq 0$ *and* $b < 0$, *we have*

$$EDS_{a,b}(G) \leq 3^a[(n+2)^b + (2n-2)^b] + 2^a(n-2)n^b,$$

with equality if and only if G is $K_1 \oplus K_{n-3} \oplus K_1 \oplus K_1$.

Proof. We present only those parts which are different from the proof of Theorem 2. Suppose that G' is a graph of order n and diameter 3 with the maximum $EDS_{a,b}$. According to Lemma 1, adding an edge will increase $EDS_{a,b}$. Thus $G'[U_{i-1} \cup U_i]$ is a complete graph for $i = 1,2,3$. Note that $|U_3| = 1$ (otherwise, if $|U_3| \geq 2$, we can add edges to G' to obtain G'' with $E(G'') = E(G') \cup \{uu_3 : u \in U_1\}$, and by Lemma 1, $EDS_{a,b}(G'') > EDS_{a,b}(G')$). So, G' has the form $G_p = K_1 \oplus K_{n-p-2} \oplus K_p \oplus K_1$ where $1 \leq p \leq \lfloor \frac{n}{2} \rfloor - 1$. Since $b < 0$, we have $f'(p) < 0$ for $1 \leq p < \lfloor \frac{n}{2} \rfloor - 1$ and $f'(p) = 0$ for $p = \lfloor \frac{n}{2} \rfloor - 1$. Thus, $f(p)$ is decreasing for $1 \leq p \leq \lfloor \frac{n}{2} \rfloor - 1$ and $b < 0$. So, $EDS_{a,b}(G_1) > EDS_{a,b}(G_p)$, where $2 \leq p \leq \lfloor \frac{n}{2} \rfloor - 1$. Hence G' is $G_1 = K_1 \oplus K_{n-3} \oplus K_1 \oplus K_1$. $\blacksquare$

A sharp lower bound on $EDS_{a,b}(G)$ for bipartite graphs G is given in Theorem 4.

Theorem 4. *Let* G *be a bipartite graph of order* $n \geq 4$ *and diameter* 3. *Then for* $a \geq 0$ *and* $b \geq 1$, *we have*

$$EDS_{a,b}(G) \geq 3^a\left[\left(n + \left\lceil \frac{n}{2} \right\rceil\right)^b + \left(n + \left\lfloor \frac{n}{2} \right\rfloor\right)^b\right] + 2^a\left[\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(n + \left\lceil \frac{n}{2} \right\rceil - 2\right)^b + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(n + \left\lfloor \frac{n}{2} \right\rfloor - 2\right)^b\right],$$

with equality if and only if G is $K_1 \oplus \overline{K_{\lceil \frac{n}{2} \rceil - 1}} \oplus \overline{K_{\lfloor \frac{n}{2} \rfloor - 1}} \oplus K_1$.

Proof. Suppose that G' is a graph with the minimum $EDS_{a,b}$ among bipartite graphs of order n and diameter 3. Let u_0 and u_3 be any two vertices of distance 3 in G'. For $i = 0,1,2,3$, let $U_i = \{u \in V(G') : d_{G'}(u_0, u) = i\}$. Then $V(G') = U_0 \cup U_1 \cup U_2 \cup U_3$, where $U_0 = \{u_0\}$. The graph $G'[U_i]$ must be edgeless, otherwise G' would have some cycle of odd length. According to Lemma 1, adding an edge will decrease $EDS_{a,b}$. Thus, $G'[U_{i-1} \cup U_i]$ is a complete bipartite graph for $i = 1,2,3$. Note that $|U_3| = 1$ (otherwise, if $|U_3| \geq 2$, we can add the edge u_0u_3 to G' to obtain G'', so $EDS_{a,b}(G'') < EDS_{a,b}(G')$, by Lemma 1). So, G' has the form $K_1 \oplus \overline{K_{n_1}} \oplus \overline{K_{n_2}} \oplus K_1$, where $n_1 + n_2 = n - 2$.

Without loss of generality, assume that $|U_1| \geq |U_2|$. We prove that $|U_1| - |U_2| \leq 1$. Suppose to the contrary that $|U_1| - |U_2| \geq 2$. We choose $w \in U_1$. Let G''' has the same vertex set as G' while $E(G''') = \{wu: u \in U_1 \cup \{u_3\}\} \cup E(G') \setminus \{wu: u \in \{u_0\} \cup U_2\}$. So, G''' is the graph $K_1 \oplus \overline{K_{n_1-1}} \oplus \overline{K_{n_2+1}} \oplus K_1$.

For all $u \in V(G')$, we get $ecc_{G'''}(u) = ecc_{G'}(u)$. We have

$$D_{G'}(w) = 2|U_1| + |U_2| + 1 \quad \text{and} \quad D_{G'''}(w) = |U_1| + 2|U_2| + 2.$$

Since $|U_1| - |U_2| \geq 2$, we obtain $D_{G'}(w) - D_{G'''}(w) > 0$. Thus

$$[ecc_{G'}(w)]^a [D_{G'}(w)]^b - [ecc_{G'''}(w)]^a [D_{G'''}(w)]^b > 0.$$

We obtain

$$\begin{aligned} D_{G'}(u_0) &= |U_1| + 2|U_2| + 1, \ D_{G'''}(u_0) = |U_1| + 2|U_2| + 2, \\ D_{G'}(u_3) &= 2|U_1| + |U_2| + 1, \ D_{G''}(u_3) = 2|U_1| + |U_2|, \\ D_{G'}(u) &= 2|U_1| + |U_2| + 1, \ D_{G'''}(u) = 2|U_1| + |U_2|, \\ D_{G'}(v) &= |U_1| + 2|U_2| + 1, \ D_{G'''}(v) = |U_1| + 2|U_2| + 2, \end{aligned}$$

where $u \in U_1 \setminus \{w\}$ and $v \in U_2$. Note that

$$ecc_{G'}(u) = ecc_{G'}(v) = 2 \quad \text{and} \quad ecc_{G'}(u_0) = ecc_{G'}(u_3) = 3.$$

Then

$$\begin{aligned} EDS_{a,b}(G') - EDS_{a,b}(G''') &= [ecc_{G'}(u_0)]^a([D_{G'}(u_0)]^b - [D_{G'''}(u_0)]^b) \\ &+ [ecc_{G'}(u_3)]^a([D_{G'}(u_3)]^b - [D_{G''}(u_3)]^b) \\ &+ \textstyle\sum_{v \in U_2} [ecc_{G'}(v)]^a([D_{G'}(v)]^b - [D_{G'''}(v)]^b) \\ &+ \textstyle\sum_{u \in U_1 \setminus \{w\}} [ecc_{G'}(u)]^a([D_{G'}(u)]^b - [D_{G'''}(u)]^b) \\ &+ [ecc_{G'}(w)]^a([D_{G'}(w)]^b - [D_{G'''}(w)]^b) \\ &> 3^a([D_{G'}(u_0)]^b - [D_{G'}(u_0) + 1]^b \\ &+ [D_{G'}(u_3)]^b - [D_{G'}(u_3) - 1]^b) \\ &+ 2^a[|U_2|([D_{G'}(v)]^b - [D_{G'}(v) + 1]^b) \\ &+ (|U_1| - 1)([D_{G'}(u)]^b - [D_{G'}(u) - 1]^b)] \\ &> 3^a([D_{G'}(u_0)]^b - [D_{G'}(u_0) + 1]^b + [D_{G'}(u_3)]^b \\ &- [D_{G'}(u_3) - 1]^b) + 2^a|U_2|([D_{G'}(v)]^b - [D_{G'}(v) + 1]^b \\ &+ [D_{G'}(u)]^b - [D_{G'}(u) - 1]^b) \geq 0, \end{aligned}$$

because for $b = 1$,

$$[D_{G'}(u_0)]^b - [D_{G'}(u_0) + 1]^b + [D_{G'}(u_3)]^b - [D_{G'}(u_3) - 1]^b = 0,$$

and

$$[D_{G'}(v)]^b - [D_{G'}(v) + 1]^b + [D_{G'}(u)]^b - [D_{G'}(u) - 1]^b = 0,$$

and for $b > 1$, by Lemma 2,

$$[D_{G'}(u_3)]^b - [D_{G'}(u_3) - 1]^b > [D_{G'}(u_0) + 1]^b - [D_{G'}(u_0)]^b$$

and

$$[D_{G'}(u)]^b - [D_{G'}(u) - 1]^b > [D_{G'}(v) + 1]^b - [D_{G'}(v)]^b,$$

since $D_{G'}(u_3) > D_{G'}(u_0) + 1$ and $D_{G'}(u) > D_{G'}(v) + 1$. Thus $EDS_{a,b}(G') > EDS_{a,b}(G''')$ for $a \geq 0$ and $b \geq 1$, a contradiction. So, $|U_1| - |U_2| \leq 1$. Hence G' is $K_1 \oplus \overline{K_{\lceil \frac{n}{2} \rceil - 1}} \oplus \overline{K_{\lfloor \frac{n}{2} \rfloor - 1}} \oplus K_1$ and

$$EDS_{a,b}(G') = 3^a\left[\left(n+\left\lceil\frac{n}{2}\right\rceil\right)^b + \left(n+\left\lfloor\frac{n}{2}\right\rfloor\right)^b\right]$$
$$+\,2^a\left[\left(\left\lceil\frac{n}{2}\right\rceil-1\right)\left(n+\left\lceil\frac{n}{2}\right\rceil-2\right)^b + \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n+\left\lfloor\frac{n}{2}\right\rfloor-2\right)^b\right],$$

3. Results for Trees

For integers $l \geq 2$ and $n_1 \geq n_2 \geq 1$, $P_l(n_1, n_2)$ is a tree obtained from the path P_l by joining one end vertex of P_l to n_1 new vertices and the other end vertex of P_l to n_2 pendant vertices; see Figure 1. The tree $P_l(n_1, n_2)$ has $n_1 + n_2$ pendant vertices.

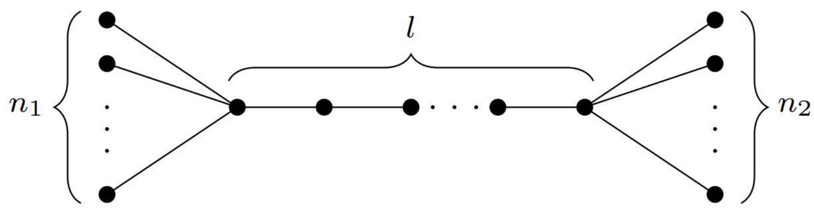

Figure 1: Tree $P_l(n_1, n_2)$.

In Theorems 5 and 6, we compare $EDS_{a,b}$ of these trees if l and the order are fixed. Theorem 5 is used in the proofs of Theorems 7 and 10. For $a = b = 1$, the following theorem was presented in [6].

Theorem 5. *Let* $2 \leq l \leq n-2$. *For* $a, b \in \mathbb{R}$, *where* $0 < b \leq 1$,

$$EDS_{a,b}(P_l(1, n-l-1)) < EDS_{a,b}(P_l(2, n-l-2)) < \cdots$$
$$< EDS_{a,b}\left(P_l\left(\left\lfloor\frac{n-l}{2}\right\rfloor, \left\lceil\frac{n-l}{2}\right\rceil\right)\right).$$

Proof. In the tree $T_1 = P_l(n_1, n_2)$, where $n_1 \geq n_2 \geq 2$, let $u_1u_2 \dots u_l$ be the path which does not contain pendant vertices of $P_l(n_1, n_2)$. We denote the pendant vertices adjacent to u_1 by $v_1, v_2, \dots, v_{n_1}$ and the pendant vertices adjacent to u_l by $v'_1, v'_2, \dots, v'_{n_2}$.

Let $V(T_2) = V(T_1)$ and $E(T_2) = \{u_1v'_{n_2}\} \cup E(T_1)\backslash\{u_lv'_{n_2}\}$. Note that T_2 is the tree $P_l(n_1+1, n_2-1)$. To prove Theorem 5, it suffices to show that $EDS_{a,b}(T_2) < EDS_{a,b}(T_1)$.

For any $v \in V(T_1)$, we obtain $ecc_{T_1}(v) = ecc_{T_2}(v)$. Note that for $i = 1,2, \dots, \lfloor\frac{l+1}{2}\rfloor$,

$$ecc_{T_1}(u_i) = ecc_{T_1}(u_{l+1-i}) = l+1-i, \quad \text{and} \quad ecc_{T_1}(v) = l+1$$

for all the pendant vertices $v \in V(T_1)$.

For $j = 1,2, \dots, n_1$ and $k = 1,2, \dots, n_2 - 1$,

- in T_1, there are $n_1 - 1$ pendant vertices of distance 2 from v_j and n_2 pendant vertices of distance $l+1$ from v_j,
- in T_2, there are n_1 pendant vertices of distance 2 from v_j and $n_2 - 1$ pendant vertices of distance $l+1$ from v_j,
- in T_1, there are $n_2 - 1$ pendant vertices of distance 2 from v'_k and n_1 pendant vertices of distance $l+1$ from v'_k,
- in T_2, there are $n_2 - 2$ pendant vertices of distance 2 from v'_k and $n_1 + 1$ pendant vertices of distance $l+1$ from v'_k,

thus
$$D_{T_2}(v_j) < D_{T_1}(v_j) \leq D_{T_1}(v'_k) < D_{T_2}(v'_k), \tag{1}$$
where
$$D_{T_1}(v_j) - D_{T_2}(v_j) = D_{T_2}(v'_k) - D_{T_1}(v'_k) = l - 1. \tag{2}$$
Similarly,
$$D_{T_2}(v'_{n_2}) < D_{T_1}(v'_{n_2}), \quad thus \;\; [D_{T_2}(v'_{n_2})]^b < [D_{T_1}(v'_{n_2})]^b \tag{3}$$
for $0 < b \leq 1$.

For $i = 1,2,\ldots,\lfloor\frac{l}{2}\rfloor$,

- in T_1, there are n_1 pendant vertices of distance i from u_i and n_2 pendant vertices of distance $l+1-i$ from u_i,
- in T_2, there are $n_1 + 1$ pendant vertices of distance i from u_i and $n_2 - 1$ pendant vertices of distance $l+1-i$ from u_i,
- in T_1, there are n_2 pendant vertices of distance i from u_{l+1-i} and n_1 pendant vertices of distance $l+1-i$ from u_{l+1-i},
- in T_2, there are $n_2 - 1$ pendant vertices of distance i from u_{l+1-i} and $n_1 + 1$ pendant vertices of distance $l+1-i$ from u_{l+1-i},

thus
$$D_{T_2}(u_i) < D_{T_1}(u_i) \leq D_{T_1}(u_{l+1-i}) < D_{T_2}(u_{l+1-i}), \tag{4}$$
where
$$D_{T_1}(u_i) - D_{T_2}(u_i) = D_{T_2}(u_{l+1-i}) - D_{T_1}(u_{l+1-i}) = l + 1 - 2i. \tag{5}$$
Note that if l is odd, then $D_{T_2}(u_{\frac{l+1}{2}}) = D_{T_1}(u_{\frac{l+1}{2}})$.

We have
$$\begin{aligned} EDS_{a,b}(T_1) - EDS_{a,b}(T_2) &= \textstyle\sum_{v\in V(T_1)} [ecc_{T_1}(v)]^a([D_{T_1}(v)]^b - [D_{T_2}(v)]^b) \\ &= \textstyle\sum_{j=1}^{n_1} [ecc_{T_1}(v_j)]^a([D_{T_1}(v_j)]^b - [D_{T_2}(v_j)]^b) \\ &\quad + \textstyle\sum_{k=1}^{n_2} [ecc_{T_1}(v'_k)]^a([D_{T_1}(v'_k)]^b - [D_{T_2}(v'_k)]^b) \\ &\quad + \textstyle\sum_{i=1}^{l} [ecc_{T_1}(u_i)]^a([D_{T_1}(u_i)]^b - [D_{T_2}(u_i)]^b). \end{aligned}$$
By (2) and (3),
$$\textstyle\sum_{j=1}^{n_1} [ecc_{T_1}(v_j)]^a([D_{T_1}(v_j)]^b - [D_{T_2}(v_j)]^b) + \sum_{k=1}^{n_2} [ecc_{T_1}(v'_k)]^a([D_{T_1}(v'_k)]^b - [D_{T_2}(v'_k)]^b)$$

$= (l+1)^a[n_1([D_{T_1}(v_1)]^b - [D_{T_2}(v_1)]^b)$
$+ (n_2-1)([D_{T_1}(v'_1)]^b - [D_{T_2}(v'_1)]^b) + [D_{T_1}(v'_{n_2})]^b - [D_{T_2}(v'_{n_2})]^b]$
$> (l+1)^a[n_1([D_{T_2}(v_1)+l-1]^b - [D_{T_2}(v_1)]^b) + (n_2-1)([D_{T_1}(v'_1)]^b$
$- [D_{T_1}(v'_1)+l-1]^b)]$
$> (l+1)^a(n_2-1)([D_{T_2}(v_1)+l-1]^b - [D_{T_2}(v_1)]^b + [D_{T_1}(v'_1)]^b$
$- [D_{T_1}(v'_1)+l-1]^b) \geq 0,$
since $(l+1)^a > 0$, for $b=1$,

$$[D_{T_2}(v_1)+l-1]^b - [D_{T_2}(v_1)]^b + [D_{T_1}(v'_1)]^b - [D_{T_1}(v'_1)+l-1]^b = 0,$$

and for $0 < b < 1$, by (1) and Lemma 2,

$$[D_{T_2}(v_1)+l-1]^b - [D_{T_2}(v_1)]^b > [D_{T_1}(v'_1)+l-1]^b - [D_{T_1}(v'_1)]^b.$$

By (5),
$\sum_{i=1}^{l} [ecc_{T_1}(u_i)]^a([D_{T_1}(u_i)]^b - [D_{T_2}(u_i)]^b)$
$= \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} [ecc_{T_1}(u_i)]^a([D_{T_1}(u_i)]^b - [D_{T_2}(u_i)]^b)$
$+ [ecc_{T_1}(u_{l+1-i})]^a([D_{T_1}(u_{l+1-i})]^b - [D_{T_2}(u_{l+1-i})]^b)$
$= \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} (l+1-i)^a([D_{T_2}(u_i)+l+1-2i]^b - [D_{T_2}(u_i)]^b$
$+ [D_{T_1}(u_{l+1-i})]^b - [D_{T_1}(u_{l+1-i})+l+1-2i]^b) \geq 0,$
since $(l+1-i)^a > 0$, for $b=1$, $[D_{T_2}(u_i)+l+1-2i]^b - [D_{T_2}(u_i)]^b + [D_{T_1}(u_{l+1-i})]^b$ $-[D_{T_1}(u_{l+1-i})+l+1-2i]^b = 0$, and for $0<b<1$, by (4) and Lemma 2,

$$[D_{T_2}(u_i)+l+1-2i]^b - [D_{T_2}(u_i)]^b > [D_{T_1}(u_{l+1-i})+l+1-2i]^b - [D_{T_1}(u_{l+1-i})]^b.$$

Hence $EDS_{a,b}(T_1) - EDS_{a,b}(T_2) > 0$. ■

Theorem 6 is used in the proof of Theorem 8.

Theorem 6. *Let* $2 \leq l \leq n-2$. *For* $a, b \in \mathbb{R}$, *where* $b < 0$,

$$EDS_{a,b}(P_l(1, n-l-1)) > EDS_{a,b}(P_l(2, n-l-2)) > \cdots > EDS_{a,b}\left(P_l\left(\left\lfloor \frac{n-l}{2} \right\rfloor, \left\lceil \frac{n-l}{2} \right\rceil\right)\right).$$

Proof. We present those parts of the proof of Theorem 6, which differ from the proof of Theorem 5. We show that $EDS_{a,b}(T_1) < EDS_{a,b}(T_2)$, where $T_1 = P_l(n_1, n_2)$, $T_2 = P_l(n_1+1, n_2-1)$ and $n_1 \geq n_2 \geq 2$,

For $j = 1,2,\ldots,n_1$ and $k = 1,2,\ldots,n_2-1$, we have

$$D_{T_2}(v_j) < D_{T_1}(v_j) \leq D_{T_1}(v'_k) < D_{T_2}(v'_k), \tag{6}$$

where

$$D_{T_1}(v_j) - D_{T_2}(v_j) = D_{T_2}(v'_k) - D_{T_1}(v'_k) = l-1. \tag{7}$$

Similarly,

$$D_{T_2}(v'_{n_2}) < D_{T_1}(v'_{n_2}), \quad \text{so} \quad [D_{T_2}(v'_{n_2})]^b > [D_{T_1}(v'_{n_2})]^b \tag{8}$$

for $b < 0$.

For $i = 1,2, \dots, \lfloor \frac{l}{2} \rfloor$, we have

$$D_{T_2}(u_i) < D_{T_1}(u_i) \leq D_{T_1}(u_{l+1-i}) < D_{T_2}(u_{l+1-i}), \tag{9}$$

where

$$D_{T_1}(u_i) - D_{T_2}(u_i) = D_{T_2}(u_{l+1-i}) - D_{T_1}(u_{l+1-i}) = l + 1 - 2i. \tag{10}$$

By (7) and (8),

$\sum_{j=1}^{n_1} [ecc_{T_1}(v_j)]^a([D_{T_1}(v_j)]^b - [D_{T_2}(v_j)]^b) + \sum_{k=1}^{n_2} [ecc_{T_1}(v'_k)]^a([D_{T_1}(v'_k)]^b - [D_{T_2}(v'_k)]^b)$
$= (l+1)^a[n_1([D_{T_1}(v_1)]^b - [D_{T_2}(v_1)]^b)$
$+ (n_2 - 1)([D_{T_1}(v'_1)]^b - [D_{T_2}(v'_1)]^b) + [D_{T_1}(v'_{n_2})]^b - [D_{T_2}(v'_{n_2})]^b]$
$< (l+1)^a[n_1([D_{T_2}(v_1) + l - 1]^b - [D_{T_2}(v_1)]^b)$
$+ (n_2 - 1)([D_{T_1}(v'_1)]^b - [D_{T_1}(v'_1) + l - 1]^b)]$
$< (l+1)^a(n_2 - 1)([D_{T_2}(v_1) + l - 1]^b - [D_{T_2}(v_1)]^b$
$+ [D_{T_1}(v'_1)]^b - [D_{T_1}(v'_1) + l - 1]^b) < 0,$

since $(l+1)^a > 0$ and by (6) and Lemma 2, for $b < 0$,

$$[D_{T_2}(v_1) + l - 1]^b - [D_{T_2}(v_1)]^b < [D_{T_1}(v'_1) + l - 1]^b - [D_{T_1}(v'_1)]^b.$$

By (10),

$\sum_{i=1}^{l} [ecc_{T_1}(u_i)]^a([D_{T_1}(u_i)]^b - [D_{T_2}(u_i)]^b)$
$= \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} [ecc_{T_1}(u_i)]^a([D_{T_1}(u_i)]^b - [D_{T_2}(u_i)]^b)$
$+ [ecc_{T_1}(u_{l+1-i})]^a([D_{T_1}(u_{l+1-i})]^b - [D_{T_2}(u_{l+1-i})]^b)$
$= \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} (l+1-i)^a([D_{T_2}(u_i) + l + 1 - 2i]^b - [D_{T_2}(u_i)]^b$
$+ [D_{T_1}(u_{l+1-i})]^b - [D_{T_1}(u_{l+1-i}) + l + 1 - 2i]^b) < 0,$

since $(l+1-i)^a > 0$ and for $b < 0$, by (9) and Lemma 2,

$$[D_{T_2}(u_i) + l + 1 - 2i]^b - [D_{T_2}(u_i)]^b < [D_{T_1}(u_{l+1-i}) + l + 1 - 2i]^b - [D_{T_1}(u_{l+1-i})]^b.$$

Hence $EDS_{a,b}(T_1) - EDS_{a,b}(T_2) < 0$. ∎

Let us present sharp bounds on $EDS_{a,b}(T)$ for trees T of given order and diameter 3.

Theorem 7. *Let T be a tree of order $n \geq 4$ and diameter 3. Let $a, b \in \mathbb{R}$ where $0 < b \leq 1$. Then*

$$EDS_{a,b}(P_2(n-3,1)) \leq EDS_{a,b}(T) \leq EDS_{a,b}\left(P_2\left(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right),$$

with equalities if and only if T is $P_2(n-3,1)$ and $P_2(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1)$, respectively.

Proof. Every tree of order n and diameter 3 has the form $P_2(n-k,k)$, where $1 \le k \le \lfloor \frac{n}{2} \rfloor$. For $0 < b \le 1$, by Theorem 5, $P_2(n-3,1)$ and $P_2(\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1)$ are the unique trees with the smallest and largest $EDS_{a,b}$, respectively.

Similarly, using Theorem 6, we obtain the following bounds for negative b.

Theorem 8. *Let T be a tree of order $n \ge 4$ and diameter 3. Let $a, b \in \mathbb{R}$ where $b < 0$. Then*
$$EDS_{a,b}\left(P_2\left(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right) \le EDS_{a,b}(T) \le EDS_{a,b}\big(P_2(n-3,1)\big),$$
with equalities if and only if T is $P_2(\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1)$ and $P_2(n-3,1)$, respectively.

We present the values of $EDS_{a,b}(P_2(n-3,1))$ and $EDS_{a,b}(P_2(\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1))$. We have
$$EDS_{a,b}(P_2(n-3,1)) = 3^a[(n-3)(2n-2)^b + (3n-6)^b] + 2^a[(2n-4)^b + n^b]$$
and
$$EDS_{a,b}\left(P_2\left(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right) = 3^a\left[\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(2n + \left\lfloor \frac{n}{2} \right\rfloor - 4\right)^b + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2n + \left\lceil \frac{n}{2} \right\rceil - 4\right)^b\right] + 2^a\left[\left(n + \left\lfloor \frac{n}{2} \right\rfloor - 2\right)^b + \left(n + \left\lceil \frac{n}{2} \right\rceil - 2\right)^b\right].$$

Let us compare $EDS_{a,b}$ of $P_l(n_1, n_2)$ and $P_{l+1}(n_1 - 1, n_2)$, which are trees having the same order, but different number of pendant vertices (and different diameter). Theorem 9 is used in the proof of Theorem 10.

Theorem 9. *Let $l \ge 2$, $n_1 \ge 2$ and $n_2 \ge 1$, where $n_1 \ge n_2$. Then for $a \ge 0$ and $b \ge 1$,*
$$EDS_{a,b}(P_l(n_1, n_2)) < EDS_{a,b}(P_{l+1}(n_1 - 1, n_2)).$$

Proof. In the tree $T_1 = P_l(n_1, n_2)$, let $u_1 u_2 \ldots u_l$ be the path which does not contain pendant vertices of $P_l(n_1, n_2)$. We denote the pendant vertices adjacent to u_1 by $v_1, v_2, \ldots, v_{n_1}$ and the pendant vertices adjacent to u_l by $v'_1, v'_2, \ldots, v'_{n_2}$. Let $V(T_2) = V(T_1)$ and $E(T_2) = \{v_1 v_2, v_1 v_3, \ldots, v_1 v_{n_1}\} \cup E(T_1) \setminus \{u_1 v_2, u_1 v_3, \ldots, u_1 v_{n_1}\}$. Note that T_2 is the tree $P_{l+1}(n_1 - 1, n_2)$. For any $v \in V(T_1)$, we obtain $ecc_{T_2}(v) \ge ecc_{T_1}(v)$. For any $v \in V(T_1) \setminus \{v_1\}$, we have $D_{T_2}(v) > D_{T_1}(v)$, thus for $a \ge 0$ and $b \ge 1$,
$$[ecc_{T_2}(v)]^a [D_{T_2}(v)]^b > [ecc_{T_1}(v)]^a [D_{T_1}(v)]^b.$$
For v_1, we get $D_{T_2}(v_1) = D_{T_1}(v_1) - n_1 + 1$.

We use v_1 and v'_1 to compare $EDS_{a,b}(T_1)$ and $EDS_{a,b}(T_2)$. For v'_1, we have $D_{T_2}(v'_1) = D_{T_1}(v'_1) + n_1 - 1$. Since $n_1 \ge n_2$, we obtain $D_{T_1}(v'_1) \ge D_{T_1}(v_1)$. Note that

$$ecc_{T_1}(v_1) = ecc_{T_2}(v_1) = ecc_{T_1}(v'_1) = l+1 \quad \text{and} \quad ecc_{T_2}(v'_1) = l+2.$$

We obtain

$$\begin{aligned} EDS_{a,b}(T_2) - EDS_{a,b}(T_1) &> [ecc_{T_2}(v_1)]^a [D_{T_2}(v_1)]^b - [ecc_{T_1}(v_1)]^a [D_{T_1}(v_1)]^b \\ &\quad + [ecc_{T_2}(v'_1)]^a [D_{T_2}(v'_1)]^b - [ecc_{T_1}(v'_1)]^a [D_{T_1}(v'_1)]^b \\ &= (l+2)^a [D_{T_1}(v_1) - n_1 + 1]^b - (l+1)^a [D_{T_1}(v_1)]^b \\ &\quad + (l+1)^a [D_{T_1}(v'_1) + n_1 - 1]^b - (l+1)^a [D_{T_1}(v'_1)]^b \\ &> (l+1)^a ([D_{T_1}(v_1) - n_1 + 1]^b - [D_{T_1}(v_1)]^b \\ &\quad + [D_{T_1}(v'_1) + n_1 - 1]^b - [D_{T_1}(v'_1)]^b) \ \geq 0, \end{aligned}$$

because for $b = 1$,

$$[D_{T_1}(v_1) - n_1 + 1]^b - [D_{T_1}(v_1)]^b + [D_{T_1}(v'_1) + n_1 - 1]^b - [D_{T_1}(v'_1)]^b = 0,$$

and for $b > 1$, by Lemma 2,

$$[D_{T_1}(v'_1) + n_1 - 1]^b - [D_{T_1}(v'_1)]^b > [D_{T_1}(v_1)]^b - [D_{T_1}(v_1) - n_1 + 1]^b,$$

since $D_{T_1}(v'_1) \geq D_{T_1}(v_1) > D_{T_1}(v_1) - n_1 + 1$. Hence, $EDS_{a,b}(T_2) > EDS_{a,b}(T_1)$.

A dominating set in a graph G is a set $\Gamma \subseteq V(G)$ such that every vertex not in Γ is adjacent to a vertex in Γ. The cardinality of a smallest dominating set is the domination number of G. Let us give an upper bound on $EDS_{a,b}(T)$ for trees T with given order and domination number 2 if $b = 1$. For $a = b = 1$, the tree of given order and domination number 2 having the largest $EDS_{a,b}$ was given in [6].

Theorem 10. *Let T be a tree of order $n \geq 6$ and domination number 2. Then for $a \geq 0$,*

$$EDS_{a,1}(T) \leq \left(2n^2 + 6\left\lfloor \frac{n^2}{4} \right\rfloor - 20n + 24\right) 5^a + (5n-8)4^a + (5n-12)3^a,$$

with equality if and only if T is $P_4(\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor)$.

Proof. Any tree of order n and domination number 2 has the form $P_l(n_1, n_2)$, where $2 \leq l \leq 4$ and $l + n_1 + n_2 = n$. By Theorem 5, a tree with the largest $EDS_{a,1}$ among trees of order n and domination number 2 is $P_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)$ or $P_3(\lceil \frac{n-3}{2} \rceil, \lfloor \frac{n-3}{2} \rfloor)$ or $P_4(\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor)$. By Theorem 9,

$$\begin{aligned} EDS_{a,1}\left(P_2\left(\left\lceil \frac{n-2}{2} \right\rceil, \left\lfloor \frac{n-2}{2} \right\rfloor\right)\right) &< EDS_{a,1}\left(P_3\left(\left\lceil \frac{n-3}{2} \right\rceil, \left\lfloor \frac{n-3}{2} \right\rfloor\right)\right) \\ &< EDS_{a,1}\left(P_4\left(\left\lceil \frac{n-4}{2} \right\rceil, \left\lfloor \frac{n-4}{2} \right\rfloor\right)\right), \end{aligned}$$

thus $P_4(\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor) = P_4(\lceil \frac{n}{2} \rceil - 2, \lfloor \frac{n}{2} \rfloor - 2)$ is the tree with the largest $EDS_{a,1}$ among trees of order n and domination number 2. We have $EDS_{a,1}\left(P_4\left(\left\lceil \frac{n}{2} \right\rceil - 2, \left\lfloor \frac{n}{2} \right\rfloor - 2\right)\right) = \left(2n^2 + 6\left\lfloor \frac{n^2}{4} \right\rfloor - 20n + 24\right) 5^a + (5n-8)4^a + (5n-12)3^a$. ■

4. Open Problems

Let us state several problems open for further research. In Theorem 1, we presented bounds on $EDS_{a,b}(G_1 \oplus G_2)$ for the join of two graphs G_1 and G_2. We suggest studying other graph products.

Problem 1. *Study $EDS_{a,b}$ for the Cartesian product, tensor product or lexicographic product of two graphs.*

In Theorems 2, 3, 4, 7 and 8, we obtained bounds on $EDS_{a,b}$ for general graphs (for $a \geq 0$, $0 < b < 1$ and $a \leq 0$, $b < 0$), bipartite graphs (for $a \geq 0$, $b \geq 1$) and trees (for $a \in \mathbb{R}, 0 < b \leq 1$ and $a \in \mathbb{R}, b < 0$) of diameter 3. We recommend studying graphs of larger diameters.

Problem 2. *Find upper or lower bounds on $EDS_{a,b}(G)$ for trees, bipartite graphs or general graphs G with given order and diameter greater than* 3.

In Theorem 10, we presented an upper bound on $EDS_{a,b}(T)$ of trees T only for domination number 2 and $b = 1$. We suggest studying related problems if both a and b are general.

Problem 3. *Find upper or lower bounds on $EDS_{a,b}(G)$ for trees or graphs G with given order and domination number, where both a and b are general.*

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References

1. A. Abdussakir, E. Susanti, N. Hidayati and N. M. Ulya, Eccentric distance sum and adjacent eccentric distance sum index of complement of subgroup graphs of dihedral group, *J. Phys. Conf. Ser*. **1375** (1) (2019) 012065.

2. Y. Alizadeh and S. Klavžar, On the difference between the eccentric connectivity index and eccentric distance sum of graphs, *Bull. Malays. Math. Sci. Soc*. **44** (2) (2021) 1123 – 1134.
3. M. Azari and A. Iranmanesh, Computing the eccentric-distance sum for graph operations, *Discrete Appl. Math.* **161** (18) (2013) 2827 – 2840.
4. J. I. N. Chen, L. He and Q. I. N. Wang, Eccentric distance sum of Sierpiński gasket and Sierpiński network, *Fractals* **27** (2) (2019) 1950016.
5. H. Chen and R. Wu, On extremal bipartite graphs with given number of cut edges, *Discrete Math. Algorithms Appl*. **12** (2) (2020) 2050015.
6. X. Geng, S. Li and M. Zhang, Extremal values on the eccentric distance sum of trees, *Discrete Appl. Math*. **161** (16–17) (2013) 2427 – 2439.
7. M. Hemmasi, A. Iranmanesh and A. Tehranian, Computing eccentric distance sum for an infinite family of fullerenes, *MATCH Commun. Math. Comput. Chem.* **71** (2) (2014) 417 – 424.
8. H. Hua and H. Bao, The eccentric distance sum of connected graphs, *Util. Math*. **100** (2016) 65 – 77.
9. H. Hua, K. Xu and S. Wen, A short and unified proof of Yu et al.'s two results on the eccentric distance sum, J. Math. Anal. Appl. **382** (1) (2011) 364 – 366.
10. A. Ilić, G. Yu and L. Feng, On the eccentric distance sum of graphs, *J. Math. Anal. Appl*. **381** (2011) 590 – 600.
11. S. Li and Y. Wu, On the extreme eccentric distance sum of graphs with some given parameters, *Discrete Appl. Math.* **206** (2016) 90 – 99.
12. S. C. Li, Y. Y. Wu and L. L. Sun, On the minimum eccentric distance sum of bipartite graphs with some given parameters, *J. Math. Anal. Appl*. **430** (2) (2015) 1149 – 1162.
13. P. Padmapriya and V. Mathad, The eccentric-distance sum of some graphs, *Electron. J. Graph Theory Appl.* (EJGTA) **5** (1) (2017) 51 – 62.
14. L. Pei and X. Pan, The minimum eccentric distance sum of trees with given distance k-domination number, *Discrete Math. Algorithms Appl.* **12** (4) (2020) 2050052.
15. L. Song, H. Liu and Z. Tang, Some properties of the leap eccentric connectivity index of graphs, *Iranian J. Math. Chem*. **11** (4) (2020) 227 – 237.
16. T. Vetrík, General eccentric distance sum of graphs, *Discrete Math. Algorithms Appl*. **13** (4) (2021) 2150046.
17. T. Vetrík and M. Masre, General eccentric connectivity index of trees and unicyclic graphs, *Discrete Appl. Math*. **284** (2020) 301 – 315.
18. L. Wei, H. Bian, H. Yu and X. Yang, The Gutman index and Schultz index in the random phenylene chains, *Iranian J. Math. Chem*. **12** (2) (2021) 67 – 78.
19. Y. -T. Xie and S. -J. Xu, On the maximum value of the eccentric distance sums of cubic transitive graphs, *Appl. Math. Comput*. **359** (2019) 194–201.
20. G. Yu, L. Feng and A. Ilić, On the eccentric distance sum of trees and unicyclic graphs, *J. Math. Anal. Appl*. **375** (1) (2011) 99 – 107.