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A New Notion of Energy of Digraphs

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ABSTRACT

The eigenvalues of a digraph are the eigenvalues of its adjacency matrix. Let $z_1, \dots z_n$ be the eigenvalues of an n-vertex digraph D. Here, we give a new notion of energy of digraphs defined by $\mathrm{E_p}(D) = \sum_{k=1}^{n} |\Re(\mathrm{z_k})\Im(\mathrm{z_k})|$, where $\Re(\mathrm{z_k})$ and $\Im(\mathrm{z_k})$ are real and imaginary parts of $\mathrm{z_k}$, respectively. We call it p-energy of the digraph D and compute the p-energy formulas for directed cycles. For $n \geq 12$, it is shown that p-energy of directed cycles increases monotonically with respect to their order. The unicyclic digraphs with smallest and largest p-energy are obtained and counter examples will be presented to show that the p-energy of a digraph does not possess increasing-property with respect to quasi-order relation over the set $D_{n,h}$, where $D_{n,h}$ is the set of n-vertex digraphs with cycles of length h. Also, an upper bound for the p-energy is presented and give all digraphs which attain this bound.

1. INTRODUCTION

Gutman [6] was the first who introduced the notion of energy of graphs. He defined the energy of a simple graph as the sum of the absolute values of its eigenvalues. For applications of graph energy in chemistry, we refer to [7]. A directed graph (or digraph) D consists of two finite sets (V, A) where V denotes the vertex-set and A represents the set of arcs. For two vertices u and v, an arc from u to v is denoted by uv. Two vertices u and v are said to be adjacent if either $uv \in A$ or $vu \in A$. A directed path (respectively, directed cycle) on n vertices is denoted by P_n (respectively, C_n). Let $\{v_1, \dots, v_n\}$ be the vertex-set of

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a directed path P_n and also of a directed cycle C_n. Then the sets $\{v_k v_{k+1} \mid k = 1, \dots, n-1\}$ and $\{ v_k v_{k+1} \mid k = 1, \dots, n-1 \} \cup \{v_n v_1\}$ are the arc-sets of P_n and C_n, respectively.

For a vertex $v \in V(D)$, the set of in-neighbors of v is defined as $N_{D^-}(v) = \{u \in V(D) \mid uv \in A(D)\}$. Similarly, $N_{D^+}(v) = \{u \in V(D) \mid vu \in A(D)\}$ is the set of out-neighbors of u. The in-degree and out-degree of v are the cardinality of $N_{D^-}(v)$ and $N_{D^+}(v)$, respectively. A digraph with this property that $|N_{D^-}(v)| = |N_{D^+}(v)| = 1$, for each $v \in V(D)$, is called a linear-digraph. A class of digraphs U_n is said to be a class of unicyclic digraphs if each digraph in U_n contains a unique directed cycle. If we remove the direction of each arc in D then the resulting graph is called underlying graph of D, which we denote it by D_U. A digraph D is said to be weakly connected if D_U is connected. Similarly, a digraph $D = (V, A)$ is called strongly connected if for every pair of vertices u and v it contains P_{uv} as a subdigraph, where P_{uv} is a directed path from u to v. If a digraph is not connected then its maximal connected subdigraphs are called components. A strongly connected component of a digraph is called strong component.

The Cartesian product of two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ is denoted by $D_1 \times D_2$ and is defined as a digraphs $D = (V, A)$ with vertex-set $V = V_1 \times V_2$ and the arc set A in which $A = \{(x, u)(y, v) \mid [xy \in A_1 \; and \; u = v] \; or \; [x = y \; and \; uv \in A_2]\}$. The adjacency matrix $A(D) = \left[a_{ij}\right]_{n\times n}$ of an n-vertex digraph $D = (V, A)$ can be defined as:

$$a_{ij} = \begin{cases} 1 & v_i v_j \in A, \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial $\Phi_D(x) = \det(xI - A(D))$ of the adjacency matrix $A(D)$ of a digraph D is called the characteristic polynomial of D and its eigenvalues are called the eigenvalues of D. From definition of adjacency matrix of D, we observe that $A(D)$ is not necessarily symmetric. Thus, the eigenvalues of a digraph may be complex number.

The concept of graph energy was extended to digraph by Peña and Rada [12] and Adiga et al. [1]. Since the zeros of $\Phi_D(x)$ are not necessarily real, the authors in [12] defined energy for an n-vertex digraph as $\mathrm{E}(D) = \sum_{k=1}^{n} |\Re(\mathrm{z_k})|$, where $z_1, \dots, z_n$ are the zeros of $\Phi_D(x)$ and $\Re\,(z_k)$ is the real part of z_k. Along with some other useful results the authors had found digraphs in the set U_n with minimal and maximal energy.

Khan et al. [9] defined a new notion of energy of digraph which they called iota energy. They defined iota energy of an n-vertex digraph D as $\mathrm{E}(D) = \sum_{k=1}^{n} |\Im(\mathrm{z_k})|$, where z_k, $k = 1, \dots, n$ are the zeros of $\Phi_D(x)$ and $\Im\,(z_k)$ is the imaginary part of z_k. We refer to [5,8,11], for the extremal and iota energies of bicyclic digraphs.

In this paper, we give a new notion of digraph energy that will be called p-energy. This is defined as the sum of the absolute values of the product of real and imaginary parts of its eigenvalues. We find the smallest and the largest p-energy among all n-vertex digraphs in the set U_n, $n \geq 12$. It is shown that $D_{n,h}$, which is the set of n-vertex digraphs with cycles of length h does not possess the increasing property with respect to the quasi order relation. An upper bound for the p-energy of digraphs were presented. At the end, we will find few families of p-equienergetic digraphs.

2. p–ENERGY OF DIGRAPHS

Since the adjacency matrix of a digraph D need not to be symmetric, the eigenvalues of D are not necessarily real. Let $z_1, \dots, z_n$ be the eigenvalues of D. The p-energy of D is defined as follows:

$$E_p = \sum_{k=1}^{n} |\Re(z_k)\Im(z_k)|, \tag{2.1}$$

where $\Re(z_k)$ and $\Im(z_k)$ are respectively, the real and imaginary parts of z_k. The characteristic polynomial of a digraph is given in the following theorem.

Theorem 2.1 ([Theorem 1.2, 3]) Let D be a digraph on n vertices and $\Phi_D(x) = x^n + \sum_{k=1}^{n} b_k x^{n-k}$ be its characteristic polynomial. Then $b_k = \sum_{L \in L_k} (-1)^{comp(L)}$, where L_k is the set of all linear subdigraphs L of D with exactly k vertices and $comp(L)$ is the number of components in L.

Example 2.2. Let D be an acyclic digraph on n vertices. Then by above theorem the characteristic polynomial of D is given by $\Phi_D(x) = x^n$. This gives that $E_p(D) = 0$.

A digraph D is said to be a symmetric digraph if and only if its adjacency matrix is equal to the adjacency matrix of its underlying graph. This implies that all eigenvalues of a symmetric digraph are real. Thus a symmetric digraph has zero p-energy. The converse is not necessarily true, that is, we have non-symmetric digraphs with zero p-energy. The following is an example of digraph which supports our assertion.

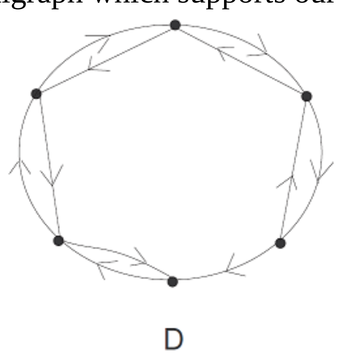

Figure 1: A non-symmetric digraph.

Example 2.3. Let D be a digraph with 6 vertices, see Figure 1. Then by applying Theorem 2.1, we get $\Phi_D(x) = x^6 - 5x^4 + 6x^2 - 2 = (x-1)(x+1)(x^4 - 4x^2 + 2)$. This gives $Spec(D) = \left\{\pm 1,\ \pm\sqrt{2+\sqrt{2}},\ \sqrt{2-\sqrt{2}}\ \right\}$. Thus $E_p(D) = 0$.

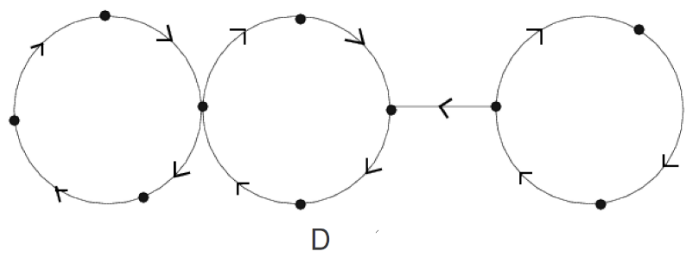

Figure 2: A digraph with three directed cycles.

The following is an example of a digraph with non-zero p-energy.

Example 2.4. Let D be a digraph of order 10, Figure 2. Then applying Theorem 2.1, we get $\Phi_D(x) = x^{10} - x^7 - 2x^6 + 2x^3 = x^3(x^3-1)(x^4-2)$. This gives $Spec(D) = \left\{0^3, \pm 2^{\frac{1}{4}}, \pm i 2^{\frac{1}{4}}, \frac{-1\pm i\sqrt{3}}{2}, 1\right\}$. Thus $E_p(D) = \frac{\sqrt{3}}{2}$.

The following formulas are well known and will be useful in finding p-energy formulas of directed cycles.

$$\sum_{k=0}^{n-1} \left|sin\frac{2k\pi}{n}\right| = \begin{cases} \cot\frac{\pi}{2n} & if\ n \equiv 1(mod\ 2), \\ 2\cot\frac{\pi}{n} & if\ n \equiv 0(mod\ 2), \end{cases} \tag{2.2}$$

$$\sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}| = \begin{cases} 2\cot\frac{\pi}{n} & if\ n \equiv 0(mod\ 4), \\ \cot\frac{\pi}{2n} & if\ n \equiv 1(mod\ 2), \\ 2\csc\frac{\pi}{n} & if\ n \equiv 2(mod\ 4). \end{cases} \tag{2.3}$$

3. p–ENERGY OF DIRECTED CYCLES

Let C_n be a directed cycle of order n, where $n \geq 2$. Then by using Theorem 2.1, we get the characteristic polynomial of C_n as $\Phi_{C_n}(x) = x^n - 1$. Thus the spectrum of C_n is given by:

$$Spec(C_n) = \left\{\cos\frac{2k\pi}{n} + i\ \sin\frac{2k\pi}{n} \mid k = 0,1,\dots,n-1\right\}. \tag{3.4}$$

This gives:

$$E_p(C_n) = \sum_{k=0}^{n-1} \left|\left(\cos\frac{2k\pi}{n}\right)\left(\sin\frac{2k\pi}{n}\right)\right| = \frac{1}{2}\sum_{k=0}^{n-1}\left|\sin\frac{4k\pi}{n}\right|. \tag{3.5}$$

We derive the p-energy formulas of C_n, $n \geq 2$, considering the following three cases:

Case 1. If n is an integral multiple of 4 then from (3.5), we have

$$
\begin{aligned}
E_p(C_n) &= \sum_{k=0}^{\frac{n}{2}-1}\left|\sin\frac{4k\pi}{n}\right| = 2\sum_{k=0}^{\frac{n}{4}-1}\left|\sin\frac{4k\pi}{n}\right| \\
&= 2\left(\sin\left(\left(\frac{n}{4}-1\right)\frac{2\pi}{n}\right)\sin\left(\left(\frac{n}{4}-1+1\right)\ \frac{2\pi}{n}\ \right)\csc\frac{2\pi}{n}\right) \\
&= 2\cot\frac{2\pi}{n}.
\end{aligned}
$$

Case 2. If n is a positive integer of the form $n \equiv 2\ (mod\ 4)$ then

$$
\begin{aligned}
E_p(C_n) &= \sum_{k=0}^{\frac{n}{2}-1}\left|\sin\frac{4k\pi}{n}\right| = 2\sum_{k=0}^{\frac{n-2}{4}}\sin\frac{2k\pi}{n} \\
&= 2\left(\sin\left(\frac{n-2}{4}\cdot\frac{\pi}{n}\right)\sin\left(\left(\frac{n-2}{4}+1\right)\ \frac{\pi}{n}\ \right)\csc\frac{\pi}{n}\right) \\
&= \cot\frac{\pi}{n}.
\end{aligned}
$$

Case 3. Finally, if n is a positive integer of the form $n \equiv 1(mod\ 2)$ then

$$
\begin{aligned}
E_p(C_n) &= \frac{1}{2}\sum_{k=0}^{n-1}\left|\sin\frac{2k\pi}{n}\right| = \frac{1}{2}\sum_{k=0}^{n-1}\sin\frac{k\pi}{n} \\
&= \frac{1}{2}\left(\sin\left(\frac{n-1}{2}\cdot\frac{\pi}{n}\right)\sin\left(\frac{n}{2}\cdot\ \frac{\pi}{n}\ \right)\csc\frac{\pi}{2n}\right) \\
&= \frac{1}{2}\cot\frac{\pi}{2n}.
\end{aligned}
$$

Briefly, we can write

$$
E_p(C_n) = \begin{cases} 2\cot\frac{2\pi}{n} & if\ n \equiv 0(mod\ 4), \\ \frac{1}{2}\cot\frac{\pi}{2n} & if\ n \equiv 1(mod\ 2), \\ \cot\frac{\pi}{n} & if\ n \equiv 2(mod\ 4). \end{cases} \tag{3.6}
$$

Let $D \in U_n$ be a digraph with cycle C_r of length r $(2 \leq r \leq n)$. Then by using Theorem 2.1, we obtain $\Phi_D(x) = x^n - x^{n-r} = x^{n-r}(x^r - 1)$. This yields

$$
E_p(D) = E_p(C_r). \tag{3.7}
$$

The following lemma will be useful in finding smallest and largest p-energy of digraphs in the set U_n. We remark that the idea of the proof is taken from Lemma 3.5 [2].

Lemma 3. 1. For $n \geq 12$, the following sequence is an increasing sequence:

$$
\langle a_n \rangle = \begin{cases} 2\cot\frac{2\pi}{n} & if\ n \equiv 0(mod\ 4), \\ \frac{1}{2}\cot\frac{\pi}{2n} & if\ n \equiv 1(mod\ 2), \\ \cot\frac{\pi}{n} & if\ n \equiv 2(mod\ 4). \end{cases}
$$

Proof. This lemma can be proved easily by proving the following inequalities:

$$
2\cot\frac{2\pi}{n} < \frac{1}{2}\cot\frac{\pi}{2n+2} < \cot\frac{\pi}{n+2} < \frac{1}{2}\cot\frac{\pi}{2n+6} < 2\cot\frac{2\pi}{n+6}\ . \tag{3.8}
$$

□

Theorem 3.2. Among all digraphs in the set U_n, the digraphs which contain a cycle C_4 or C_2 has smallest p-energy. For maximal p-energy, we have the following two cases:

(1) If $n \in \{4,8,12\}$ then a digraph in the set U_n containing a directed cycle of length $n-1$ together with a pendent vertex has largest p-energy.

(2) If n is not in the set $\{4,8,12\}$ then C_n has largest p-energy.

Proof. Let D be a digraph in the set U_n and C_r be its unique directed cycle, where $2 \leq r \leq n$. Then (3.7) implies that $E_p(D) = E_p(C_r)$. From (3.6), it is easy to see that $E_p(C_2) = E_p(C_4) = 0$. Thus the digraph in the set U_n which contains a directed cycle of length 2 or 4 have smallest p-energy. For largest p-energy, (1) and (2) follow from (3.6) and Lemma 3.1.

Next, we calculate p-energy formulas for a polynomial of the form $\phi(x) = x^h - (a+ib)$. The zeros of $\phi(x)$ are easy to find and are given by:

$$x = r^{\frac{1}{h}}\left(\cos\frac{\theta+2k\pi}{h} + \iota \sin\frac{\theta+2k\pi}{h}\right),$$

where $r = \sqrt{a^2+b^2}, a = rcos\theta, b = rsin\,\theta$ and θ is the principal argument and $k \in \{0,1,\dots,h-1\}$. This gives:

$$\begin{aligned} E_p(\phi(x)) &= r^{\frac{2}{h}} \textstyle\sum_{k=0}^{h-1} \left|\left(\cos\frac{\theta+2k\pi}{h}\right)\left(\sin\frac{\theta+2k\pi}{h}\right)\right| \\ &= \frac{1}{2} r^{\frac{2}{h}} \textstyle\sum_{k=0}^{h-1} \left|\cos\frac{2\theta+4k\pi}{h}\right|. \end{aligned} \quad (3.9)$$

If $h \equiv 0\ (mod\ 4)$, then (3.9) gives

$$E_p(\phi(x)) = r^{\frac{2}{h}} \textstyle\sum_{k=0}^{\frac{h}{2}-1} \left|\sin\frac{2\theta+4k\pi}{h}\right| = 2\, r^{\frac{2}{h}} \textstyle\sum_{k=0}^{\frac{h}{4}-1} \left|\sin\frac{2\theta+4k\pi}{h}\right|. \quad (3.10)$$

Using geometric sum formula and some basic trigonometric identities, we get

$$\begin{aligned} E_p(\phi(x)) &= 2r^{\frac{2}{h}}\left(\sin\left(\frac{2\theta}{h}+\frac{h}{4}\cdot\frac{2\pi}{h}\right)\sin\left(\frac{h-4}{4}\cdot\frac{2\pi}{h}\right)\csc\frac{2\pi}{h} + \left|\sin\frac{2\theta}{h}\right|\right) \\ &= 2r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \cos\frac{2\theta}{h}\cot\frac{2\pi}{h}\right) \\ &= 2r^{\frac{2}{h}}\left|\sin\frac{2\theta}{h}\right| + r^{\frac{2}{h}}\cos\frac{2\theta}{h} E_p(C_h). \end{aligned}$$

If $h \equiv 2\ (mod 4)$, then (3.9) gives

$$E_p(\phi(x)) = r^{\frac{2}{h}} \textstyle\sum_{k=0}^{\frac{h}{2}-1} \left|\sin\frac{2\theta+4k\pi}{h}\right| = r^{\frac{2}{h}} \textstyle\sum_{k=0}^{\frac{h}{2}-1} \left|\sin\frac{2\theta+2k\pi}{h}\right|. \quad (3.11)$$

Using the geometric sum formula and some basic trigonometric identities, we get

$$\begin{aligned} E_p(\phi(x)) &= r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \sin\left(\frac{2\theta}{h}+\frac{h}{2}\cdot\frac{\pi}{h}\right)\sin\left(\frac{h-2}{2}\cdot\frac{\pi}{h}\right)\csc\frac{\pi}{h}\right) \\ &= r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \cos\frac{2\theta}{h}\cot\frac{\pi}{h}\right) \end{aligned}$$

$$= 2r^{\frac{2}{h}}\left|\sin\frac{2\theta}{h}\right| + r^{\frac{2}{h}}\cos\frac{2\theta}{h}E_p(C_h).$$

If $h \equiv 1\ (mod2)$, then (3.9) gives

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\sum_{k=0}^{h-1}\left|\sin\frac{2\theta+2k\pi}{h}\right| = \frac{1}{2}\ r^{\frac{2}{h}}\sum_{k=0}^{h-1}\left|\sin\ \frac{2\theta+k\pi}{h}\right|. \tag{3.12}$$

If $-\pi \le \theta \le -\frac{\pi}{2}$ then (3.12) becomes:

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\sin\left(\frac{2\theta}{h}+\frac{\pi}{h}\right)\right| + \sum_{k=2}^{h-1}\left|\sin\ \frac{2\theta+k\pi}{h}\right|\right). \tag{3.13}$$

Using geometric sum formula and some basic trigonometric identities, we get

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\sin\left(\frac{2\theta}{h}+\frac{\pi}{h}\right)\right| + \sin\left(\frac{2\theta}{h}+\frac{h+1}{2}\cdot\frac{\pi}{h}\right)\sin\left(\frac{h-2}{2}\cdot\frac{\pi}{h}\right)\csc\frac{\pi}{2h}\right)$$
$$= \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\sin\left(\frac{2\theta}{h}+\frac{\pi}{h}\right)\right| + \cos\left(\frac{2\theta}{h}+\frac{\pi}{2h}\right)\cos\frac{\pi}{h}\csc\frac{\pi}{2h}\right).$$

If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ then (3.12) becomes:

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \sin\left(\frac{2\theta}{h}+(h-1+1)\cdot\frac{\pi}{2h}\right)\sin\left((h-1)\cdot\frac{\pi}{2h}\right)\csc\frac{\pi}{2h}\right)$$
$$= \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \cos\frac{2\theta}{h}\cot\frac{\pi}{h}\right).$$

If $\frac{\pi}{2} \le \theta \le \pi$ then (3.12) becomes:

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\sin\left(\frac{2\theta}{h}+\frac{(h-1)\pi}{h}\right)\right| + \sum_{k=1}^{h-2}\left|\sin\ \frac{2\theta+k\pi}{h}\right|\right).$$

Using geometric sum formula and some basic trigonometric identities, we get

$$E_p(\phi(x)) = \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\cos\left(\frac{2\theta}{h}-\frac{\pi}{h}\right)\right| + \sin\left(\frac{2\theta}{h}+(h-1)\cdot\frac{\pi}{2h}\right)\sin\left((h-2)\cdot\frac{\pi}{2h}\right)\csc\frac{\pi}{2h}\right)$$
$$= \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + \left|\cos\left(\frac{2\theta}{h}-\frac{\pi}{h}\right)\right| + \cos\left(\frac{2\theta}{h}-\frac{\pi}{2h}\right)\cos\frac{\pi}{2h}\csc\frac{\pi}{2h}\right).$$

In summary, we can write $E_p(\phi(x))$ by the following formula:

$$\begin{cases} 2r^{\frac{2}{h}}\left|\sin\frac{2\theta}{h}\right| + r^{\frac{2}{h}}\cos\frac{2\theta}{h}E_p(C_h) & if\ h \equiv 0\ (mod\ 4), \\ r^{\frac{2}{h}}\left|\sin\frac{2\theta}{h}\right| + r^{\frac{2}{h}}\cos\frac{2\theta}{h}E_p(C_h) & if\ h \equiv 2\ (mod\ 4), \\ \frac{1}{2}r^{\frac{2}{h}}\left(u + 2\cos\frac{\pi}{h}\cos\frac{2\theta}{h}E_p(C_h)\right) & if\ h \equiv 1\ (mod\ 2)\ and\ \ -\pi \le \theta \le -\frac{\pi}{2}, \\ \frac{1}{2}r^{\frac{2}{h}}\left(\left|\sin\frac{2\theta}{h}\right| + 2\cos\frac{2\theta}{h}E_p(C_h)\right) & if\ h \equiv 1\ (mod\ 2)\ and\ \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \\ \frac{1}{2}r^{\frac{2}{h}}\left(v + 2\cos\frac{\pi}{h}\cos\frac{2\theta}{h}E_p(C_h)\right) & if\ h \equiv 1\ (mod\ 2)\ and\ \ \frac{\pi}{2} \le \theta \le \pi. \end{cases} \tag{3.14}$$

where $u = \left|\sin\frac{2\theta}{h}\right| + \left|\sin\ \left(\frac{2\theta}{h}+\frac{\pi}{h}\right)\right| - \sin\frac{2\theta}{h}\cos\frac{\pi}{h}$ and $v = \left|\sin\frac{2\theta}{h}\right| + \left|\cos\ \left(\frac{2\theta}{h}-\frac{\pi}{h}\right)\right| + \sin\frac{2\theta}{h}\cos\frac{\pi}{h}$. □

4. INTEGRAL REPRESENTATION AND INCREASING PROPERTY OF P−ENERGY

Another representation of graph energy and digraph energy is the integral representation. This representation is useful as one can find graph energy and digraph energy without finding the zeros of its characteristic polynomial. In this section, we study the integral representation of p-energy of digraphs. For this we will denote the principal value of the improper integral $\int_{-\infty}^{\infty} F(x)dx$ by $p.v \int_{-\infty}^{\infty} F(x)dx$. The following theorem is important and will be useful to represent p-energy of a digraph in integral form. The proof is similar to the proof of Theorem 3.3 [4].

Theorem 4.1. Let $z_1, \dots, z_n$ be the zeros of a real monic nth degree polynomial $\phi(x)$ and $\Im(z_k)$ be the imaginary part of z_k. Then

$$\sum_{k=1}^{n} |\Im(z_k)| = \sum_{k=1}^{n} \operatorname{sgn}\big(\Im(z_k)\big)\, z_k = \frac{1}{\pi}\ p.v\ \int_{-\infty}^{\infty} \left(n - \frac{x\phi\prime(x)}{\phi(x)}\right) dx.$$

Let A be the adjacency matrix of a digraph D and $z = x + \dot{\iota} y$ be its eigenvalue. Then it is easy to see that $z^2 = x^2 - y^2 + 2\,\dot{\iota}\ xy$ is an eigenvalue of A^2 (square of the adjacency matrix A). That is, $\Re(z)\Im(z) = \frac{1}{2}\Im(z^2)$. This gives $E_p(D) = \frac{1}{2}\sum_{k=1}^{n}|\Im(z_k')|$, where $z_1', \dots, z_n'$ are the eigenvalues of A^2.

The following theorem is the integral definition of p-energy of digraphs. The proof is similar to the proof of Theorem 3.3 [4].

Theorem 4.2. Let A be the adjacency matrix of a digraph D. Then

$$E_p(D) = \frac{1}{2\pi}\ p.v\ \int_{-\infty}^{\infty} \left(n - \frac{x\psi\prime(x)}{\psi(x)}\right) dx,$$

where $\psi(x)$ is the characteristic polynomial of A^2.

Next we study the increasing property of digraphs in the set $D_{n,h}$, where $D_{n,h}$ is the set of n-vertex digraphs which contain cycles of order h. The following theorem is well-known.

Theorem 4.3 (Pena and Rada [13]). Let D be a digraph in the set $D_{n,h}$. Then the characteristic polynomial of D is:

$$\Phi_D(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (-1)^k\, c(D, kh) x^{n-kh},$$

where $c(D, kh)$ denotes the number of linear subdigraphs of order kh, $k = 1,2, \dots, \lfloor \frac{n}{h} \rfloor$.

The following is the definition of quasi-order relation over $D_{n,h}$.

Definition 4.1 (Pena and Rada [13]). Let D_1 and D_2 be two digraphs in the set $D_{n,h}$. Then, $D_1 \preccurlyeq D_2$ if for every $k = 1,2,\dots,\lfloor\frac{n}{h}\rfloor$ we have $c(D_1,kh) \leq c(D_2,kh)$. If $D_1 \preccurlyeq D_2$ and there exists an integer k such that $c(D_1,kh) < c(D_2,kh)$, then we say that $D_1 \prec D_2$. It is obvious to see that $\preccurlyeq$ satisfies reflexive and transitive property over the set $D_{n,h}$.

The p-energy formulae of directed cycle are known. Next, we find p-energy formulae for digraph in the set $D_{n,h}$.

Theorem 4.4. Let D be a digraph in the set $D_{n,h}$. Then

$$E_p(D) = \begin{cases} f_1(h)E_p(C_h) + f_2(h) & if \ \ h \equiv 2 \ (mod \ 4), \\ f_3(h)E_p(C_h) + f_4(h) & if \ \ h \equiv 1 \ (mod \ 2), \\ f_1(h)E_p(C_h) + 2f_2(h) & if \ \ h \equiv 0 \ (mod \ 4), \end{cases}$$

where $f_i(h)$, for $i = 1,2,3$ are functions which depend on the eigenvalues of D.

Proof. Let D be a digraph in the set $D_{n,h}$ be a digraph. Then applying Theorem 4.3 we get $\Phi_D(x) = x^n + \sum_{k=1}^{\lfloor\frac{n}{h}\rfloor}(-1)^k c(D,kh)x^{n-kh}$, where it is clear from the context that we write c_{kh} instead of $c(D,kh)$. Next, let $s = \lfloor\frac{n}{h}\rfloor$ and put $\Phi_D(x) = 0$. Then we obtain $x^{n-sh}(x^{sh} + \sum_{k=1}^{s}(-1)^k c_{kh}x^{sh-kh}) = 0$. This gives $x^{sh} + \sum_{k=1}^{s}(-1)^k c_{kh}x^{sh-kh} = 0$. Rearranging the equation we get:

$$(x^h)^s + \sum_{k=1}^{s}(-1)^k c_{kh}(x^h)^{s-k} = 0. \tag{4.15}$$

For $j = 1,\dots,s$ let $x_j^h = a_j + ib_j$ be the zeros of (4.15) and its polar form is:

$$x_j = r^{\frac{1}{h}}\left(\cos\theta_j + i\sin\theta_j\right)^{\frac{1}{h}},$$

where $a_j = r_j\cos\theta_j$, $b_j = r_j\sin\theta_j$, $r_j = (a^2+b^2)^{\frac{1}{2}}$ and θ_j is the principal argument, $j = 1,\dots,s$. Thus, for each $j = 1,\dots,s$ and $k = 0,1,2,\dots, h-1$ it holds that

$$x_j = r^{\frac{1}{h}}\left(\cos\left(\frac{\theta_j+2k\pi}{h}\right) + i\sin\left(\frac{\theta_j+2k\pi}{h}\right)\right). \tag{4.16}$$

Using the above equation, we calculate p-energy of the digraph D as follows:

$$\begin{aligned} E_p(D) &= \sum_{j=1}^{s}\sum_{k=0}^{h-1} r_j^{\frac{2}{h}}\left|\cos\left(\frac{\theta_j+2k\pi}{h}\right)\sin\left(\frac{\theta_j+2k\pi}{h}\right)\right| \\ &= \frac{1}{2}\sum_{j=1}^{s}\sum_{k=0}^{h-1} r_j^{\frac{2}{h}}\left|\sin\left(\frac{2\theta_j+4k\pi}{h}\right)\right|. \end{aligned} \tag{4.17}$$

If $h \equiv 0\ (mod 4)$ then by using (3.14), we get

$$E_p(D) = \sum_{j=1}^{s} r_j^{\frac{2}{h}}\left(\cos\frac{2\theta_j}{h}E_p(C_h) + 2\left|\sin\frac{2\theta_j}{h}\right|\right). \tag{4.18}$$

Next, let $h \equiv 2\ (mod 4)$. Then (4.17) and (3.14) imply that:

$$E_p(D) = \sum_{j=1}^{s} r_j^{\frac{2}{h}}\left(\cos\frac{2\theta_j}{h}E_p(C_h) + \left|\sin\frac{2\theta_j}{h}\right|\right). \tag{4.19}$$

Finally, if $h \equiv 1\ (mod\ 2)$ then we divide the roots of (4.15) into three types. Type 1 are the roots $x_{j_1}^h = a_{j_1} + ib_{j_1},\ j_1 = 1, \dots, s_1$ of (4.15) for which the principal argument of $a_{j_1} + ib_{j_1}$ lies in $\left(-\pi, -\frac{\pi}{2}\right)$. Type 2: are the roots $x_{j_2}^h = a_{j_2} + ib_{j_2}, j_2 = 1, \dots, s_2$ of (4.15) for which the principal argument of $a_{j_2} + ib_{j_2}$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Type 3 are the roots $x_{j_3}^h = a_{j_3} + ib_3, j_3 = 1, \dots, s_3$ of (4.15) for which the principal argument of $a_{j_3} + ib_{j_3}$ lies in $\left(\frac{\pi}{2}, \pi\right)$. Here $s = s_1 + s_2 + s_3$.

$$E_p(D) = \frac{1}{2}\ (\textstyle\sum_{j_1=1}^{s_1} \sum_{k=0}^{h-1} r_{j_1}^{\frac{2}{h}} \left|\sin\left(\frac{2\theta_{j_1} + 4k\pi}{h}\right)\right| + \sum_{j_2=1}^{s_2} \sum_{k=0}^{h-1} r_{j_2}^{\frac{2}{h}} \left|\sin\left(\frac{2\theta_{j_2} + 4k\pi}{h}\right)\right|$$
$$+ \textstyle\sum_{j_3=1}^{s_3} \sum_{k=0}^{h-1} r_{j_3}^{\frac{2}{h}} \left|\sin\left(\frac{2\theta_{j_3} + 4k\pi}{h}\right)\right|).$$

Using (3.15), we get

$$E_p(D) = \frac{1}{2}(\textstyle\sum_{j_!}^{s_1} r_{j_1}^{\frac{2}{h}} \left(u_{j_1} + 2\cos\frac{\pi}{h}\cos\frac{2\theta_{j_1}}{h} E_p(C_h)\right)$$
$$+ \textstyle\sum_{j_2}^{s_2} r_{j_2}^{\frac{2}{h}} \left(\left|\sin\frac{2\theta_{j_2}}{h}\right| + 2\cos\frac{\pi}{h}\cos\frac{2\theta_{j_2}}{h} E_p(C_h)\right)$$
$$+ \textstyle\sum_{j_3}^{s_3} r_{j_3}^{\frac{2}{h}} \left(v_{j_3} + 2\cos\frac{\pi}{h}\cos\frac{2\theta_{j_3}}{h} E_p(C_h)\right)).$$

This gives:

$$E_p(D) = (\ \textstyle\sum_{j_!}^{s_1} r_{j_1}^{\frac{2}{h}}\ \cos\frac{\pi}{h}\cos\frac{2\theta_{j_1}}{h} E_p(C_h) + \sum_{j_2}^{s_2} r_{j_2}^{\frac{2}{h}}\ \cos\frac{\pi}{h}\cos\frac{2\theta_{j_2}}{h} E_p(C_h)$$
$$+ \textstyle\sum_{j_3}^{s_3} r_{j_3}^{\frac{2}{h}}\ \cos\frac{\pi}{h}\cos\frac{2\theta_{j_3}}{h} E_p(C_h))\ E_p(C_h)$$
$$+ \frac{1}{2}\left(\textstyle\sum_{j_!}^{s_1} r_{j_1}^{\frac{2}{h}} u_{j\_1} + \sum_{j_2}^{s_2} r_{j_2}^{\frac{2}{h}} \left|\sin\frac{2\theta_{j_2}}{h}\right| + \sum_{j_3}^{s_3} r_{j_3}^{\frac{2}{h}} v_{j_3}\right). \tag{4.20}$$

In summary, we write

$$E_p(D) = \begin{cases} f_1(h)E_p(C_h) + f_2(h) & if\ \ h \equiv 2(mod\ 4), \\ f_3(h)E_p(C_h) + f_4(h) & if\ \ h \equiv 1(mod\ 2), \\ f_1(h)E_p(C_h) + 2f_2(h) & if\ \ h \equiv 0(mod\ 4), \end{cases}$$

where the functions $f_1(h), f_2(h), f_3(h)$ and $f_4(h)$ are given below:

$$f_1(h) = \textstyle\sum_j^s r_j^{\frac{2}{h}} \cos\frac{2\theta_j}{h}, \quad f_2(h) = \sum_j^s r_j^{\frac{2}{h}} \left|\sin\frac{2\theta_j}{h}\right|,$$
$$f_3(h) = \textstyle\sum_{j_!}^{s_1} r_{j_1}^{\frac{2}{h}} \cos\frac{\pi}{h}\cos\frac{2\theta_{j_1}}{h} + \sum_{j_2}^{s_2} r_{j_2}^{\frac{2}{h}} \cos\frac{\pi}{h}\cos\frac{2\theta_{j_2}}{h} + \sum_{j_3}^{s_3} r_{j_3}^{\frac{2}{h}} \cos\frac{\pi}{h}\cos\frac{2\theta_{j_3}}{h},$$
$$f_4(h) = \frac{1}{2}\left(\textstyle\sum_{j_!=1}^{s_1} r_{j_1}^{\frac{2}{h}} u_{j_1} + \sum_{j_2=1}^{s_2} r_{j_2}^{\frac{2}{h}} \left|\sin\frac{2\theta_{j_2}}{h}\right| + \sum_{j_3=1}^{s_3} r_{j_3}^{\frac{2}{h}} v_{j_3}\right),$$

where

$$u_{j_1} = \left|\sin\left(\frac{2\theta_{j_1}}{h} + \frac{\pi}{h}\right)\right| + \left|\sin\left(\frac{2\theta_{j_1}}{h}\right)\right| - \sin\left(\frac{2\theta_{j_1}}{h}\right)\cos\frac{\pi}{h},$$
$$v_{j_3} = \left|\sin\left(\frac{2\theta_{j_3}}{h}\right)\right| + \left|\cos\left(\frac{2\theta_{j_3}}{h} - \frac{\pi}{h}\right)\right| - \sin\left(\frac{2\theta_{j_3}}{h}\right)\cos\frac{\pi}{h},$$

This completes the proof. □

Next, we study the increasing property of digraphs in the set $D_{n,h}$. For this we show that the digraphs does not satisfy the increasing property with respect to quasi-order relation.

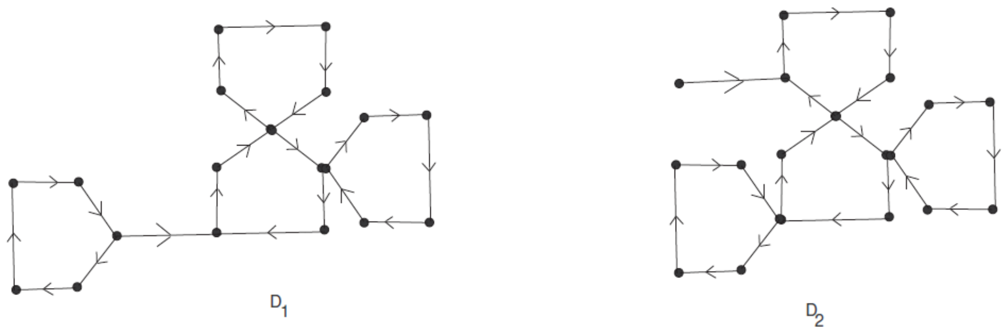

Figure 3: D_1, $D_2 \in D_{18,5}$.

Example 4.5. Consider the digraphs $D_1, D_2 \in D_{18,5}$ in Figure 3. Note that $h \equiv 1(mod\ 2)$. Applying Theorem 4.3, we obtain the characteristic polynomials of D_1 and D_2 which are given by $\Phi_{D_1}(x) = x^{18} - 4x^{13} + 4x^8 - x^3$ and $\Phi_{D_2}(x) = x^{18} - 4x^{13} + 3x^8 - x^3$. For $k = 1,2,3$, it is obvious to see that $c(D_2, kh) \leq c(D_1, kh)$ but their p-energies are given by $E_p(D_1) = 4.8474$ and $E_p(D_2) = 5.0062$, that is, $E_p(D_1) < E_p(D_2)$. Hence it is not necessary that the p-energy of digraphs will increase with respect to quasi-order relation defined over $D_{n,h}$, when $h \equiv 1\ (mod\ 2)$.

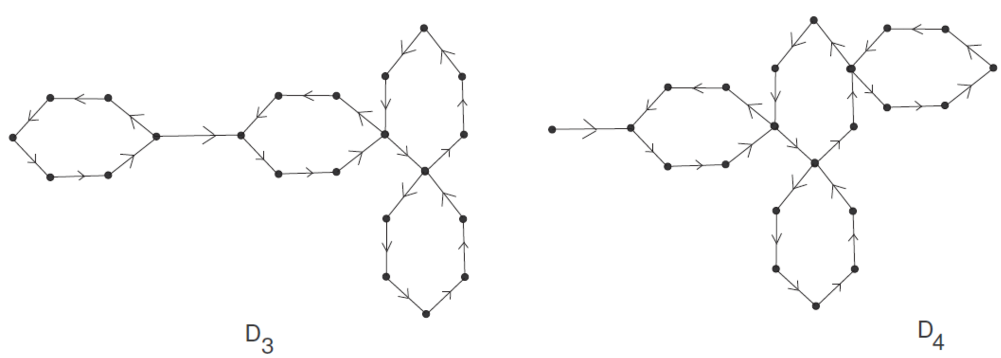

Figure 4: $D_3, D_4 \in D_{22,6}$.

Example 4.6. Let $D_3, D_4 \in D_{22,6}$ be digraphs in Figure 4. Then we see that $h \equiv 2(mod\ 4)$. Applying Theorem 4.3, we obtain $\Phi_{D_3} = x^{22} - 4x^{16} + 4x^{10} - x^4$ and $\Phi_{D_4} = x^{22} - 4x^{16} + 3x^{10} - x^4$. It is easy to see that $c(D_3, kh) \geq c(D_4, kh)$ for $k \in \{1,2,3\}$ and their p-energies are $E_p(D_3) = 5.3759$ and $E_p(D_4) = 5.7087$, that is, $E_p(D_3) < E_p(D_4)$. This shows that p-energy does not possess increasing-property when $h \equiv 2(mod\ 4)$.

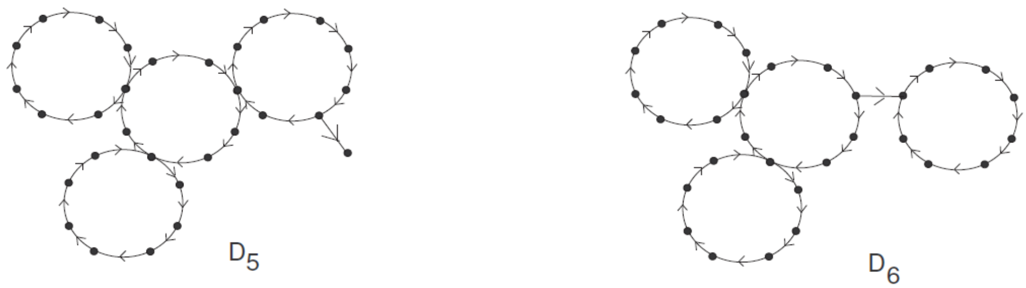

Figure 5: $D_5, D_6 \in D_{30,8}$.

Example 4.7. Let $D_5, D_6 \in D_{30,8}$ be digraphs in Figure 5. Observe that $h \equiv 0(mod 4)$. Then Theorem 4.3 imply that $\Phi_{D_5} = x^{30} - 4x^{22} + 3x^{14} - x^6$ and $\Phi_{D_6} = x^{30} - 4x^{22} + 4x^{14} - x^6$. For $k \in \{1,2,3\}$ it is obvious to see that $c(D_6, kh) \geq c(D_5, kh)$. Moreover, $E_p(D_5) = 6.6900$ and $E_p(D_6) = 6.1163$. This gives $E_p(D_6) < E_p(D_5)$.

5. UPPER BOUND AND p–EQUIENERGETIC DIGRAPHS

Let D be a digraph of order n and $\{z_1, \dots, z_n\}$ be its spectrum. Then it is known in [3, Theorem 1.9] that $\sum_{k=1}^{n} z_k^s = c_s$, where c_s is the number of closed walks in D of length s. The following result is important and will be used in proving a few results in this section.

Lemma 5.1 (Rada [13]). Let $\{z_1, \dots, z_n\}$ be the spectrum of an n-vertex digraph D and a be the number of its arcs. Then

(1) $\sum_{k=1}^{n}(\Re(z_k))^2 - \sum_{k=1}^{n}(\Im(z_k))^2 = c_2$,

(2) $\sum_{k=1}^{n}(\Re(z_k))^2 + \sum_{k=1}^{n}(\Im(z_k))^2 \leq a$.

From Lemma 5.1, we get

$$\sum_{k=1}^{n}(\Re(z_k))^2 \leq \frac{a+c_2}{2}. \tag{5.21}$$

Let D be any digraph. Then nD is a digraph with n copies of D. Next, we find upper bound for the p-energy of an arbitrary digraph. We remark that the idea of the proof is taken from Theorem 2.3 [14].

Theorem 5.2. Let D be a digraph of order n and a be the number of its arcs. Then $E_p(D) \leq \frac{1}{2}\sqrt{a^2 - c_2^2}$. Moreover, equality holds if $D = \frac{n}{2}C_2$, $\frac{n}{2}$ copies of C_2.

Proof. *Let* $\{z_1, \dots, z_n\}$ be the spectrum of a digraph D. Then by applying Cauchy-Schwarz inequality to the vectors $X = (|\Re(z_1)|, |\Re(z_2)|, \dots, |\Re(z_n)|)$ and $Y = (|\Im(z_1)|, |\Im(z_2)|, \dots, |\Im(z_n)|)$, we obtain:

$$E_p(D) = \sum_{k=1}^{n} |\Re(z_k) \cdot \Im(z_k| = \sum_{k=1}^{n} |\Re(z_k)||\Im(z_k|$$
$$\leq \sqrt{\sum_{k=1}^{n} (\Re(z_k)^2} \sqrt{\sum_{k=1}^{n} (\Im(z_k)^2} .$$

Using Lemma 5.1, we get

$$E_p(D) \leq \sqrt{\frac{a+c_2^2}{2}} \sqrt{\sum_{k=1}^{n} (\Re(z_k)^2 - c_2} \leq \sqrt{\frac{a+c_2}{2}} \sqrt{\frac{a+c_2}{2} - c_2} \leq \frac{\sqrt{a^2-c_2^2}}{2}.$$

For the second part, clearly the digraph $D = \frac{n}{2} C_2$ have n vertices and $c_2 = n$ arcs. Thus, it holds that $E_p(D) = 0$. On the other hand, it is obvious to see that $Spec(D) = \left\{1^{\binom{n}{2}}, -1^{\binom{n}{2}}\right\}$. This gives $Ep(D) = \frac{n}{2}(0) = 0$. This completes the proof. □

Two digraphs of same order are cospectral if they have same spectrum, otherwise non-cospectral. Trivial examples of cospectral digraphs are the isomorphic digraphs. For more study on cospectral digraphs we refer [10]. Two digraphs with same number of vertices are said to be p-equienergetic if they have same p-energy. It is clear that cospectral digraphs are always p-equienergetic. In this section, we are interested in constructing a few classes of p-equienergetic non-cospectral digraphs. The following is an easy lemma whose proof is similar to the proof of Lemma 5.2 [9].

Lemma 5.3. For $n \geq 6$, we have $E_p(C_n) = 2E_p(C_{\frac{n}{2}})$ if and only if $n \equiv 2 \ (mod\ 4)$.

The following theorems give a class of non-cospectral p-equienergetic digraphs. Since the proof is similar to the proof of Theorem 5.3 [9], thus we omit it.

Theorem 5.4. For $n \geq 6$, let D be a digraph of order n which contains m vertex-disjoint directed cycles of lengths $s_1, s_2, \dots, s_m$, where $s_k \equiv 2 \ (mod\ 4)$, $k = 1,2, \dots, m$. Take another digraph H with n vertices and contains $2m$ vertex-disjoint directed cycles of lengths $\frac{s_1}{2}, \frac{s_1}{2}, \frac{s_2}{2}, \frac{s_2}{2}, \dots, \frac{s_k}{2}, \frac{s_k}{2}$. Then the two digraphs H and D are non-cospectral p-equienergetic digraphs.

Lemma 5.5. Let D be a digraph of order n and $x_k + iy_k$ be its eigenvalues satisfying $|x_k|, \ |y_k| \leq 1, k = 1,2, \dots, n$. Then $E_p(D \times C_2) = 2\sum_{k=1}^{n} |y_k|$ and $E_p(D \times C_4) = 2\sum_{k=1}^{n} (|x_k| + |y_k|)$.

Proof. It is easy to see that $Spec(C_2) = \{-1,1\}$ and $Spec(C_4) = \{-1,1,-i,i\}$. Next, let D be a digraph of order n and $x_k + iy_k$ be its eigenvalues satisfying $|x_k|, |y_k| \leq 1, k = 1,2, \dots, n$. Then $Spec(D \times C_2)\& = \{ (x_k + 1) + iy_k, (x_k - 1) + iy_k \mid k = 1, \dots, n \}$ and $Spec(D \times C_4)\& = \{(x_k + 1) + iy_k, (x_k - 1) + iy_k, x_k + i(y_k + 1), \ x_k + i(y_k - 1) \mid k = 1, \dots, n\}$. Consequently, the p-energies of $D \times C_2$ and $D \times C_4$ are given by:

$$E_p(D \times C_2) = 2\sum_{k=1}^{n} |y_k(x_k - 1)| + |y_k(x_k + 1)|, \quad (5.22)$$

$$E_p(D\times C_4) = 2\sum_{k=1}^{n}(|y_k(x_k-1)| + |y_k(x_k+1)| + |x_k(y_k-1)| + |x_k(y_k+1)|). \tag{5.23}$$

Since $|x_k|, |y_k| \le 1$, the equations (5.22) and (5.23) become $E_p(D\times C_2) = 2\sum_{k=1}^{n}|y_k|$ and $E_p(D\times C_4) = 2\sum_{k=1}^{n}(|x_k| + |y_k|)$. The proof is complete. □

Corollary 5.6. For $n \ge 4$, we have

$$E_p(C_n\times C_2) = \begin{cases} 2\cot\frac{\pi}{2n} & if\ n \equiv 1(mod\ 2),\\ 4\cot\frac{\pi}{n} & if\ n \equiv 0(mod\ 2).\end{cases}$$

$$E_p(C_n\times C_4) = \begin{cases} 2\cot\frac{\pi}{4n} & if\ n \equiv 1(mod\ 2),\\ 4\cot\frac{\pi}{2n} & if\ n \equiv 2(mod\ 4),\\ 8\cot\frac{\pi}{n} & if\ n \equiv 0(mod\ 4).\end{cases}$$

Proof. From (3.4), the eigenvalues of C_n are $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0,1,\dots,n-1$. Clearly, $|\cos\frac{2k\pi}{n}| \le 1$ and $|\sin\frac{2k\pi}{n}| \le 1$. By Lemma 5.5, it holds that

$$E_p(C_n\times C_2) = 2\sum_{k=1}^{n}\left|\sin\frac{2k\pi}{n}\right|,$$

$$E_p(C_n\times C_4) = 2\sum_{k=1}^{n}\left(\left|\cos\frac{2k\pi}{n}\right| + \left|\sin\frac{2k\pi}{n}\right|\right).$$

Using (2.2) and (2.3), we get the required result. □

Using Corollary 5.6, we give few pairs of non-cospectral p-equienergetic digraphs in the following example.

Example 5.7. For any integer n, where $n \ge 4$, we have the following:

(1) If $n \equiv 1(mod\ 2)$ then $E_p\big(2(C_n\times C_4)\big) = E_p(C_{4n}\times C_2)$.

(2) If $n \equiv 0(mod\ 4)$ then $E_p\big(2(C_n\times C_2)\big) = E_p(C_n\times C_4)$.

(3) If $n \equiv 2(mod\ 4)$ then $E_p\big(2(C_{2n}\times C_2)\big) = E_p(C_n\times C_4)$.

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