M–Polynomial of some Graph Operations and Cycle Related Graphs

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ABSTRACT
In this paper, we obtain M-polynomial of some graph operations and cycle related graphs. As an application, we compute M-polynomial of some nanostructures viz., TUC₄C₈[p,q] nanotube, TUC₄C₈[p,q] nanotorus, line graph of subdivision graph of TUC₄C₈[p,q] nanotube and TUC₄C₈[p,q] nanotorus, V-tetracenic nanotube and V-tetracenic nanotorus. Further, we derive some degree based topological indices from the obtained polynomials.

1. INTRODUCTION

Let G be a simple, connected, undirected graph of order n and size m with vertex set V(G) and edge set E(G). The degree dₓ(v) of a vertex v ∈ V(G) is the number of edges incident to it in G. An isolated vertex or singleton graph is a vertex with degree zero. Let {v₁,v₂,⋯,vₙ} be the vertices of G and let d₁ = dₓ(v₁). The subdivision graph S(G) [24] of a graph G is the graph obtained by inserting a new vertex onto each edge of G. Let G₁ and G₂ be two graphs of order n₁, n₂ and size m₁, m₂ respectively. The union [24] of G₁ and G₂ is the graph with vertex set V₁ ∪ V₂ and edge set E₁ ∪ E₂ is denoted by G₁ ∪ G₂ and |V(G₁ ∪ G₂)| = n₁ + n₂, |E(G₁ ∪ G₂)| = m₁ + m₂. The join [24] G₁ + G₂ of G₁ and G₂ is the graph obtained from G₁ ∪ G₂ by joining each vertex of G₁ with every vertex of G₂ by an edge. Order and size of G₁ + G₂ are n₁ + n₂ and m₁ + m₂ + n₁n₂, respectively. The

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corona [24] \( G_1 \circ G_2 \) of two graphs \( G_1 \) and \( G_2 \) of order \( n_1 \) and \( n_2 \) respectively, is defined as the graph obtained by taking one copy of \( G_1 \) and \( n_1 \) copies of \( G_2 \) and then joining the \( i^{th} \) vertex of \( G_1 \) to every vertex in the \( i^{th} \) copy of \( G_2 \). For undefined graph theoretic terminologies and notions refer [24].

Several topological indices have been defined in the literature. Among them some standard topological indices are first Zagreb index [22], second Zagreb index [23], modified second Zagreb index [10], Randić’ index [36], harmonic index [16], symmetric division index [10] and inverse sum index [10]. The general form of these degree-based topological indices of a graph is given by

\[
TI(G) = \sum_{e=uv \in E(G)} f(d_G(u),d_G(v)),
\]

where \( f = f(x,y) \) is a function appropriately chosen for the computation. Table 1 gives the standard topological indices defined by \( f(x,y) \). For more details on degree-based and distance based topological indices refer [1−7,12,13,18,19,21,32,39−41,43,45].

It would be interesting that, if all these topological indices are obtained from a single expression. This role is played by polynomials. In fact there are several graph polynomials like PI polynomial [3], Tutte polynomial [14], matching polynomial [15,20], Schultz polynomial [25], Zang-Zang polynomial [46], etc., Among them, the Hosoya polynomial [26] is the best and well-known polynomial which plays a vital role in determining distance-based topological indices such as Wiener index [44], hyper Wiener index [9] of graphs. Similarly, M-polynomial which was introduced in 2015 by Deutsch and Klavžar in [10], which is useful in determining many degree-based topological indices (listed in Tables 1 and 2). This motivates us to study M-polynomial of some graph operations and some cycle related graphs. Recently, the study of M-polynomial are reported in [8,11,28,33−35,37].

<table>
<thead>
<tr>
<th>Notation</th>
<th>Topological Index</th>
<th>( f(x,y) )</th>
<th>Derivation from ( M(G;x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1(G) )</td>
<td>First Zagreb</td>
<td>( x + y )</td>
<td>( (D_x + D_y)(M(G;x,y))</td>
</tr>
<tr>
<td>( M_2(G) )</td>
<td>Second Zagreb</td>
<td>( xy )</td>
<td>( (D_xD_y)(M(G;x,y))</td>
</tr>
<tr>
<td>( M_2^m(G) )</td>
<td>Second modified</td>
<td>( \frac{1}{xy} )</td>
<td>( (S_xS_y)(M(G;x,y))</td>
</tr>
<tr>
<td>( S_D(G) )</td>
<td>Symmetric division</td>
<td>( \frac{x^2 + y^2}{xy} )</td>
<td>( (D_xS_y + D_yS_x)(M(G;x,y))</td>
</tr>
<tr>
<td>( H(G) )</td>
<td>Harmonic</td>
<td>( \frac{2}{x + y} )</td>
<td>( 2S_x(M(G;x,y))</td>
</tr>
<tr>
<td>( I_n(G) )</td>
<td>Inverse sum</td>
<td>( \frac{xy}{x + y} )</td>
<td>( S_xD_xD_y(M(G;x,y))</td>
</tr>
</tbody>
</table>

Table 1. [10] Operators to derive degree-based topological indices from M-polynomial.
where, \( D_x = x \frac{\partial f(x,y)}{\partial x}, D_y = y \frac{\partial f(x,y)}{\partial y} \), \( S_x = \int_0^x f(t,y) \, dt \), \( S_y = \int_0^y f(x,t) \, dt \) and \( f(x,y) = f(x,x) \) are the operators. Along with these operators, we also mention two more operators in Table 2 to calculate general sum connectivity index and first general Zagreb index.

**Definition 1.** [10] Let \( G \) be a graph. Then \( M \)-polynomial of \( G \) is defined as
\[
M(G; x, y) = \sum_{i \leq j} m_{ij}(G)x^iy^j,
\]
where \( m_{ij}, i, j \geq 1 \), is the number [19] of edges \( uv \) of \( G \) such that \( \{d_G(u), d_G(v)\} = \{i, j\} \).

**Table 2:** New operators to derive degree-based topological indices from \( M \)-polynomial.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Topological Index</th>
<th>( f(x,y) )</th>
<th>Derivation from ( M(G; x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_a(G) )</td>
<td>General sum connectivity [21]</td>
<td>((x + y)^a)</td>
<td>( D_x^a(f(M(G; x,y)))</td>
</tr>
<tr>
<td>( M_1^a(G) )</td>
<td>First general Zagreb [31]</td>
<td>( x^{a-1} + y^{a-1} )</td>
<td>( (D_x^{a-1} + D_y^{a-1})(M(G; x,y))</td>
</tr>
</tbody>
</table>

Note 1: Hyper Zagreb index is obtained by taking \( \alpha = 2 \) in general sum connectivity index.
Note 2: Taking \( \alpha = 2,3 \) in first general Zagreb index, first Zagreb and forgotten topological indices are obtained respectively.

2. **M-POLYNOMIAL OF SOME GRAPH OPERATIONS**

In this section, we obtain M-polynomial of some graph operations.

**Lemma 2.1.** For any \( r \)-regular graph \( G \) of order \( n \) and size \( m \), the \( M \)-polynomial of \( G \) is given by \( M(G; x, y) = mx^r y^r \).

**Proof.** Since \( G \) is a \( r \)-regular graph with \( m \) edges and every edge is incident on vertex of degree \( r \), the proof follows.

The product [24] \( G \times H \) of graphs \( G \) and \( H \) has the vertex set \( V(G \times H) = V(G) \times V(H) \) and \((a,x)(b,y)\) is an edge of \( G \times H \) if and only if \([a = b \text{ and } xy \in E(H)] \text{ or } [x = y \text{ and } ab \in E(G)] \).

**Theorem 2.2.** Let \( G \) be an \( r_1 \)-regular graph of order \( n_1 \) and \( H \) be an \( r_2 \)-regular graph of order \( n_2 \). Then \( M(G \times H; x, y) = n_1 n_2 x^{r_1+r_2} y^{r_1+r_2} \).

**Proof.** Since the graphs \( G \) and \( H \) are regular graphs of degree \( r_1 \) and \( r_2 \) respectively. Therefore the graph obtained by product of \( G \) and \( H \) is a regular graph of degree \( r_1 + r_2 \) with \( n_1 n_2 \) vertices. Hence the result follows from Lemma 2.1.
The composition \([24] G[H]\) of graphs \(G\) and \(H\) with disjoint vertex sets \(V(G)\) and \(V(H)\) and edge sets \(E(G)\) and \(E(H)\) is the graph with vertex set \(V(G[H]) = V(G) \times V(H)\) and \((a,x)(b,y)\) is an edge of \(G[H]\) if and only if \(a\) is adjacent to \(b\) in \(G\) or \(a = b\) and \(x\) is adjacent to \(y\) in \(H\).

**Figure 1.** Some cycle related graphs.
Theorem 2.3. Let $G$ be an $r_1$-regular graph of order $n_1$ and $H$ be an $r_2$-regular graph of order $n_2$. Then, $M(G[H]; x, y) = n_1 n_2 x^{n_2 r_1} + r_2 y^{n_2 r_1 + r_2}$.

Proof. Since $G$ and $H$ are regular graphs of degree $r_1$ and $r_2$ respectively. The graph obtained by the composition of two graphs $G$ and $H$ is a regular graph of degree $n_2 r_1 + r_2$ with $n_1 n_2$ vertices. Hence the result follows from Lemma 2.1.

3. M-Polynomial of Cycle Related Graphs

In this section, we obtain M-polynomial of some cycle related graphs, Figure 1. Definitions 2-10 can be found in [17], definition 11 is in [42] and definitions 12-16 can be found in [30, 38]. We also derive some topological indices (mentioned in Tables 1 and 2) of these graphs from the respective M-polynomials. For more details on wheel related graphs refer [17,27,38,42] and references cited there in.

Definition 2. The fan graph $F_n, (n \geq 3)$ is defined as the graph $K_1 + P_n$, where $K_1$ is singleton graph and $P_n$ is the path on $n$ vertices.

Theorem 3.1. Let $F_n$ be a fan of order $n + 1$ and size $2n - 1$. Then, $M(F_n; x, y) = 2x^2 y^3 + 2x^2 y^n + (n - 3)x^3 y^3 + (n - 2)x^3 y^n$.

Proof. The fan $F_n$ has $n + 1$ vertices and $2n - 1$ edges. It is easy to see that $|m_{[2, 3]}| = 2, |m_{[2n]}| = 2$ and the remaining edge partition of $F_n$ is as follows:

- $|E_{[3, 3]}| = |uc \in E(F_n): d_u = 3 \text{ and } d_c = 3| = (n - 3)$,
- $|E_{[3, n]}| = |uc \in E(F_n): d_u = 3 \text{ and } d_c = n| = (n - 2)$,

proving the result.

Corollary 3.2. If $F_n$ is a Fan, then

1. $M_1(F_n) = n^2 + 9n - 10$,
2. $M_2(F_n) = 3n^2 + 7n - 15$,
3. $M_3(F_n) = \frac{n^2 + 3n + 3}{9n}$,
4. $S_F(F_n) = \frac{n^3 + 7n^2 + 4n - 6}{3n}$,
5. $H(F_n) = \frac{n^2 + 2n + 12}{3(n + 2)} + \frac{9n - 23}{5(n + 3)}$,
6. $I_n(F_n) = \frac{3n(n - 2)}{n + 3} + \frac{3}{10} + \frac{4n}{n + 2}$,
7. $X_d(F_n) = 2 \cdot 5^a + 2(n + 2)^a + (n - 3) \cdot 6^a + (n - 2)(n - 3)^a$,
8. $M_d(F_n) = 2^{a+2} + 3^a(2n - 5) + 3^a(n - 1) + n^{a+1}$. 


Proof. The M-polynomial for fan $F_n$ is given by
\[ M(F_n;x,y) = 2x^2y^3 + 2x^2y^n + (n-3)x^3y^3 + (n-2)x^3y^n. \]
Using the expressions from Tables 1 and 2, we have
\[
D_x = x \frac{\partial f(x,y)}{\partial x} = 4x^2y^n + 4x^2y^3 + 3(n-3)x^3y^3 + 3(n-2)x^3y^n
\]
\[
D_y = y \frac{\partial f(x,y)}{\partial y} = 2nx^2y^n + 6x^2y^3 + 3(n-3)x^3y^2 + n(n-2)x^3y^n
\]
\[
S_x = \int_0^x f'(t) \, dt = x^2y^n + x^2y^3 + \frac{(n-3)}{3}x^3y^3 + \frac{(n-2)}{3}x^3y^n
\]
\[
S_y = \int_0^y f'(t) \, dt = \frac{2}{n}x^2y^n + \frac{2}{3}x^2y^3 + \frac{(n-3)}{3}x^3y^3 + \frac{(n-2)}{n}x^3y^n.
\]
Therefore,
\[
M_1(F_n) = (D_x + D_y)(M(F_n;x,y))|_{x=y=1} = n^2 + 9n - 10,
\]
\[
M_2(F_n) = (D_xD_y)(M(F_n;x,y))|_{x=y=1} = 3n^2 + 7n - 15,
\]
\[
M_2^n(F_n) = (S_xS_y)(M(F_n;x,y))|_{x=y=1} = \frac{n^3}{3} + \frac{n+3}{9},
\]
\[
S_D(F_n) = (D_xS_y + D_yS_x)(M(F_n;x,y))|_{x=y=1} = \frac{n^3+7n^2+4n-6}{3n},
\]
\[
H(F_n) = 2S_x J(M(F_n;x,y))|_{x=1} = \frac{n^2+2n+12}{3(n+2)} + \frac{9n-23}{5(n+3)},
\]
\[
I_n(F_n) = S_x J D_x D_y (M(F_n;x,y))|_{x=1} = \frac{3n(n-2)}{n+3} + \frac{3(5n-7)}{10} + \frac{4n}{n+2},
\]
\[
\chi_a(F_n) = D_x^a J(M(F_n;x,y))|_{x=1} = 2 \cdot 5^a + 2(n+2)^a + (n-3) \cdot 6^a + (n-2)(n-3)^a,
\]
\[
M_1^a(F_n) = (D_x^a + D_y^a)(M(F_n;x,y))|_{x=y=1} = 2^{a+2} + 3^a(2n-5) + 3^a(n-1) + n^{a+1}. \]

Definition 3. The wheel $W_n = C_n + K_1$ is a graph with $n+1$ vertices and $2n$ edges, where the vertex $c$ with degree $n$ is called the central vertex while the vertices on the cycle $C_n$ are called rim vertices.

Theorem 3.3. Let $W_n$ be a wheel of order $n+1$ and size $2n$. Then,
\[ M(W_n;x,y) = nx^3y^3(1 + y^{n-3}). \]

Proof. The wheel $W_n$ has $n+1$ vertices and $2n$ edges. The edge set of $W_n$ can be partitioned as,
\[
|E_{[3,3]}| = |uv \in E(W_n) : d_u = 3 \text{ and } d_v = 3| = n,
\]
\[
|E_{[3,n]}| = |uc \in E(W_n) : d_u = 3 \text{ and } d_c = n|
\]
\[
= |E(W_n) - |E_{[3,3]}| = n.
\]

Corollary 3.4. If $W_n$ is a wheel, then
1. $M_1(W_n) = n^2 + 9n$,
2. $M_2(W_n) = 3n^2 + 9n$.\]
\[ M_2^m(W_n) = \frac{n+3}{9}, \]
\[ S_D(W_n) = \frac{n^2+6n+9}{3}, \]
\[ H(W_n) = \frac{n^2+9n}{3(n+3)}, \]
\[ I_n(W_n) = \frac{3n}{2} + \frac{3n^2}{n+3}, \]
\[ \chi_a(W_n) = n(6^a + (n+3)^a), \]
\[ M_1^a(W_n) = 3^{a+1} + n^a. \]

**Proof.** Let \( M(W_n; x,y) = \sum_{i \leq j} m_{ij}(W_n)x^iy^j = nx^3y^3(1 + y^{n-3}). \) Using the expressions from Tables 1 and 2, we have
\[
D_x = x \frac{\partial f(x,y)}{\partial x} = 3nx^3y^3 + 3nx^3y^n
\]
\[
D_y = y \frac{\partial f(x,y)}{\partial y} = 3nx^3y^3 + n^2x^3y^n
\]
\[
S_x = \int_0^x f(t,y) \frac{dt}{t} = \frac{nx^3y^3}{3} + \frac{nx^3y^n}{3}
\]
\[
S_y = \int_0^y f(x,t) \frac{dt}{t} = \frac{nx^3y^3}{3} + x^3y^n.
\]

Thus we get,
\[
M_1(W_n) = (D_x + D_y)(M(W_n; x,y))|_{x=y=1} = n^2 + 9n,
\]
\[
M_2(W_n) = (D_xD_y)(M(W_n; x,y))|_{x=y=1} = 3n^2 + 9n,
\]
\[
M_2^m(W_n) = (S_xS_y)(M(W_n; x,y))|_{x=y=1} = \frac{n}{9} + \frac{1}{3},
\]
\[
S_D(W_n) = (D_xS_y + D_yS_x)(M(W_n; x,y))|_{x=y=1} = \frac{n^2+6n+9}{3},
\]
\[
H(W_n) = 2S_xf(M(W_n; x,y))|_{x=1} = \frac{n}{3} + \frac{2n}{n+3},
\]
\[
I_n(W_n) = S_xD_xD_y(M(W_n; x,y))|_{x=1} = \frac{3n}{2} + \frac{3n^2}{n+3},
\]
\[
\chi_a(W_n) = D_x^a \left( f(M(W_n; x,y)) \right)|_{x=1} = n(6^a + (n+3)^a),
\]
\[
M_1^a(W_n) = (D_x^a + D_y^a)(M(W_n; x,y))|_{x=y=1} = 3^{a+1} + n^a.
\]

**Definition 4.** The gear graph \( G_n \) is a wheel graph with a vertex added between each pair adjacent vertices of the outer circle.

**Theorem 3.5.** Let \( G_n \) be a gear graph. Then \( M(G_n; x,y) = 2nx^2y^3 + nx^3y^n. \)

**Proof.** Let \( G_n \) is a graph having \((2n + 1)\) vertices and \(3n\) edges. The edge partition of \( G_n \) is given by,
\[ |E_{[2,3]}| = |uv \in E(G_n) : d_u = 2 \text{ and } d_v = 3| = 2n, \]
\[ |E_{[3,n]}| = |uv \in E(G_n) : d_u = 3 \text{ and } d_v = n| \]
\[ = |E(G_n)| - |E_{[2,3]}| = n. \]

Using definition of M-polynomial and above edge partitions, we get the desired result. \( \square \)

**Corollary 3.6.** If \( G_n \) is a gear graph, then
1. \( M_1(G_n) = n^2 + 13n, \)
2. \( M_2(G_n) = 3n^2 + 12n, \)
3. \( M_2^m(G_n) = \frac{n+1}{3}, \)
4. \( S_D(G_n) = \frac{n^2}{3} + \frac{13n}{3} + 3, \)
5. \( H(G_n) = \frac{4n}{5} + \frac{n}{n+3}, \)
6. \( I_n(G_n) = \frac{12n}{5} + \frac{3n^2}{n+3}, \)
7. \( \chi_a(G_n) = 2n5^a + n(n + 3)^a, \)
8. \( M_1^a(G_n) = n(2^{a+1} + 3^{a+1} + n^a). \)

**Definition 5.** The helm \( H_n \) is a graph obtained from a wheel \( W_n \) with central vertex \( c \), by attaching a pendant edge to each rim vertex of \( W_n \). A closed helm \( CH_n \) is the graph with central vertex \( c \), obtained from a helm by joining each pendant vertex to form a cycle.

**Theorem 3.7.** Let \( H_n \) be a helm. Then \( M(H_n; x, y) = nx^4 + nx^4y^4 + nx^4y^n. \)

**Proof.** Let \( H_n \) is a graph having \((2n + 1)\) vertices and \(3n\) edges. The edge partition of \( H_n \) is given by,
\[ |E_{[1,4]}| = |uv \in E(H_n) : d_u = 1 \text{ and } d_v = 4| = n, \]
\[ |E_{[4,4]}| = |uv \in E(H_n) : d_u = 4 \text{ and } d_v = 4| = n, \]
\[ |E_{[4,n]}| = |uv \in E(H_n) : d_u = 4 \text{ and } d_v = n| \]
\[ = |E(H_n)| - |E_{[1,4]}| - |E_{[4,4]}| = n. \]

**Corollary 3.8.** If \( H_n \) is a helm graph, then
1. \( M_1(H_n) = n^2 + 17n, \)
2. \( M_2(H_n) = 4n^2 + 20n, \)
3. \( M_2^m(H_n) = \frac{5n+4}{16}, \)
4. \( S_D(H_n) = \frac{n(n+1)}{4} + 6n + 4, \)
5. \( H(H_n) = \frac{2n}{5} + \frac{n}{4} + \frac{2n}{n+4}, \)
6. \( I_n(H_n) = \frac{n^2}{n+4} + \frac{14n}{5}, \)
7. \( \chi_a(H_n) = n(5^a + 8^a + (n + 4)^a) \),
8. \( M_1^a(H_n) = n(4^{a+1} + n^a) \).

**Theorem 3.9.** Let \( CH_n \) be a closed helm. Then
\[ M(CH_n; x, y) = nx^3 y^3 + nx^3 y^4 + nx^4 y^4 + nx^4 y^n. \]

**Proof.** Let \( CH_n \) is a graph having \((2n + 1)\) vertices and \(4n\) edges. The edge partition of \( CH_n \) is given by,
\[
\begin{align*}
|E_{[3,3]}| &= |uv \in E(CH_n): d_u = 3 \text{ and } d_v = 3| = n, \\
|E_{[3,4]}| &= |uv \in E(CH_n): d_u = 3 \text{ and } d_v = 4| = n, \\
|E_{[4,4]}| &= |uv \in E(CH_n): d_u = 4 \text{ and } d_v = 4| = n, \\
|E_{[4,n]}| &= |uv \in E(CH_n): d_u = 4 \text{ and } d_v = n| = n.
\end{align*}
\]

**Corollary 3.10.** If \( CH_n \) is a gear graph, then
1. \( M_1(CH_n) = n^2 + 25n \),
2. \( M_2(CH_n) = 4n^2 + 37n \),
3. \( M_2^m(CH_n) = \frac{37n+36}{144} \),
4. \( S_D(CH_n) = \frac{73n+3}{12} \),
5. \( H(CH_n) = \frac{n}{3} + \frac{n}{4} + \frac{2n}{7} + \frac{2n}{n+4} \),
6. \( I_n(CH_n) = \frac{3n}{2} + \frac{12n}{7} + \frac{4n^2}{n+4} + 2n \),
7. \( \chi_a(CH_n) = n(6^a + 7^a + 8^a + (n + 4)^a) \),
8. \( M^a(CH_n) = n(3^{a+1} + 4^{a+1} + n^a) \).

**Definition 6.** The flower \( Fl_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to the central vertex \( c \) of the helm.

**Theorem 3.11.** Let \( Fl_n \) be a flower. Then
\[ M(Fl_n; x, y) = nx^2 y^4 + nx^2 y^{2n} + nx^4 y^4 + nx^4 y^{2n}. \]

**Proof.** Let flower \( Fl_n \) is a graph having \((2n + 1)\) vertices and \(4n\) edges. The edge partition of \( Fl_n \) is given by,
\[
\begin{align*}
|E_{[2,4]}| &= |uv \in E(Fl_n): d_u = 2 \text{ and } d_v = 4| = n, \\
|E_{[2,2n]}| &= |uv \in E(Fl_n): d_u = 2 \text{ and } d_v = 2n| = n, \\
|E_{[4,4]}| &= |uv \in E(Fl_n): d_u = 4 \text{ and } d_v = 4| = n, \\
|E_{[4,2n]}| &= |uv \in E(Fl_n): d_u = 4 \text{ and } d_v = 2n| \\
&= |E(Fl_n)| - |E_{[2,4]}| - |E_{[2,2n]}| - |E_{[4,4]}| = n.
\end{align*}
\]
Corollary 3.12. If $F_{l_n}$ is a flower graph, then
1. $M_1(F_{l_n}) = 4n(n+5),$
2. $M_2(F_{l_n}) = 12n(n+2),$
3. $M_2^m(F_{l_n}) = \frac{3n+6}{16},$
4. $S_D(F_{l_n}) = \frac{3n^2}{2} + \frac{5n}{2} + 3,$
5. $H(F_{l_n}) = \frac{n}{n+1} + \frac{n}{n+2} + \frac{7n}{8},$
6. $I_n(F_{l_n}) = \frac{4n}{3} + \frac{2n^2}{n+1} + \frac{4n^2}{n+2} + 2n,$
7. $\chi_a(F_{l_n}) = n(6^a + 8^a + (2n+2)^a + (2n+4)^a),$
8. $M_1^a(F_{l_n}) = n(2^{a+1} + 4^{a+1} + n^{a2^{a+1}}).$

Definition 7. The sunflower graph $S_{F_n}$ is a graph obtained from a wheel with central vertex $c$, $n$-cycle $v_0,v_1,\ldots,v_{n-1}$ and additional $n$ vertices $w_0,w_1,\ldots,w_{n-1}$ where $w_i$ is joined by edges to $v_i,v_{i+1}$ for $i = 0,1,\ldots,n-1$ where $i+1$ is taken modulo $n$.

Theorem 3.13. Let $S_{F_n}$ be a sunflower. Then $M(S_{F_n};x,y) = 2nx^2y^5 + nx^5y^5 + nx^5y^n$.

Proof. The sunflower graph $S_{F_n}$ is a graph having $(2n+1)$ vertices and $4n$ edges. The edge partition of $S_{F_n}$ is given by,

$|E_{[2.5]}| = |uv \in E(S_{F_n}): d_u = 2 \text{ and } d_v = 5| = 2n,$

$|E_{[5.5]}| = |uv \in E(S_{F_n}): d_u = 5 \text{ and } d_v = 5| = n,$

$|E_{[5,n]}| = |uv \in E(S_{F_n}): d_u = 5 \text{ and } d_v = n|$

$= |E(S_{F_n})| - |E_{[2.5]}| - |E_{[5.5]}| = n.$

Corollary 3.14. If $S_{F_n}$ is a sunflower graph, then
1. $M_1(S_{F_n}) = n^2 + 29n,$
2. $M_2(S_{F_n}) = 5n(n+9),$ 
3. $M_2^m(S_{F_n}) = \frac{n}{5} + \frac{n}{25} + \frac{1}{5},$
4. $S_D(S_{F_n}) = \frac{n^2 + 39n + 25}{5},$
5. $H(S_{F_n}) = \frac{4n}{7} + \frac{n}{5} + \frac{2n}{n+5},$
6. $I_n(S_{F_n}) = \frac{5n^2}{n+5} + \frac{5n}{2} + \frac{20n}{7},$
7. $\chi_a(S_{F_n}) = n(2 \cdot 7^a + 10^a + (n+5)^a),$
8. $M_1^a(S_{F_n}) = n(2^{a+1} + 5^{a+1} + n^a).$
Definition 8. The friendship graph $f_n$ is a collection of $n$-triangles with a common vertex. Friendship graph can also be obtained from a wheel $W_{2n}$ with cycle $C_{2n}$ by deleting alternate edges of the cycle. That is $f_n = K_1 + nK_2$.

Theorem 3.15. Let $f_n$ be a friendship graph. Then $M(f_n; x, y) = nx^2y^2 + 2nx^2y^{2n}$.

Proof. Let friendship graph $f_n$ is a graph having $(2n + 1)$ vertices and $3n$ edges. The edge partition of $f_n$ is given by,

$$|E_{[2,2]}| = |uv \in E(f_n): d_u = 2 \text{ and } d_v = 2| = n,$$
$$|E_{[2,2n]}| = |uv \in E(f_n): d_u = 2 \text{ and } d_v = 2n|$$
$$= |E(f_n)| - |E_{[2,2]}| = 2n.$$

Corollary 3.16. If $f_n$ is a flower graph, then

1. $M_1(f_n) = 4n(n + 2),$  
2. $M_2(f_n) = 4n(2n + 1),$  
3. $M_2^m(f_n) = \frac{n+2}{4},$  
4. $S_D(f_n) = 2(n^2 + n + 1),$  
5. $H(f_n) = \frac{n}{2} + \frac{2n}{n+1},$  
6. $I_n(f_n) = n + \frac{4n^2}{n+1},$  
7. $\chi_a(f_n) = n(4^a + 2^a(n + 1)^a),$  
8. $M_a^a(f_n) = n2^{a+1}(n + 2).$

Definition 9. A web graph is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t,n)$ is the generalized web with $t$ cycles each of order $n$.

Theorem 3.17. Let $W(t,n)$ be a generalized web. Then

$$M(W(t,n); x, y) = nxy^4 + n(2t - 1)x^4y^4 + nx^4y^n.$$  

Proof. Let generalized web $W(t,n)$ is a graph having $(tn + n + 1)$ vertices and $n(2t + 1)$ edges. The edge partition of $W(t,n)$ is given by,

$$|E_{[1,4]}| = |uv \in E(W(t,n)): d_u = 1 \text{ and } d_v = 4| = n,$$
$$|E_{[4,4]}| = |uv \in E(W(t,n)): d_u = 4 \text{ and } d_v = 4| = n(2t - 1),$$
$$|E_{[4,n]}| = |uv \in E(W(t,n)): d_u = 4 \text{ and } d_v = n|$$
$$= |E(W(t,n))| - |E_{[1,4]}| - |E_{[4,4]}| = n.$$  

Corollary 3.18. If $W(t,n)$ be a generalized web, then
1. $M_1(W(t,n)) = n(n + 8(2t - 1) + 9),
2. $M_2(W(t,n)) = 4n(n + 4(2t - 1) + 1),
3. $M^m_2(W(t,n)) = \frac{n}{4} + \frac{n(2t-1)}{16} + \frac{1}{4^t},
4. $S_D(W(t,n)) = \frac{n^2}{2} + \frac{n}{4} + 2n(2t - 1) + 4n + 4,
5. $H(W(t,n)) = \frac{2n}{5} + \frac{n(2t-1)}{4} + \frac{2n}{n+4},
6. $I_n(W(t,n)) = \frac{4n}{5} + 2n(2t - 1) + \frac{4n^2}{n+4},
7. $\chi_a(W(t,n)) = n(5^a + (2t - 1)8^a + (4 + n)^a),
8. $M^a_1(W(t,n)) = 2n \cdot 4^a + 2n \cdot 4^a(2t - 1) + n^{a+1} + n.$

**Definition 10.** The crown (or sun) $CW_n$ is a corona of form $C_n \circ K_1$ where $n \geq 3$. That is crown is a helm without central vertex.

**Theorem 3.19.** Let $CW_n$ be a crown graph. Then

$$M(CW_n; x,y) = nx^3 + nx^3y^3.$$ 

**Proof.** Let $CW_n$ is a crown graph having 2n vertices and 2n edges. The edge partition of $CW_n$ is given by,

$$|E_{[1,3]}| = |uv \in E(CW_n): d_u = 1 \text{ and } d_v = 3| = n,$n

$$|E_{[3,3]}| = |uv \in E(CW_n): d_u = 3 \text{ and } d_v = 3|$$

$$= |E(CW_n)| - |E_{[1,3]}| = n.$$

**Corollary 3.20.** If $CW_n$ is a flower graph, then

1. $M_1(CW_n) = 10n,$
2. $M_2(CW_n) = 12n,$
3. $M^m_2(CW_n) = \frac{4n}{9},$
4. $S_D(CW_n) = \frac{10n}{3},$
5. $H(CW_n) = \frac{n}{2} + \frac{n}{3},$
6. $I_n(CW_n) = \frac{9n}{4},$
7. $\chi_a(CW_n) = n(4^a + 6^a),$
8. $M^a_1(CW_n) = n(3^{a+1} + 1).$

The duplication of an edge $[42] e = uv$ by a new vertex $v'$ in a graph $G$ produces a new graph $G'$ by adding a new vertex $v'$ such that $N(v') = \{u, v\}.$
Definition 11. Consider a wheel $W_n = C_n + K_1$ with $v_1, v_2, \ldots, v_n$ as its rim vertices and $c$ as its central vertex. Let $e_1, e_2, \ldots, e_n$ be the rim edges of $W_n$ which are duplicated by new vertices $w_1, w_2, \ldots, w_n$, respectively and let $f_1, f_2, \ldots, f_n$ be the spoke edges of $W_n$ which are duplicated by the vertices $u_1, u_2, \ldots, u_n$, respectively. The resultant graph is called duplication of the wheel denoted by $DuW_n$.

Theorem 3.21. Let $DuW_n$ be the duplication of the wheel. Then

$$M(DuW_n; x, y) = 3nx^2y^6 + nx^2y^{2n} + nx^6y^6 + nx^6y^{2n}.$$  

Proof. Let duplication of the wheel $DuW_n$ is a graph having $(3n + 1)$ vertices and $6n$ edges. The edge partition of $DuW_n$ is given by,

\[
|E_{[2,6]}| = |uv \in E(DuW_n) : d_u = 2 \text{ and } d_v = 6| = 3n, \\
|E_{[2,2n]}| = |uv \in E(DuW_n) : d_u = 2 \text{ and } d_v = 2n| = n, \\
|E_{[6,6]}| = |uv \in E(DuW_n) : d_u = 6 \text{ and } d_v = 6| = n, \\
|E_{[6,2n]}| = |uv \in E(DuW_n) : d_u = 6 \text{ and } d_v = 2n| = |E(DuW_n)| - |E_{[2,6]}| - |E_{[2,2n]}| - |E_{[6,6]}| = n.
\]

Corollary 3.22. If $CW_n$ be the duplication of the wheel, then

1. $M_1(DuW_n) = 4n(n + 11)$,
2. $M_2(DuW_n) = 8n(2n + 9)$,
3. $M_2^m(DuW_n) = \frac{5n+6}{18}$,
4. $S_D(DuW_n) = \frac{4n^2+17n+16}{4}$,
5. $H(DuW_n) = \frac{3n}{4} + \frac{n}{n+1} + \frac{n}{n+6} + \frac{n}{n+3}$,
6. $I_n(DuW_n) = \frac{9n}{2} + \frac{8n^2}{n+1} + 3n$,
7. $\chi_\alpha(DuW_n) = n(3 \cdot 8^\alpha + 12^\alpha + (2n+2)^\alpha + (2n+6)^\alpha)$,
8. $M_2^\alpha(DuW_n) = (4n \cdot 2^\alpha + 6n \cdot 6^\alpha + (2n)^{\alpha+1})$.

Definition 12. A uniform $n$-fan split graph $SF_{n}^r$, contains a star $S_{n-1}$ with hub at $x$ such that the deletion of $n$ edges of $S_{n-1}$ partitions the graph into $n$ independent fans $F_r^i = P_r^i + K_1, (1 \leq i \leq n)$ and a isolated vertex, Figure 2.
Corollary

Proof. The uniform $n$-fan split graph $SF_n^r$ has $(nr + n + 1)$ vertices and $2nr$ edges. The edge set of $SF_n^r$ can be partitioned as,

$$|E_{[2,3]}| = |uv \in E(SF_n^r) : d_u = 2 \text{ and } d_v = 3| = 2n,$$

$$|E_{[2r+1]}| = |uc \in E(SF_n^r) : d_u = 2 \text{ and } d_c = r + 1| = 2n,$$

$$|E_{[3,3]}| = |uc \in E(SF_n^r) : d_u = 3 \text{ and } d_c = 3| = n(r - 3),$$

$$|E_{[3r+1]}| = |uc \in E(SF_n^r) : d_u = 3 \text{ and } d_c = r + 1| = n(r - 2),$$

$$|E_{[nr+1]}| = |uc \in E(SF_n^r) : d_u = n \text{ and } d_c = r + 1| = |E(SF_n^r) - |E_{[2,3]}| - |E_{[2r+1]}| - |E_{[3,3]}| - |E_{[3r+1]}| = n.$$
**Definition 13.** The graph $SW^r_n$ contains a star $S_{n-1}$ with hub at $x$ such that the deletion of the $n$ edges of $S_{n-1}$ partitions the graph into $n$ independent wheels $W^i_r = C^i_r + K_1, (1 \leq i \leq n)$ and an isolated vertex, Figure 2.

**Theorem 3.25.** Let $SW^r_n$ be the graph having $(nr + n + 1)$ vertices and $n(2r + 1)$ edges. Then

$$M(SW^r_n; x, y) = nrx^3y^3 + nr^3x^r + nx^ny^r + 1.$$  

**Proof.** Let $SW^r_n$ be a graph having $(nr + n + 1)$ vertices and $n(2r + 1)$ edges. The edge partition of $SW^r_n$ is given by,

$$|E_{[3,3]}| = |uv \in E(SW^r_n): d_u = 3 \text{ and } d_v = 3| = nr,$$

$$|E_{[3,r+1]}| = |uv \in E(SW^r_n): d_u = 3 \text{ and } d_v = r + 1| = nr,$$

$$|E_{(n,r+1)}| = |uv \in E(SW^r_n): d_u = n \text{ and } d_v = r + 1| = |E(SW^r_n)| - |E_{[3,r+1]}| - |E_{[3,3]}| = n.$$

**Corollary 3.26.** If $SW^r_n$ graph, then

1. $M_1(SW^r_n) = n^2 + n(r + 1) + nr(r + 10),$  
2. $M_2(SW^r_n) = n^2(r + 1) + 3nr(r + 4),$  
3. $M_2^m(SW^r_n) = \frac{nr^2 + 4nr + 9}{9(r+1)},$  
4. $S_D(SW^r_n) = \frac{3n^2 + 3(r + 1)^2 + nr(r + 4)^2}{3(r+1)},$  
5. $H(SW^r_n) = \frac{2n}{n+1} + nr\left(\frac{r+10}{3(r+4)}\right),$  
6. $I_n(SW^r_n) = \frac{9nr(r+2)}{2(r+4)} + \left(\frac{n^2(r+1)}{n(r+4)}\right),$  
7. $x_a(SW^r_n) = nr \cdot 6^a + nr \cdot (r + 4)^a + n(n + r + 1)^a,$  
8. $M_1^a(SW^r_n) = 3nr \cdot 3^a + n^{a+1} + nr(r + 1)^a + n(r + 1)^a.$

**Definition 14.** Let $u_i, (1 \leq i \leq n) \text{ be the vertices of the complete graph } K_n. \text{ Let } W^i_r = C^i_r + K_1 \text{ be the wheel with hubs } w^i, (1 \leq i \leq n), \text{ respectively. Let } u_iw^i, (1 \leq i \leq n) \text{ be an edge. The graph so constructed is called uniform } n \text{-wheel split graph } KW(n, r), \text{ Figure 2.}$

**Note:** A uniform $n$-wheel split graph $KW(n, r)$ is a graph in which the deletion of $n$ edges $u_iw^i, (1 \leq i \leq n)$ partitions the graph into a complete graph and $n$ independent wheels $W_r$. This graph can be thought of as a generalization of the standard split graph in the sense that the elements of the independent sets are replaced by wheels here.
Figure 3. Graphs $SW(6,9)$ and $KDW(6,9)$.

**Theorem 3.27.** Let $KW(n,r)$ be a uniform $n$-wheel split graph. Then

$$M(KW(n,r); x, y) = nrx^3y^3 + nrx^3y^{r+1} + nx^ny^{r+1} + \binom{n}{2}x^ny^n.$$ 

**Proof.** Let $KW(n,r)$ be a uniform $n$-wheel split graph having $n(r+2)$ vertices and $\frac{n}{2}(4r+n+1)$ edges. The edge partition of $KW(n,r)$ is given by,

$$|E_{[3,3]}| = |uv \in E(KW(n,r)): d_u = 3 \text{ and } d_v = 3| = nr,$$

$$|E_{[3,r+1]}| = |uv \in E(KW(n,r)): d_u = 3 \text{ and } d_v = r+1| = nr,$$

$$|E_{[n,r+1]}| = |uv \in E(KW(n,r)): d_u = n \text{ and } d_v = r+1| = n,$$

$$|E_{[n,n]}| = |uv \in E(KW(n,r)): d_u = n \text{ and } d_v = n|$$

$$= |E(KW(n,r))| - |E_{[3,3]}| - |E_{[3,r+1]}| - |E_{[n,r+1]}| = \binom{n}{2}.$$

---

**Corollary 3.28.** If $KW(n,r)$ be a uniform $n$-wheel split graph, then

1. $M_1(KW(n,r)) = n^3 + n(r+1) + nr(r+10)$,
2. $M_2(KW(n,r)) = \frac{n^4 - n^3 + 2n^2(r+1) + 6nr(r+4)}{2},$
3. $M_2^m(KW(n,r)) = \frac{1}{18} \left( \frac{9(r+3) + 2nr(r+4)}{(r+1)} - \frac{9}{r+1} \right)$,
4. $S_D(KW(n,r)) = r - n + 1 + \frac{nr(r+4)^2}{3(r+1)} + n^2 \left( \frac{r+2}{r+1} \right)$,
5. $H(KW(n,r)) = nr \left( \frac{r+10}{3(r+4)} \right) + n \left( \frac{n+r+3}{2(n+r+1)} \right) - \frac{1}{2}$,
6. $I_n(KW(n,r)) = \frac{1}{4} n^2(n+3) + \frac{9nr}{2} - \frac{9nr}{(r+4)} - \frac{n^3}{(n+r+1)}$,
7. $\chi_a(KW(n,r)) = nr \cdot 6^a + nr \cdot (r+4)^a + n(n+r+1)^a + \left( \frac{n}{2} \right) (2n)^a$,
8. $M_1^a(KW(n,r)) = nr \cdot 3^{a+1} + n^{a+1} + n(n-1)n^a + nr(r+1)^a + n(r+1)^a$. 
**Definition 15.** Let \( u_i, (1 \leq i \leq n) \) be the vertices of a star \( S_{n-1} \) with a hub at \( x \). Let \( u_iw^i, (1 \leq i \leq n) \) be an edge. Let \( W^i = C^i + K_1 \) be wheels with hubs \( w^i, (1 \leq i \leq n) \). The graph so obtained is denoted by \( SW(n,r) \), Figure 3.

**Theorem 3.29.** Let \( SW(n,r) \) be the graph having \( n(r + 2) + 1 \) vertices and \( 2n(r + 1) \) edges. Then

\[
M(SW(n,r);x,y) = nx^2y^n + nx^2y^{r+1} + nrx^3y^3 + nrx^3y^{r+1}.
\]

**Proof.** Let \( SW(n,r) \) is a graph having \( n(r + 2) + 1 \) vertices and \( 2n(r + 1) \) edges. The edge partition of \( SW(n,r) \) is given by,

\[
|E_{(2,n)}| = |uv \in E(SW(n,r)): d_u = 2 \text{ and } d_v = n| = n,
\]

\[
|E_{(2,r+1)}| = |uv \in E(SW(n,r)): d_u = 2 \text{ and } d_v = r + 1| = n,
\]

\[
|E_{(3,3)}| = |uv \in E(SW(n,r)): d_u = 3 \text{ and } d_v = 3| = nr,
\]

\[
|E_{(3,r+1)}| = |uv \in E(SW(n,r)): d_u = 3 \text{ and } d_v = r + 1| = |E(SW(n,r)) - |E_{(2,n)}| - |E_{(2,r+1)}| - |E_{(3,3)}| = nr.
\]

**Corollary 3.30.** If \( SW(n,r) \) be a graph, then

1. \( M_1(SW(n,r)) = n^2 + n(r + 5) + nr(r + 10), \)
2. \( M_2(SW(n,r)) = 2n^2 + 2n(r + 1) + 3nr(r + 4), \)
3. \( M_2^m(SW(n,r)) = \frac{2nr^2 + 8nr + 9(n + r + 1)}{18(r + 1)}, \)
4. \( S_D(SW(n,r)) = \frac{3n^2(r + 1) + 3n^2(r^2 + 2r + 5) + 26(r + 1) + nr(r + 4) + 6(r + 1)}{6(r + 1)}, \)
5. \( H(SW(n,r)) = \frac{nr(r + 10)}{3(r + 4)} + \frac{2n(n + r + 5)}{(n + 2)(r + 3)}, \)
6. \( I_n(SW(n,r)) = \frac{2n^2}{n + 2} + \frac{2n + 1}{r + 3} + \frac{9nr(r + 2)}{2(r + 4)}, \)
7. \( \chi_a(SW(n,r)) = n(n + 2)^a + n(r + 1)^a + nr \cdot 6^a + nr(r + 4)^a, \)
8. \( M_1(a)(SW(n,r)) = n2^{a+1} + nr \cdot 3^{a+1} + n^{a+1} + n(r + 1)^a + nr(r + 1)^a. \)

**Definition 16.** Let \( x_i, (1 \leq i \leq n) \) be the vertices of the complete graph \( K_n \). Let \( W^i = C^i + K_1 \) be wheel with hub \( w^i, (1 \leq i \leq n) \). Let \( x_iw^i, (1 \leq i \leq n) \) be an edge. Subdivide each edge \( x_iw^i \) by \( u_i, (1 \leq i \leq n) \). The graph so obtained is denoted by \( KDW(n,r) \), Figure 3.

**Theorem 3.31.** Let \( KDW(n,r) \) be the graph having \( n(r + 3) \) vertices and \( \frac{n}{2}(4r + n + 3) \) edges. Then

\[
M(KDW(n,r);x,y) = nx^2y^n + nx^2y^{r+1} + nrx^3y^3 + nrx^3y^{r+1} + \left(\binom{n}{2}\right)x^ny^n.
\]
Proof. Let $KDW(n, r)$ is a graph having $n(r + 3)$ vertices and $\frac{n}{2}(4r + n + 3)$ edges. The edge partition of $KDW(n, r)$ is given by,

\begin{align*}
|E_{[2,n]}| &= |uv \in E(KDW(n, r)) : d_u = 2 \text{ and } d_v = n| = n, \\
|E_{[2,r+1]}| &= |uv \in E(KDW(n, r)) : d_u = 2 \text{ and } d_v = r + 1| = n, \\
|E_{[3,3]}| &= |uv \in E(KDW(n, r)) : d_u = 3 \text{ and } d_v = 3| = nr, \\
|E_{[3,r+1]}| &= |uv \in E(KDW(n, r)) : d_u = 3 \text{ and } d_v = r + 1| = nr, \\
|E_{[n,n]}| &= |uv \in E(KDW(n, r)) : d_u = n \text{ and } d_v = n| \\
&= |E(KDW(n, r))| - |E_{[2,n]}| - |E_{[2,r+1]}| - |E_{[3,3]}| - |E_{[3,r+1]}| = \binom{n}{2}.
\end{align*}

\[\square\]

Corollary 3.32. If $KDW(n, r)$ be a graph, then

1. $M_1(KW(n, r)) = n^3 + n(r + 5) + nr(r + 10),$
2. $M_2(KW(n, r)) = \frac{n(n(n - n - 1) + 4) + nr(3r + 14)}{2},$
3. $M_2^m(KW(n, r)) = \frac{9n^2 - 9(r + 1) + 2n(9r + 1) + nr(r + 4)}{18n(r + 1)},$
4. $S_D(KW(n, r)) = \frac{3(r + 1)(3n^2 + 4) + 3n(r^2 + 3) + 2nr(r + 4)^2}{6(r + 1)},$
5. $H(KW(n, r)) = n\left(\frac{1}{2} + \frac{2}{n + 2} + \frac{2}{r + 3}\right) + nr\left(\frac{1}{3} + \frac{2}{r + 4}\right) - \frac{1}{2},$
6. $I_n(KW(n, r)) = \frac{n^3}{4} + n^2\left(\frac{2}{n + 2} - \frac{1}{4}\right) + \frac{2n(r + 1)}{r + 3} + \frac{9nr(2 + r)}{2(r + 4)},$
7. $\chi_a(KW(n, r)) = n \cdot (n + 2)^a + n(r + 3)^a + nr \cdot 6^a + nr \cdot (r + 4)^a + \binom{n}{2}(2n)^a,$
8. $M_2^a(KW(n, r)) = n \cdot 2^{a + 1} + nr \cdot 3^{a + 1} + n^{a + 1} + n(r + 1)^a + nr(r + 1)^a + (n - 1)n^{a + 1}.$

4. M–POLYNOMIAL OF SOME NANOSTRUCTURES

In science and technology, nanostructures play a vital role in small electronic devices to big satellites, pharmaceutical and medical treatments, communication and information, food science and so on. Among these, M-polynomial of dendrimers were studied in [33], V-phenylenic nanotubes and nanotori in [29] titania nanotubes in [34], Armchair polyhex nanotube and zig-zag polyhex nanotubes were encountered in [35]. In this paper, we consider $TUC_4C_8[p, q]$ nanotube, $TUC_4C_8[p, q]$ nanotorus, line graph of the subdivision graph of $TUC_4C_8[p, q]$ nanotube and $TUC_4C_8[p, q]$ nanotorus, V-tetracenic nanotube and V-tetracenic nanotorus and compute M-polynomial.

Let $p$ and $q$ denote the number of squares in a row and the number of rows of squares, respectively in nanotube and nanotorus of $TUC_4C_8[p, q]$.

The nanotube and nanotorus of $TUC_4C_8[4, 3]$ is shown in Figure 4 (a), (b) respectively. The line graph of subdivision graph of $TUC_4C_8[4, 3]$ nanotube is given in Figure 5 (b). The line graph of
subdivision graph of $TUC_4C_8[4,2]$ nanotorus is given in Figure 6 (b). The structures V-tetracenic nanotube and V-tetracenic nanotorus are given in Figures 7 and 8, respectively.

![Figure 4](image1.png) ![Figure 5](image2.png) ![Figure 6](image3.png)

Figure 4. (a) $TUC_4C_8[4,3]$ nanotube; (b) $TUC_4C_8[4,3]$ nanotorus.

Figure 5. (a) Subdivision graph of $TUC_4C_8[4,3]$ of nanotube; (b) line graph of the subdivision graph of $TUC_4C_8[4,3]$ of nanotube.

Figure 6. (a) Subdivision graph of $TUC_4C_8[4,2]$ of nanotorus; (b) line graph of the subdivision graph of $TUC_4C_8[4,2]$ of nanotorus.

We now obtain M-polynomial of these nanostructures as follows.

**Theorem 4.1.** Let $A = TUC_4C_8[p,q]$ nanotube. Then

\[
M(A; x, y) = 4px^2y^3 + (6pq - 5p)x^3y^3.
\]

**Proof.** The $TUC_4C_8[p,q]$ nanotube has $4pq$ vertices and $6pq - p$ edges. The edge set of $TUC_4C_8[p,q]$ nanotube can be partitioned as,
Theorem 4.2. Let $B = TUC_4C_8[p, q]$ nanotorus. Then, $M(B; x, y) = 6px^3y^3$.

Proof. The $TUC_4C_8[p, q]$ nanotorus is a 3-regular graph with $6pq$ edges. Thus, from Lemma 2.1, M-polynomial of $TUC_4C_8[p, q]$ nanotorus is $M(B; x, y) = 6px^3y^3$.

Theorem 4.3. Let $C$ be the line graph of subdivision graph of $TUC_4C_8[p, q]$ nanotube. Then

$$M(C; x, y) = 2px^2y^2 + 4px^2y^3 + 6pqx^2y^3 - 5px^3y^3.$$

Proof. The line graph of subdivision graph of $TUC_4C_8[p, q]$ nanotube has $12pq - 2p$ vertices and $18pq - 5p$ edges. The edge partition of line graph of subdivision graph of $TUC_4C_8[p, q]$ nanotube is given by,

$$|E_{[2,2]}| = |uv \in E(C): d_u = 2 \text{ and } d_v = 2| = 2p,$$
$$|E_{[2,3]}| = |uv \in E(C): d_u = 2 \text{ and } d_v = 3| = 4p,$$
$$|E_{[3,3]}| = |uv \in E(C): d_u = 3 \text{ and } d_v = 3| = 6pq - 11p.$$

Theorem 4.4. Let $D$ be the line graph of subdivision graph of $TUC_4C_8[p, q]$ nanotorus. Then $M(D; x, y) = 18px^3y^3$.

Proof. The line graph of subdivision graph of $TUC_4C_8[p, q]$ nanotorus is a 3-regular graph with $18pq$ edges. Thus, from Lemma 2.1 we have, $M(D; x, y) = 18px^3y^3$.

![Figure 7. V-tetracenic nanotube $G[p, q]$.](image)

Theorem 4.5. Let $H$ be the V-tetracenic nanotube. Then

$$M(H; x, y) = 16px^2y^3 + (27q - 20)px^3y^3.$$
**Proof.** The V-tetracenic nanotube has $18pq$ vertices and $27pq - 4p$ edges. The edge partition of V-tetracenic nanotube is obtained as,

\[ |E_{2,3}| = |uv \in E(H): d_u = 2 \text{ and } d_v = 3| = 16p, \]
\[ |E_{3,3}| = |uv \in E(H): d_u = 3 \text{ and } d_v = 3| = |E(H)| - |E_{2,3}| = 27pq - 20p. \]

**Figure 8.** V-tetracenic nanotorus $G[p, q]$.

**Theorem 4.6.** Let $I$ be the V-tetracenic nanotorus. Then $M(I; x, y) = 27pqx^3y^3$.

**Proof.** The proof follows from Lemma 2.1 as V-tetracenic nanotorus is 3-regular graph with $27pq$ edges.

We skip calculating topological indices of these nanostructures as it is routine work.

5. **CONCLUDING REMARKS**

In this paper, we have proposed new operators to derive general sum connectivity index and first general Zagreb index of a graph from the respective M-polynomial. Further, we have obtained M-polynomials of some graph operations and cycle related graphs. In addition, some degree based topological indices of these graphs are derived. The advantage of M-polynomial is that, from that one expression we can obtain several degree-based topological indices. It is very challenging to obtain new operators to derive all the degree-based topological indices from M-polynomial.

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