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Extremal Cacti with respect to Sombor Index

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ABSTRACT

Recently, a novel topological index, Sombor index, was introduced by Gutman, defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

where d_u denotes the degree of vertex u. In this paper, we first determine the maximum Sombor index among cacti with n vertices and t cycles, then determine the maximum Sombor index among cacti with perfect matchings. We also characterize corresponding maximum cacti.

1. INTRODUCTION

In this paper, all notations and terminologies can refer to Bondy and Murty [2]. The Sombor index and reduced Sombor index are defined as [10]

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}.$$

Immediately after, R. Cruz et al. [5] studied the extremal Sombor index among unicyclic graphs and bicyclic graphs. The same author also [4] determined the extremal Sombor index of chemical graphs. At the same time, using different methods, Deng et al. [7] also determined molecular trees with extremal values of Sombor indices. Wang et al. [24] obtain the relations between Sombor and other degree-based indices. Liu et al. [17] ordered the chemical graphs by their Sombor index. Redšepović [22] studied chemical

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applicability of Sombor indices. Other results can be found in [1,3,6,9,11–13,18,19,21,23,25,26].

Cactus is a graph that any two cycles have at most one common vertex. Denote by $\mathcal{H}(n,t)$, $\mathcal{C}(2\beta,t)$, the collection of cacti with n vertices and t cycles, the collection of cacti with perfect matchings, 2β vertices and t cycles, respectively. There are a lot of research about topological indices of graphs with perfect matching or given matching numbers, we can refer to [8,15,16,20,25] and references cited therein. In this paper, we first determine the maximum Sombor index among cacti $\mathcal{H}(n,t)$, then determine the maximum Sombor index among cacti $\mathcal{C}(2\beta,t)$. We also characterize corresponding maximum cacti.

2. PRELIMINARIES

In the following, we give a few important lemmas which will be useful in the main results.

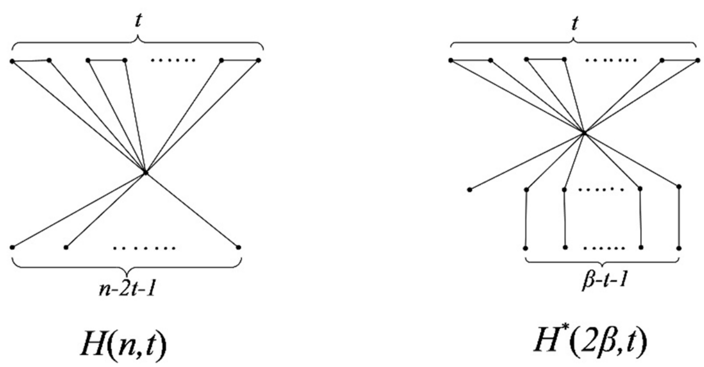

Figure 1. Cacti $H(n,t)$ and $H^*(2\beta,t)$.

Lemma 2.1. [10] Let $G \in \mathcal{H}(n,0)$. Then

$$SO(G) \leq (n-1)\sqrt{(n-1)^2+1},$$

with equality if and only if $G \cong H(n,0)$.

Lemma 2.2. [5] Let $G \in \mathcal{H}(n,1)$. Then

$$SO(G) \leq (n-3)\sqrt{(n-1)^2+1} + 2\sqrt{(n-1)^2+4} + 2\sqrt{2},$$

with equality if and only if $G \cong H(n,1)$.

Lemma 2.3. *Let* $f_1(x) = \sqrt{x^2+d^2} - \sqrt{x^2+(d-r)^2}$, *d, r* are constants and $0 < r \leq d$. Then $f_1(x)$ is a monotonically decreasing function.

Proof. $f_1'(x) = \frac{x}{\sqrt{x^2+d^2}} - \frac{x}{\sqrt{x^2+(d-r)^2}} < 0$, thus $f_1(x)$ is a monotonically decreasing function. ∎

Lemma 2.4. Let $f_2(x) = r\sqrt{x^2+1^2} + (x-r)\sqrt{x^2+2^2} + (x-r)\sqrt{(x-r)^2+2^2}$, r is constants and $0 \le r \le x$. Then $f_2(x)$ is a monotonically increasing function.

Proof. $f_2'(x) = \frac{rx}{\sqrt{x^2+1}} + \sqrt{x^2+4} + \frac{x(x-r)}{\sqrt{x^2+4}} + \sqrt{(x-r)^2+4} + \frac{(x-r)^2}{\sqrt{(x-r)^2+4}} > 0$, thus $f_2(x)$ is a monotonically increasing function. ∎

Lemma 2.5. Let $f_3(x) = x\sqrt{x^2+2^2} - (x-2)\sqrt{(x-2)^2+2^2}$. *Then* $f_3(x)$ *is a* monotonically increasing function.

Proof. $f_3'(x) = \sqrt{x^2+4} - \sqrt{(x-2)^2+4} + \frac{x^2}{\sqrt{x^2+4}} + \frac{(x-2)^2}{\sqrt{(x-2)^2+4}} > 0$, thus $f_3(x)$ is a monotonically increasing function. ∎

Lemma 2.6. [3,25] Let $G \in \mathcal{C}(2\beta, 0)$. Then

$$SO(G) \le \sqrt{5}(\beta-1) + \sqrt{\beta^2+1} + (\beta-1)\sqrt{\beta^2+4},$$

with equality if and only if $G \cong H^*(2\beta, 0)$.

Lemma 2.7. [25] Let $G \in \mathcal{C}(2\beta, 1)$. Then

$$SO(G) \le \beta\sqrt{(\beta+1)^2+4} + \sqrt{(\beta+1)^2+1} + \sqrt{5}(\beta-1) + 2\sqrt{2},$$

with equality if and only if $G \cong H^*(2\beta, 1)$.

Lemma 2.8. Let $f(x) = (x-1)\sqrt{x^2+2^2} + \sqrt{x^2+1^2}$ *and* $g(x) = f(x) - f(x+1)$. Then $g(x)$ $(x \ge 1)$ is a monotonically decreasing function.

Proof. Note that $f'(x) = \sqrt{x^2+4} + \frac{x(x-1)}{\sqrt{x^2+4}} + \frac{x}{\sqrt{x^2+1}}$ and $f''(x) = \frac{3x-1}{\sqrt{x^2+4}} - \frac{x^2(x-1)}{(x^2+4)^{\frac{3}{2}}} + \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{2x(x^2+6)-4}{(x^2+4)^{\frac{3}{2}}} + \frac{1}{(x^2+1)^{\frac{3}{2}}} > 0$ *for* $x \ge 1$, thus $g(x)$ $(x \ge 1)$ is a monotonically decreasing function. ∎

3. Maximum Sombor Index among Cacti $\mathcal{H}(n,t)$

Theorem 3.1. Let $G \in \mathcal{H}(n,t)$ $(n \ge 5)$. Then

$$SO(G) \le (n-2t-1)\sqrt{(n-1)^2+1} + 2t\sqrt{(n-1)^2+4} + 2\sqrt{2}t,$$

with equality if and only if $G \cong H(n,t)$.

Proof. For convenience, we denote

$$Q(n,t) \triangleq (n-2t-1)\sqrt{(n-1)^2+1} + 2t\sqrt{(n-1)^2+4} + 2\sqrt{2}t.$$

In the following we make inductive assumptions about $n+t$.

By Lemmas 2.1 and 2.2, the conclusion holds if $t=1$ or $t=2$. If $n=5$, the conclusion holds clearly. So we only consider $n \geq 6$ and $t \geq 2$ in the following. $PV(G)$ denotes the set of pendent vertices in G. We call the vertices connected with pendent vertices support vertices, denoted by $Supp(G)$.

Case 1. $PV(G) \neq \emptyset$. Let v be a pendent vertex, $N(v)=w$, $N(w)=\{v,x_1,x_2,\dots,x_{d-1}\}$. $d_{x_i}=1$ for $1 \leq i \leq r-1$, $d_{x_i} \geq 2$ for $r \leq i \leq d-1$. Let $G^* = G - v - x_1 - x_2 - \cdots - x_{r-1}$, then $G^* \in \boldsymbol{\mathcal{H}}(n-r,t)$. By Lemma 2.3, 2.4 and 2.5, we have

$$SO(G) = SO(G^*) + r\sqrt{d^2+1^2} + \sum_{i=r}^{d-1}\left(\sqrt{d^2+d_{x_i}^2} - \sqrt{(d-r)^2+d_{x_i}^2}\right)$$

$$\leq Q(n-r,t) + r\sqrt{d^2+1^2} + \sum_{i=r}^{d-1}\left(\sqrt{d^2+d_{x_i}^2} - \sqrt{(d-r)^2+d_{x_i}^2}\right)$$

$$\leq Q(n,t) + 2t\sqrt{(n-r-1)^2+2^2} - 2t\sqrt{(n-1)^2+2^2}$$
$$+(n-r-2t-1)\sqrt{(n-r-1)^2+1^2} - (n-2t-1)\sqrt{(n-1)^2+1^2}$$
$$+r\sqrt{d^2+1^2} + (d-r)\left(\sqrt{d^2+2^2} - \sqrt{(d-r)^2+2^2}\right)$$

$$\leq Q(n,t) + 2t\sqrt{(n-r-1)^2+2^2} - 2t\sqrt{(n-1)^2+2^2}$$
$$+(n-r-2t-1)\sqrt{(n-r-1)^2+1^2} - (n-2t-1)\sqrt{(n-1)^2+1^2}$$
$$+r\sqrt{(n-1)^2+1^2} + (n-r-1)\left(\sqrt{(n-1)^2+2^2} - \sqrt{(n-r-1)^2+2^2}\right)$$

$$= Q(n,t) + (n-2t-r-1)\left[\left(\sqrt{(n-1)^2+2^2} - \sqrt{(n-r-1)^2+2^2}\right)\right.$$
$$\left.-\left(\sqrt{(n-1)^2+1^2} - \sqrt{(n-r-1)^2+1^2}\right)\right.$$

$\leq Q(n,t)$, with equality if and only if $G \cong H(n,t)$.

Case 2. $PV(G) = \emptyset$. Suppose that there exists vertices v_0, v_1, v_2 on a cycle of G, v_0v_1, $v_1v_2 \in E(G)$, $d(v_1) = d(v_2) = 2$, $d(v_0) \triangleq d \geq 3$. In the following, we classify the circle lengths.

Subcase 2.1. The circle length greater than or equal to 4. Let $G^* = G - v_1 + v_0v_2$, then $G^* \in \mathcal{H}(n-1, t)$. Then

$$SO(G) = SO(G^*) + \sqrt{d^2+2^2} + \sqrt{2^2+2^2} - \sqrt{d^2+2^2}$$
$$\leq Q(n-1,t) + 2\sqrt{2}$$
$$= Q(n,t) + 2t\sqrt{(n-2)^2+2^2} - 2t\sqrt{(n-1)^2+2^2}$$
$$+(n-2t-2)\sqrt{(n-2)^2+1^2} - (n-2t-1)\sqrt{(n-1)^2+1^2} + 2\sqrt{2}$$
$$= Q(n,t) + 2t[\sqrt{(n-2)^2+2^2} - \sqrt{(n-1)^2+2^2}]$$
$$+(n-2t-2)[\sqrt{(n-2)^2+1^2} - \sqrt{(n-1)^2+1^2}]$$
$$-\sqrt{(n-1)^2+1^2} + 2\sqrt{2} < Q(n,t)$$

Subcase 2.2. The circle length equals 3. Let $G^* = G - v_1 - v_2$, then $G^* \in \mathcal{H}(n-2, t-1)$. By Lemma 2.3, 2.5, we have

$$O(G) = SO(G^*) + 2\sqrt{d^2+2^2} + \sqrt{2^2+2^2} + \sum_{i=1}^{d-2}\left(\sqrt{d^2+d_{x_i}^2} - \sqrt{(d-2)^2+d_{x_i}^2}\right)$$
$$\leq Q(n-2,t-1) + 2\sqrt{d^2+2^2} + \sqrt{2^2+2^2} + (d-2)(-\sqrt{(d-2)^2+2^2})$$
$$= Q(n,t) + (2t-2)\sqrt{(n-3)^2+2^2} - 2t\sqrt{(n-1)^2+2^2}$$
$$+ (n-2t-1)\sqrt{(n-3)^2+1^2} - (n-2t-1)\sqrt{(n-1)^2+1^2}$$
$$+ 2\sqrt{d^2+2^2} + (d-2)(\sqrt{d^2+2^2} - \sqrt{(d-2)^2+2^2})$$
$$\leq Q(n,t) + (2t-2)\sqrt{(n-3)^2+2^2} - 2t\sqrt{(n-1)^2+2^2}$$
$$+ (n-2t-1)\sqrt{(n-3)^2+1^2} - (n-2t-1)\sqrt{(n-1)^2+1^2}$$
$$+ (n-1)\sqrt{(n-1)^2+2^2} - (n-3)\sqrt{(n-3)^2+2^2}$$
$$= Q(n,t) + (n-2t-1)[\left(\sqrt{(n-1)^2+2^2} - \sqrt{(n-3)^2+2^2}\right)$$
$$-\left(\sqrt{(n-1)^2+1^2} - \sqrt{(n-3)^2+1^2}\right)] \leq Q(n,t),$$

with equality if and only if $G \cong H(n,t)$. This completes the proof. ■

Using a similar way, for the reduced Sombor index, we also have similar result. We omit the proof.

Theorem 3.2. Let $G \in \mathcal{H}(n,t)$ $(n \geq 5)$. Then

$$SO_{red}(G) \leq (n-2t-1)(n-2) + 2t\sqrt{(n-2)^2+1} + \sqrt{2}t,$$

with equality if and only if $G \cong H(n,t)$.

4. MAXIMUM SOMBOR INDEX AMONG CACTI $\mathcal{C}(2\beta, t)$

For convenience, we denote $\Phi(\beta,t) \triangleq SO(H^*(2\beta,t)) = (\beta+t-1)\sqrt{(\beta+t)^2+4} + \sqrt{(\beta+t)^2+1} + \sqrt{5}(\beta-t-1) + 2\sqrt{2}t$. Note that the definition of function $g(x), f(x)$ has introduced in Lemma 2.8. Suppose M is a perfect matching of $G \in \mathcal{C}(2\beta,t)$.

In the following Lemmas 4.1, 4.2 and 4.3, we make inductive assumptions about $\beta+t$. By Lemmas 2.6 and 2.7, these conclusions of Lemmas 4.1, 4.2 and 4.3 hold if $t=0$ or $t=1$. If $\beta=3$, $t=2$, the conclusion holds clearly. So, we only consider $\beta \geq 4, t \geq 2$ in the following.

Lemma 4.1. Let $G \in \mathcal{C}(2\beta,t)$ $(\beta \geq 2)$, $\delta(G) \geq 2$. Then $SO(G) < \Phi(\beta,t)$.

Proof. Suppose the cycle $C = v_1v_2 \dots v_\lambda v_1$ where $3 \leq d_{v_1} \triangleq d \leq \beta+t$, $d_{v_i}=2$ for $i=2,3,\dots,\lambda$. $N(v_1)=\{v_2, v_\lambda, x_1, x_2, \dots x_{d-2}\}$. Let $G^* = G - v_1v_2$, then $G^* \in \mathcal{C}(2\beta, t-1)$ and $SO(G^*) \in \Phi(\beta, t-1)$. By Lemmas 2.3 and 2.8, we have

$$SO(G) = SO(G^*) + \sqrt{d^2+2^2} + (\sqrt{2^2+2^2} - \sqrt{2^2+1^2})$$

$$+ \sum_{i=1}^{d-1}(\sqrt{d^2+d_{x_i}^2} - \sqrt{(d-1)^2+d_{x_i}^2})$$

$$\leq \Phi(\beta,t) + (\beta+t-2)\sqrt{(\beta+t-1)^2+2^2} + \sqrt{(\beta+t-1)^2+1^2}$$

$$-(\beta+t-1)\sqrt{(\beta+t)^2+2^2} - \sqrt{(\beta+t)^2+1^2}$$

$$+ t\sqrt{t^2+2^2} - (t-1)\sqrt{(t-1)^2+2^2}$$

$$= \Phi(\beta,t) + g(\beta+t-1) - g(d-1) + [\left(\sqrt{t^2+2^2} - \sqrt{(t-1)^2+2^2}\right)$$

$$-(\sqrt{t^2+1^2} - \sqrt{(t-1)^2+1^2})] < \Phi(\beta,t).$$

This completes the proof. ■

The support vertices are the vertices connected with pendent vertices.

Lemma 4.2. Let $G \in \mathcal{C}(2\beta,t)$ $(\beta \geq 2)$, $\delta(G)=1$ and there exists a support vertex u with degree 2 of the corresponding pendent vertex v. Then $SO(G) \leq \Phi(\beta,t)$, with equality if and only if $G \cong H^*(2\beta,t)$.

Proof. Let $N(u)=\{v,w\}$, $N(w)=\{u, x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_{d-1}\}$. $d_{x_i}=1$ for $1 \leq i \leq r$, $d_{x_i} \geq 2$ for $r+1 \leq i \leq d-1$. For convenience, denote $d_w = d$. Let $G^* = G - v - u$, then $G^* \in \mathcal{C}(2\beta-2, t)$. By Lemmas 2.3 and 2.8, we have

$$
\begin{aligned}
SO(G) &= SO(G^*) + \sum_{i=r+1}^{d-1}\left(\sqrt{d^2+d_{x_i}^2} - \sqrt{(d-1)^2+d_{x_i}^2}\right) \\
&+ r(\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2}) + \sqrt{d^2+2^2} + \sqrt{2^2+1^2} \\
&\leq \Phi(\beta-1,t) + (d-r-1)(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) \\
&+ r(\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2}) + \sqrt{d^2+2^2} + \sqrt{2^2+1^2} \\
&= \Phi(\beta,t) + (\beta+t-2)\sqrt{(\beta+t-1)^2+2^2} + \sqrt{(\beta+t-1)^2+1^2} \\
&- (\beta+t-1)\sqrt{(\beta+t)^2+2^2} - \sqrt{(\beta+t)^2+1^2} \\
&+ (d-r-1)(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) \\
&+ r(\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2}) + \sqrt{d^2+2^2} \\
&= \Phi(\beta,t) + g(\beta+t-1) + r(\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2}) \\
&+ (d-r-1)(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) + \sqrt{d^2+2^2}.
\end{aligned}
$$

Since $G \in \mathcal{C}(2\beta,t)$, then $r \leq 1$. We consider the following two cases.

Case 1. $r = 0$.

$$
\begin{aligned}
SO(G) &\leq \Phi(\beta,t) + g(\beta+t-1) + (d-1)(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) \\
&+ \sqrt{d^2+2^2} \\
&= \Phi(\beta,t) + g(\beta+t-1) - g(d-1) + [(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) \\
&-(\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2})] < \Phi(\beta,t).
\end{aligned}
$$

Case 2. $r = 1$.

$$
\begin{aligned}
SO(G) &\leq \Phi(\beta,t) + g(\beta+t-1) + (\sqrt{d^2+1^2} - \sqrt{(d-1)^2+1^2}) \\
&+(d-2)(\sqrt{d^2+2^2} - \sqrt{(d-1)^2+2^2}) + \sqrt{d^2+2^2} \\
&= \Phi(\beta,t) + g(\beta+t-1) - g(d-1) \leq \Phi(\beta,t),
\end{aligned}
$$

with equality if and only if $G \cong H^*(2\beta,t)$. This completes the proof. ■

Lemma 4.3. Let $G \in \mathcal{C}(2\beta,t)$ $(\beta \geq 2)$, $\delta(G) = 1$ and the degrees of all support vertices are at least 3. Then $SO(G) \leq \Phi(\beta,t)$, with equality if and only if $G \cong H^*(2\beta,t)$.

Proof. Suppose $C = v_1v_2 \dots v_\lambda v_1$ is such a cycle that $v_i (2 \leq i \leq \lambda)$ does not on other cycles of G, i.e., $d_{v_i} = 2\ or\ 3$ for $2 \leq i \leq \lambda$. $3 \leq d_{v_1} \triangleq d \leq \beta + t$. Without loss of generality, suppose $v_1v_2 \notin M$. In the following, we classify the circle lengths.

Case 1. The circle length equals 3. Let $N(v_1) = \{v_2, v_3, x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_{d-2}\}$ where $r = 0\ or\ 1$, $d_{x_i} = 1$ for $1 \leq i \leq r$, $d_{x_i} \geq 2$ for $r+1 \leq i \leq d-2$.

Subcase 1.1. $d_{v_2} = d_{v_3} = 2$. Let $G^* = G - v_2 - v_3$, then $G^* \in \mathcal{C}(2\beta-2, t-1)$ and $SO(G^*) \leq \Phi(\beta-1, t-1)$.

$$SO(G) = SO(G^*) + \sum_{i=r+1}^{d-2}\left(\sqrt{d^2 + d_{x_i}^2} - \sqrt{(d-2)^2 + d_{x_i}^2}\right)$$
$$+ r\left(\sqrt{d^2 + 1^2} - \sqrt{(d-2)^2 + 1^2}\right) + 2\sqrt{d^2 + 2^2} + \sqrt{2^2 + 2^2}$$
$$\leq \Phi(\beta, t) + (\beta + t - 3)\sqrt{(\beta + t - 2)^2 + 2^2} + \sqrt{(\beta + t - 2)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2}$$
$$+ \sum_{i=r+1}^{d-2}\left(\sqrt{d^2 + d_{x_i}^2} - \sqrt{(d-2)^2 + d_{x_i}^2}\right) + r\left(\sqrt{d^2 + 1^2}\right.$$
$$\left.- \sqrt{(d-2)^2 + 1^2}\right) + 2\sqrt{d^2 + 2^2}.$$

Subcase 1.11. $r = 0$. By Lemma 2.3 and 2.8, we have

$$SO(G) \leq \Phi(\beta, t) + (\beta + t - 3)\sqrt{(\beta + t - 2)^2 + 2^2} + \sqrt{(\beta + t - 2)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2}$$
$$+ (d - 2)\left(\sqrt{d^2 + 2^2} - \sqrt{(d-2)^2 + 2^2}\right) + 2\sqrt{d^2 + 2^2}$$
$$= \Phi(\beta, t) + [f(\beta + t - 2) - f(\beta + t)] - [f(d - 2) - f(d)]$$
$$+ \left[\left(\sqrt{d^2 + 2^2} - \sqrt{(d-2)^2 + 2^2}\right) - \left(\sqrt{d^2 + 1^2} - \sqrt{(d-2)^2 + 1^2}\right)\right]$$
$$< \Phi(\beta, t) + [g(\beta + t - 2) + g(\beta + t - 1)] - [g(d - 2) + g(d - 1)]$$
$$= \Phi(\beta, t) + [g(\beta + t - 2) - g(d - 2)] + [g(\beta + t - 1) - g(d - 1)]$$
$$\leq \Phi(\beta, t).$$

Subcase 1.12. $r = 1$. By Lemma 2.3 and 2.8, we have

$$SO(G) \leq \Phi(\beta, t) + (\beta + t - 3)\sqrt{(\beta + t - 2)^2 + 2^2} + \sqrt{(\beta + t - 2)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2} + \left(\sqrt{d^2 + 1^2} - \sqrt{(d-2)^2 + 1^2}\right)$$
$$+ (d - 3)\left(\sqrt{d^2 + 2^2} - \sqrt{(d-2)^2 + 2^2}\right) + 2\sqrt{d^2 + 2^2}$$
$$= \Phi(\beta, t) + [f(\beta + t - 2) - f(\beta + t)] - [f(d - 2) - f(d)]$$
$$= \Phi(\beta, t) + [g(\beta + t - 2) + g(\beta + t - 1)] - [g(d - 2) + g(d - 1)]$$
$$= \Phi(\beta, t) + [g(\beta + t - 2) - g(d - 2)] + [g(\beta + t - 1) - g(d - 1)]$$
$$\leq \Phi(\beta, t),$$

with equality if and only if $G \cong H^*(2\beta, t)$.

Subcase 1.2. $d_{v_2} = d_{v_3} = 3$.

Let $N(v_2) = \{v_1, v_3, v_2'\}$ and $N(v_3) = \{v_1, v_2, v_3'\}$. Let $G^* = G - v_2' - v_3'$, then $G^* \in \mathcal{C}(2\beta - 2, t)$ and $SO(G^*) \leq \Phi(\beta - 1, t)$. Note that $\beta \geq 4$, $t \geq 2$.

$$SO(G) = SO(G^*) + 2\left(\sqrt{d^2 + 3^2} - \sqrt{d^2 + 2^2}\right) + \sqrt{3^2 + 3^2} + 2\sqrt{3^2 + 1^2} - \sqrt{2^2 + 2^2}$$

$$\leq \Phi(\beta-1,t)+2\left(\sqrt{d^2+3^2}-\sqrt{d^2+2^2}\right)+2\sqrt{10}+\sqrt{2}$$
$$=\Phi(\beta,t)+g(\beta+t-1)+2\left(\sqrt{d^2+3^2}-\sqrt{d^2+2^2}\right)-\sqrt{5}+2\sqrt{10}+\sqrt{2}$$
$$\leq \Phi(\beta,t)+g(5)+2\left(\sqrt{3^2+3^2}-\sqrt{3^2+2^2}\right)-\sqrt{5}+2\sqrt{10}+\sqrt{2}$$
$$=\Phi(\beta,t)+4\sqrt{5^2+2^2}+\sqrt{5^2+1^2}-5\sqrt{6^2+2^2}-\sqrt{6^2+1^2}$$
$$+2\left(\sqrt{3^2+3^2}-\sqrt{3^2+2^2}\right)-\sqrt{5}+2\sqrt{10}+\sqrt{2}<\Phi(\beta,t).$$

Subcase 1.3. $d_{v_2}=2,\ d_{v_3}=3$.

Let $N(v_3)=\{v_1,v_2,v_3'\}$. $N(v_1)=\{v_2,v_3,x_1,,\dots,x_{d-2}\}$ Let $G^*=G-v_3-v_3'$, then $G^*\in \mathcal{C}(2\beta-2,t-1)$ and $SO(G^*)\leq \Phi(\beta-1,t-1)$.

$$SO(G)=SO(G^*)+\sum_{i=1}^{d-2}\left(\sqrt{d^2+d_{x_i}^2}-\sqrt{(d-1)^2+d_{x_i}^2}\right)+\sqrt{d^2+3^2}+\sqrt{d^2+2^2}$$
$$+\sqrt{2^2+3^2}+\sqrt{1^2+3^2}-\sqrt{(d-1)^2+1^2}$$
$$\leq \Phi(\beta-1,t-1)+(d-2)\left(\sqrt{d^2+2^2}-\sqrt{(d-1)^2+2^2}\right)+\sqrt{d^2+3^2}$$
$$+\sqrt{d^2+2^2}+\sqrt{2^2+3^2}+\sqrt{1^2+3^2}-\sqrt{(d-1)^2+1^2}$$
$$\leq \Phi(\beta,t)+(\beta+t-3)\sqrt{(\beta+t-2)^2+2^2}+\sqrt{(\beta+t-2)^2+1^2}$$
$$-(\beta+t-1)\sqrt{(\beta+t)^2+2^2}-\sqrt{(\beta+t)^2+1^2}$$
$$+(d-2)\left(\sqrt{d^2+2^2}-\sqrt{(d-1)^2+2^2}\right)$$
$$+\sqrt{d^2+3^2}+\sqrt{d^2+2^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}-\sqrt{(d-1)^2+1^2}$$
$$=\Phi(\beta,t)+[f(\beta+t-2)-f(\beta+t)]$$
$$+(d-2)\left(\sqrt{d^2+2^2}-\sqrt{(d-1)^2+2^2}\right)$$
$$+\sqrt{d^2+3^2}+\sqrt{d^2+2^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}-\sqrt{(d-1)^2+1^2}$$
$$=\Phi(\beta,t)+[f(\beta+t-2)-f(\beta+t)]-[f(d-1)-f(d)]$$
$$+\sqrt{d^2+3^2}-\sqrt{d^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}$$
$$=\Phi(\beta,t)+[f(\beta+t-1)-f(\beta+t)]-[f(d-1)-f(d)]$$
$$+[f(\beta+t-2)-f(\beta+t-1)]+\sqrt{d^2+3^2}-\sqrt{d^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}$$
$$=\Phi(\beta,t)+[g(\beta+t-1)-g(d-1)]+g(\beta+t-2)$$
$$+\sqrt{d^2+3^2}-\sqrt{d^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}$$
$$\leq \Phi(\beta,t)+g(\beta+t-2)+\sqrt{d^2+3^2}-\sqrt{d^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}.$$

If $d\geq 3$, note that $\beta+t\geq 6$, then $SO(G)\leq \Phi(\beta,t)+g(4)+\sqrt{3^2+3^2}-\sqrt{3^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}=\Phi(\beta,t)+3\sqrt{4^2+2^2}+\sqrt{4^2+1^2}-4\sqrt{5^2+2^2}-\sqrt{5^2+1^2}+\sqrt{3^2+3^2}-\sqrt{3^2+1^2}+\sqrt{13}+\sqrt{10}-2\sqrt{2}<\Phi(\beta,t)$. If $d=2$, then $\beta=2$ and $t=1$. Thus $G\cong H^*(4,1)$.

Case 2. The circle length greater than or equal to 4.

Subcase 2.1. $d_{v_2}=3$.

Let $N(v_2) = \{v_1, v_3, v_2'\}$, then $d_{v_3} = 2\ or\ 3$. Let $G^* = G - v_2 - v_2' + v_1v_3$, then $G^* \in \mathcal{C}(2\beta - 2, t)$ and $SO(G^*) \le \Phi(\beta - 1, t)$.

$$SO(G) = SO(G^*)+\sqrt{d^2 + 3^2} + \sqrt{3^2 + 1^2} + \sqrt{3^2 + d_{v_3}^2} - \sqrt{d^2 + d_{v_3}^2}$$
$$\le \Phi(\beta - 1, t)+\sqrt{d^2 + 3^2} + \sqrt{3^2 + 1^2} + \sqrt{3^2 + d_{v_3}^2} - \sqrt{d^2 + d_{v_3}^2}$$
$$= \Phi(\beta, t) + (\beta + t - 2)\sqrt{(\beta + t - 1)^2 + 2^2}+\sqrt{(\beta + t - 1)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2} - \sqrt{5}$$
$$+ \sqrt{d^2 + 3^2} + \sqrt{3^2 + 1^2} + \sqrt{3^2 + d_{v_3}^2} - \sqrt{d^2 + d_{v_3}^2}.$$

Subcase 2.11. $d_{v_3} = 2$.

$$SO(G) \le \Phi(\beta, t) + (\beta + t - 2)\sqrt{(\beta + t - 1)^2 + 2^2}+\sqrt{(\beta + t - 1)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2} - \sqrt{5}$$
$$+ \sqrt{d^2 + 3^2} - \sqrt{d^2 + 2^2} + \sqrt{10} + \sqrt{13}$$
$$\le \Phi(\beta, t) + g(\beta + t - 1) + 3\sqrt{2} + \sqrt{10} - \sqrt{5}$$
$$\le \Phi(\beta, t) + g(5) + 3\sqrt{2} + \sqrt{10} - \sqrt{5}$$
$$= \Phi(\beta, t) + 4\sqrt{29} + \sqrt{26} - 5\sqrt{40} - \sqrt{37} + 3\sqrt{2} + \sqrt{10} - \sqrt{5} < \Phi(\beta, t).$$

Subcase 2.12. $d_{v_3} = 3$.

$$SO(G) \le \Phi(\beta, t) + (\beta + t - 2)\sqrt{(\beta + t - 1)^2 + 2^2}+\sqrt{(\beta + t - 1)^2 + 1^2}$$
$$- (\beta + t - 1)\sqrt{(\beta + t)^2 + 2^2} - \sqrt{(\beta + t)^2 + 1^2} - \sqrt{5}+\sqrt{10} + 3\sqrt{2}$$
$$\le \Phi(\beta, t) + g(\beta + t - 1) + 3\sqrt{2} + \sqrt{10} - \sqrt{5}$$
$$\le \Phi(\beta, t) + g(5) + 3\sqrt{2} + \sqrt{10} - \sqrt{5} < \Phi(\beta, t).$$

Subcase 2.2. $d_{v_2} = 2$.

Since $v_1v_2 \notin M$, then $v_2v_3 \in M$, so $d_{v_3} = 2$. Let $G^* = G - v_3v_4 + v_2v_4$, then $G^* \in \mathcal{C}(2\beta, t)$.

$$SO(G) = SO(G^*) + \sqrt{d^2 + 2^2} - \sqrt{d^2 + 3^2}+\sqrt{2^2 + 2^2}+\sqrt{d_{v_4}^2 + 2^2}$$
$$- \sqrt{1 + 3^2} - \sqrt{d_{v_4}^2 + 3^2}$$
$$\le SO(G^*) + \sqrt{d^2 + 2^2} - \sqrt{d^2 + 3^2}+4\sqrt{2} - \sqrt{10} - \sqrt{13}$$
$$< SO(G^*)+4\sqrt{2} - \sqrt{10} - \sqrt{13} < SO(G^*),$$

when $\lambda - 1 = 3$, by Case 1, we know $SO(G^*) \le \Phi(\beta, t)$; when $\lambda - 1 \ge 4$, by Case 2.1, we know $SO(G^*) < \Phi(\beta, t)$. Thus, we have $SO(G) < \Phi(\beta, t)$.

This completes the proof. ■

By Lemma 4.1, 4.2 and 4.3, we can obtain the maximum Sombor index among cacti $\mathcal{C}(2\beta, t)$

Theorem 4.4. Let $G \in \mathcal{C}(2\beta, t)$ $(\beta \geq 2)$. Then $SO(G) \leq \Phi(\beta, t)$, with equality if and only if $G \cong H^*(2\beta, t)$.

Using a similar way, for the reduced Sombor index, we also have similar result. We omit the proof.

Theorem 4.5. Let $G \in \mathcal{C}(2\beta, t)$ $(\beta \geq 2)$. Then $SO_{red}(G) \leq SO_{red}(H^*(2\beta, t))$, with equality if and only if $G \cong H^*(2\beta, t)$.

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