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The Maximal Total Irregularity of Some Connected Graphs

M. ELIASI

Department of Mathematics, Faculty of Khansar, University of Isfahan, I. R. Iran
Correspondence should be addressed to eliasi@math.iut.ac.ir
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ABSTRACT The total irregularity of a simple graph G is defined as $irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_u - d_v|$, where d_u denotes the degree of a vertex $u \in V(G)$. In this paper by using the Gini index, we obtain the ordering of the total irregularity index for some classes of connected graphs, with the same number of vertices.

KEYWORDS Total irregularity index • Gini index • majorization • trees • unicyclic graph • bicyclic graph

1. INTRODUCTION

Throughout this paper, we consider simple graphs (finite undirected graphs without loops and multiple edges). For $u \in V(G)$, d_u denotes the degree of u in G. An edge of G connecting the vertices u and v is denoted by uv. A graph G is regular if all of its vertices have the same degree, otherwise it is irregular. Up to now, several parameters have been proposed to characterize the regularity of a graph.

For example in [1], Albertson defined the imbalance of an edge $e = uv \in E(G)$ as $emb(e) = |d_u - d_v|$ and the irregularity of G as $irr(G) = \sum_{e\in E(G)} emb(e)$. More results on the imbalance and the irregularity of a graph G can be found in [1,2,10]. Recently, in [3] a new measure of irregularity of a simple undirected graph, so-called the total irregularity, was defined as $irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)}|d_u - d_v|$. These irregularity measures as well as other attempts to measure the irregularity of a graph were studied in several works [4,8–10]. Dimitrov and Skrekovski [6] derived relation between $irr(G)$ and $irr_t(G)$ for a connected graph G with n vertices. Abdo et al. [3], obtained the upper bound of the total irregularity among all graphs with n vertices, and they showed that the star graph S_n is the tree with the maximal total irregularity among all trees with n vertices.

You et al. [13] investigated the total irregularity of unicyclic graphs and determined the graph with the maximal total irregularity among all unicyclic graphs on n vertices. In [14], the authors introduced two transformations to study the total irregularity of bicyclic graphs and characterized the graph with the maximal total irregularity among all bicyclic graphs on n vertices. Zhu et al. [15] introduced an important transformation and investigated the minimal total irregularity of graphs, and they characterized the graph with the minimal, the second minimal, the third minimal total irregularity among trees, unicyclic or bicyclic graphs on n vertices.

The theory of majorization as a powerful tool has widely been applied to the related research areas of pure and the applied mathematics [12]. Recently some issues related to the structural properties of graphs have been explored solving suitable optimization problems via majorization technique see [7, 11].

In this paper, we use this theory to study the total irregularity of some classes of simple graphs. It let us to determine the five graphs with the first through fifth greatest total irregularity index among the class of trees of order n. We extend the previous results about the graph with the maximal, second maximal, third maximal irregularity among bicyclic graphs on n vertices. Also we do a similar work for unicyclic graphs on n vertices.

2. Preliminary Results

We begin by introducing the main mathematical theory explored the theory of majorization. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$, be two non-increasing sequences of real numbers. If they satisfy the conditions $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for $1 \leq k \leq n-1$ and $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$, then we say that $x = (x_1, x_2, \cdots, x_n)$ is ***majorized*** by $y = (y_1, y_2, \dots, y_n)$ and write $x \preccurlyeq y$. Furthermore, by $x \prec y$ we mean that $x \preccurlyeq y$ and $x \neq y$. A real-value function φ defined on a set $A \subseteq \mathrm{R}^n$ is said to be Schur-convex on A if $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ and $x \preccurlyeq y$ then $\varphi(x) \leq \varphi(y)$. If, in addition, $\varphi(x) < \varphi(y)$ where $x \prec y$, then φ is said to be strictly Schur-convex on A.

The Gini coefficient (also known as the Gini index or Gini ratio) is a measure of statistical dispersion intended to represent the income distribution of a nation's residents. For $x = (x_1, x_2, \dots, x_n)$, the Gini index can be written as $\Phi_{11}(x) = \frac{1}{n^2 \bar{x}} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|$, where $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$ [12].

Dalton in 1920 proved the following result:

Lemma 1. The Gini index is an strictly Schur-convex function.

Proof. See [5,12].

The Lemma 1 leads us to the following important corollary:

Corollary 2. Let G and H be two connected graphs with degree sequences $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, respectively such that $|E(G)| = |E(H)|$. If $x \preccurlyeq y$, then $irr_t(G) \leq irr_t(H)$. The equality holds if and only if $x = y$.

Proof. Let $|E(G) = |E(H)| = m$. Then $\bar{x} = \bar{y} = \frac{2m}{n}$. Therefore

$$\Phi_{11}(x) = \frac{1}{n^2\bar{x}} \sum_{i=1}^{n} \sum_{j=}^{n} |x_i - x_j| = \frac{1}{n^2\frac{2m}{n}} \sum_{u,v \in V(G)} |d_u - d_v| = \frac{1}{n^2\frac{2m}{n}} (2irr_t(G)) = \frac{irr_t(G)}{mn}.$$

Similarly, $\Phi_{11}(y) = \frac{irr_t(H)}{mn}$. Since $x \preccurlyeq y$, by Lemma 1 $\Phi_{11}(x)$ is an strictly Schur-convex function and so $\Phi_{11}(x) \leq \Phi_{11}(y)$. Hence $irr_t(G) \leq irr_t(H)$ and the equality holds if and only if $x = y$.

3. Main Results

Let $\mathcal{T}_n$, $\mathcal{U}_n$ and $\mathfrak{B}_n$ be the set of trees of order n, the set of connected unicyclicgraphs of order n, and the set of connected bicyclic graphs of order n, respectively. Also for a graph G, denoted by $\Delta(G)$ the maximum degree of G.

Let P_n and S_n be the path and star on n vertices, respectively. In [3] the authors showed that the star graph S_n is the tree with the maximal totalirregularity among all trees with n vertices. It has been shown that [15] the path P_n has minimal total irregularity among all trees with n vertices. Here we prove this result by a different and very short method.

Theorem 3. Let $T \in \mathcal{T}_n \backslash \{P_n, S_n\}$ be a tree with n vertices. Then

$$2n - 4 = irr_t(P_n) < irr_t(T) < irr_t(S_n) = (n-1)(n-2).$$

Proof. Note that each two trees with n vertices have the same number of edges equal to $n - 1$. Since the degree sequence $(2, \ldots, 2,1,1)$, belongs to P_n, is minimal in the class $\mathcal{T}_n$ (i.e., in the order $\preccurlyeq$) and the degree sequence $(n-1,1, \ldots, 1)$, belongs to S_n, is maximal in the class $\mathcal{T}_n$, we obtain the result by of Corollary 2.

Now we extend Theorem 3 by majorization. Let $T_1 = S_n, T_2, \cdots, T_{13}$ be the trees on n vertices as shown in Figure 1. In the following theorem, we show that the graph T_2 has the second maximal, the graph T_3 has the third maximal, the graphs T_4, T_5 and T_6 have the fourth maximal and the graphs T_7, T_8, T_9 and T_{13} have the fifth maximal total irregularity among all trees.

Theorem 4. Let $T \in \mathcal{T}_n \backslash \{T_1, T_2, \dots, T_9, T_{13}\}$ and $n \geq 13$. Then
$irr_t(T_1) > irr_t(T_2) > irr_t(T_3) > irr_t(T_4) = irr_t(T_5) = irr_t(T_6) > irr_t(T_7) = irr_t(T_8)$
$= irr_t(T_9) = irr_t(T_{13}) > irr_t(T)$.

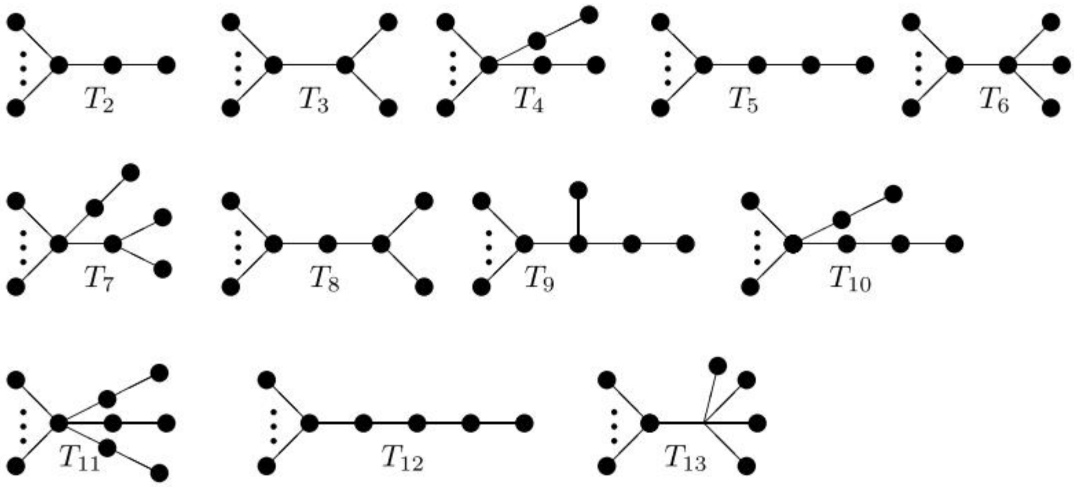

Figure 1. The trees $T_2, \dots, T_{13}$. This figure is taken from [11].

Proof. By an elementary computation, we have $irr_t(T_1) = (n-1)(n-2)$, $irr_t(T_2) = n^2 - 3n$, $irr_t(T_3) = n^2 - 3n - 2$, $irr_t(T_4) = irr_t(T_5) = irr_t(T_6) = n^2 - 3n - 4$, $irr_t(T_7) = irr_t(T_8) = irr_t(T_9) = irr_t(T_{13}) = n^2 - 3n - 6$ and $irr_t(T_{10}) = irr_t(T_{11}) = irr_t(T_{12}) = n^2 - 3n - 10$.

So we only need to show that if $T \in \mathcal{T}_n \backslash \{T_1, T_2, \dots, T_{13}\}$, then $irr_t(T_{13}) > irr_t(T)$. Clearly, T_1 is the unique tree with $\Delta = n - 1, T_2$ is the unique tree with $\Delta = n - 2, T_3$, T_4, T_5 are the all trees with $\Delta = n - 3$ and $T_6, \dots, T_{12}$ are the all trees with $\Delta = n - 4$. Since$T \in \mathcal{T}_n \backslash \{T_1, T_2, \dots, T_{13}\}$, then $\Delta(T) \leq n - 5$. Let $a = (d_1, d_2, \dots, d_n)$ be the degree sequence of T. Since the degree sequence of T_{13} is $b = (n - 5, 5, 1, \dots, 1)$, it is easy to see that $a \prec b$, because T_{13} is the unique tree with b as its degree sequence. Thus, $irr_t(T_{13}) > irr_t(T)$ follows from Corollary 1.

Let U_1, $U_2, \dots, U_{16}$ be the unicyclic graphs as shown in Figure 2. In [13], the authors investigated the total irregularity of unicyclic graphs and determined the graph with the maximal total irregularity $n^2 - n - 6$ among unicyclic graphs on n vertices. Here we determine the four unicyclic graphs with the first through fourth greatest total irregularity index among the class of unicyclic graphs of order n.

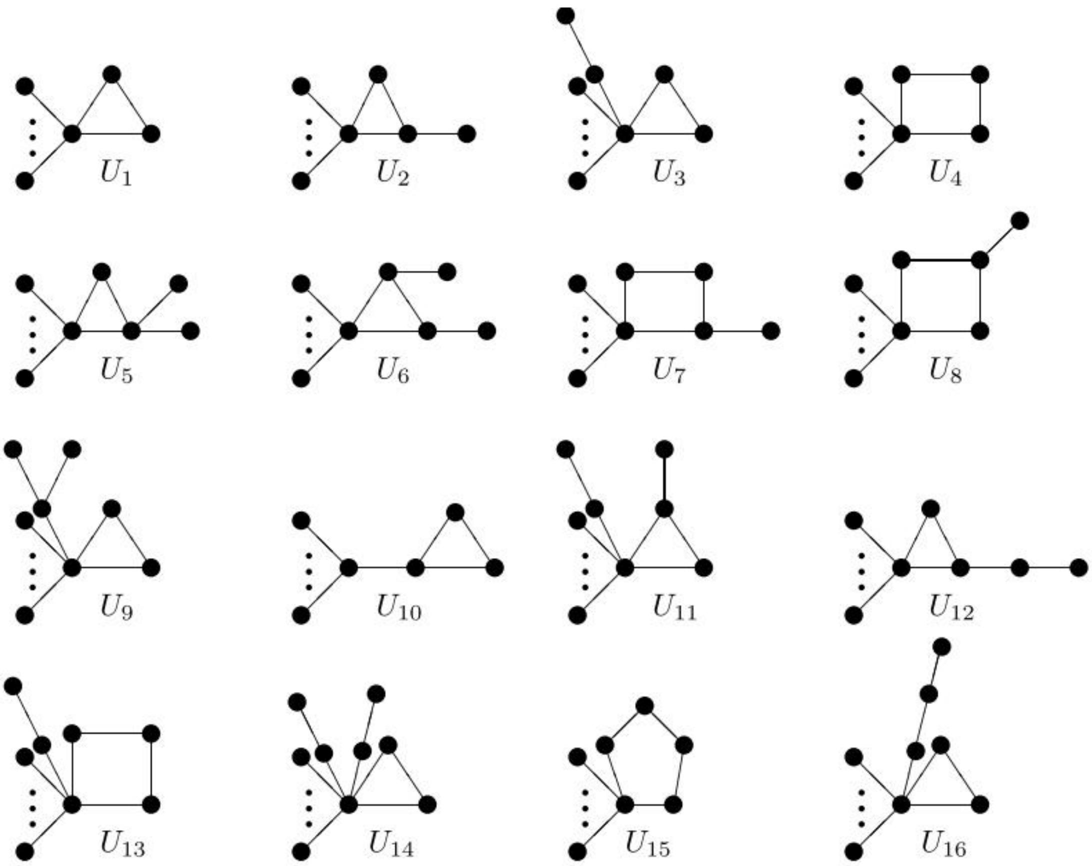

Figure 2. The unicyclic graphs $U_1, \ldots, T_{16}$. This figure is taken from [11].

Theorem 5. Let $G \in \mathcal{U}_n \backslash \{U_1, U_2, \ldots, U_6\}$ and $n \geq 13$. Then

$$irr_t(U_1) > irr_t(U_2) > irr_t(U_5) > irr_t(U_3) = irr_t(U_4) = irr_t(U_6) \geq irr_t(G).$$

Proof. By an elementary computation, we have$irr_t(U_1) = n^2 - n - 6$, $irr_t(U_2) = n^2 - n - 8$, $irr_t(U_3) = irr_t(U_4) = irr_t(U_6) = n^2 - n - 12$, $irr_t(U_5) = n^2 - n - 10$, $irr_t(U_7) = irr_t(U_8) = irr_t(U_9) = irr_t(U_{10}) = irr_t(U_{11}) = irr_t(U_{12}) = n^2 - n - 14$ and $irr_t(U_{13}) = irr_t(U_{14}) = irr_t(U_{15}) = irr_t(U_{16}) = n^2 - n - 20$. So we only need to prove that if $G \in \mathcal{U}_n \backslash \{U_1, U_2, \ldots, U_{16}\}$, then $irr_t(U_6) > irr_t(G)$.

It is easy to check that U_1 is the unique unicyclic graph with $\Delta = n - 1$, U_2, U_3, U_4 are all unicyclic graphs with $\Delta = n - 2$, and U_5, $U_6, \ldots, U_{16}$ are all unicyclic graphs with $\Delta = n - 3$. If $G \in \mathcal{U}_n \backslash \{U_1, U_2, \ldots, U_{16}\}$, then $\Delta(G) \leq n - 4$. Suppose that degree sequence of G is $a = (d_1, d_2, \ldots, d_n)$. Since $G \in \mathcal{U}_n$, then G has only exactly one cycle. This implies that $n-4 \geq d_1 \geq d_2 \geq d_3 \geq 2$. If $b=(n-4,5,2,1,\dots,1)$ then $a \preccurlyeq b$. Since each unicyclic graph with n vertices has n edges, by Corollary 2, we can conclude that

$$\begin{aligned} irr_t(G) &\leq \mathrm{n}-9+\mathrm{n}-6+(\mathrm{n}-5)(\mathrm{n}-3)+3+4(\mathrm{n}-3)+\mathrm{n}-3=\mathrm{n}^2-\mathrm{n}-12 \\ &= \mathrm{irr_t}(\mathrm{U}_6). \end{aligned}$$

This completes the proof.

Corollary 6. Let $n \geq 6$ be a positive integer and let G be a unicyclic graph on n vertices. Then, $irr_t(G) \leq n^2-n-6$ and the equality holds if and only if $G \cong U_1$.

Let $B_1, B_2, \dots, B_{11}$ be the bicyclic graphs as shown in Figure 3. In [14], the authors characterized the graph with the maximal total irregularity among all bicyclic graphs on n vertices. The next result extends this result by determining the first up to third greatest total irregularity together with the corresponding bicyclic graphs among the class of connected bicyclic graphs of order n.

Theorem 7. Let $G \in \mathfrak{B}_n \backslash \{B_1, B_2, \dots, B_5\}$ and $n \geq 12$. Then

$$irr_t(B_1) > irr_t(B_3) > irr_t(B_4) = irr_t(B_5) \geq irr_t(G).$$

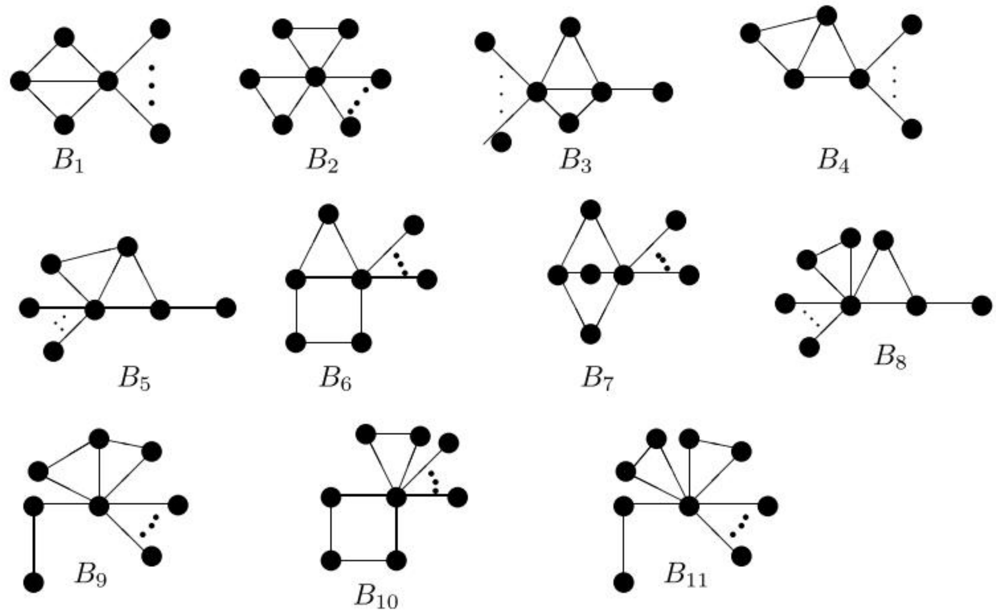

Figure 3. The bicyclic graphs $B_1, \dots, B_{11}$. This figure is taken from [11].

Proof. By an elementary computation, we have $irr_t(B_1)=n^2+n-16$, $irr_t(B_2)=n^2+n-22$, $irr_t(B_3)=n^2+n-18$, $irr_t(B_4)=irr_t(B_5)=n^2+n-20$,

$irr_t(B_6) = irr_t(B_7) = irr_t(B_8) = irr_t(B_9) = n^2 + n - 24$ and $irr_t(B_{10}) = irr_t(B_{11}) = n^2 + n - 32$. So we only need to prove that if $G \in \mathfrak{B}_n \backslash \{B_1, B_2, \dots, B_{11}\}$, then $irr_t(G) \leq irr_t(B_5)$.

It is easy to check that B_1, B_2 are all bicyclic graphs with $\Delta = n - 1$, $B_3, \dots, B_{11}$ are all bicyclic graphs with $\Delta = n - 2$. If $G \in \mathfrak{B}_n \backslash \{B_1, B_2, \dots, B_{11}\}$, then $\Delta(G) \leq n - 3$. Suppose that the degree sequence of G is $a = (d_1, d_2, \dots, d_n)$. Since $G \in \mathfrak{B}_n$,

$$n - 3 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq 2.$$

Let $b = (n - 3, 5, 2, 2, 1, \dots, 1)$. Then $a \preccurlyeq b$ and by Corollary 2, we can conclude that:

$$\begin{aligned} irr_t(G) &\leq n - 8 + n - 5 + n - 5 + (n-4)^2 + 3 + 3 + 4(n-4) + n - 4 + n - 4 \\ &= n^2 + n - 20 = irr_t(B_5). \end{aligned}$$

This completes the proof.

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