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A Note on Revised Szeged Index of Graph Operations

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ABSTRACT

Let G be a finite and simple graph with edge set $E(G)$. The revised Szeged index is defined as

$$Sz^*(G) = \sum_{e=uv\epsilon E(G)} (n_u(e|G) + \frac{n_G(e)}{2})(n_v(e|G) + \frac{n_G(e)}{2}),$$

where $n_u(e|G)$ denotes the number of vertices in Glying closer to u than to v and $n_G(e)$ is the number of equidistant vertices of e in G. In this paper, we compute the revised Szeged index of the join and corona product of graphs.

1 INTRODUCTION

Let G be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph G, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N_G(v) = N(v)$ is the set $\{u \in V(G) | uv \in E(G)\}$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. Let $u, v \in V(G)$, then the *distance* $d_G(u, v)$ between u and v is defined as the length of any shortest path in G connecting u and v. We consult [14] for notation and terminology which are not defined here.

The first and second Zagreb indices are defined as $M_1(G) = \sum_{u\in V(G)} d_G^2(u)$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, respectively. Furtula and Gutman [5] defined the

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forgotten topological index as $F(G) = \sum_{uv\in E(G)}(d_G^2(u) + d_G^2(v))$. The interested readers are referred to [3,7] for more information on this topic.

A vertex $w \in V(G)$, is said to be equidistant from the edge $e = uv$ of G if $d_G(u,w) = d_G(v,w)$. The number of equidistant vertices of e is denoted by $n_G(e)$. Let uv be an edge of G. Define the sets $N(u,G) = \{x \in V(G) | d_G(u,x) < d_G(v,x)\}$ and $N(v,G) = \{x \in V(G) | d_G(v,x) < d_G(u,x)\}$ consisting, respectively, of vertices of G lying closer to u than to v, and lying closer to v than to u. The number of such vertices is then $n_u(e|G) = |N(u,G)|$ and $n_v(e|G) = |N(v,G)|$. Note that vertices equidistant to u and v are not included into either $N(u,G)$ or $N(v,G)$. It also worth noting that $u \in N(u,G)$ and $v \in N(v,G)$, which implies that $n_u(e|G) \geq 1$ and $n_u(e|G) \geq 1$. The Szeged index $Sz(G)$ was introduced by Gutman [6]. It is defined as $Sz(G) = \sum_{e=uv\epsilon E(G)} n_u(e|G)n_v(e|G)$.

The Szeged index in graphs is well studied in the literature, see for example [9,10]. Randić [13] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named as the revised Szeged index. The revised Szeged index of a connected graph G is defined as $Sz^*(G) = \sum_{e=uv\epsilon E(G)}(n_u(e|G) + \frac{n_G(e)}{2})(n_v(e|G) + \frac{n_G(e)}{2})$. Nagarajan et al. [11] obtained the revised Szeged index of the Cartesian product of two connected graphs. In this paper we compute the revised Szeged index of the join and corona product of graphs. Readers interested in more information on computing topological indices of graph operations can be referred to [1,2,4,8,12].

2. MAIN RESULTS

In this section, we compute the revised Szeged index of the join and corona product of graphs. We let for every edge $e = uv \in E(G)$, $t_{uv}(G) = |N_G(u) \cap N_G(v)|$.

2.1. THE JOIN OF GRAPHS

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. Obviously, $|V(G)| = |V_1| + |V_2|$ and $|E(G)| = |E_1| + |E_2| + |V_1||V_2|$.

Theorem 1. Let G_1 be a graph of order n_1 and of size m_1 and let G_2 be a graph of order n_2 and of size m_2. If $G = G_1 + G_2$, then

$$Sz^*(G) = \frac{2M_2(G_1) + 2M_2(G_2) - n_2M_1(G_1) - n_1M_1(G_2) - F(G_1) - F(G_2)}{4}$$
$$+\frac{4n_1^2n_2^2 + 8m_1m_2 + m_1(n_1^2 + 6n_1n_2 - 3n_2^2) + m_2(n_2^2 + 6n_1n_2 - 3n_1^2)}{4}.$$

Proof. By definition, $Sz^*(G) = \sum_{uv\epsilon E(G)}(n_u(e|G) + \frac{n_G(e)}{2})(n_v(e|G) + \frac{n_G(e)}{2})$. We partition the edges of G in to three subset E_1, E_2 and E_3, as $E_1 = \{e = uv \mid u, v \in V(G_1)\}$, $E_2 = \{e = uv \mid u, v \in V(G_2)\}$ and $E_3 = \{e = uv \mid u \in V(G_1), v \in V(G_2)\}$.

Let $e = uv \in E_1$. If $w \in V(G_2)$ or $w \in N_{G_1}(u) \cap N_{G_1}(v)$, then $d_G(u, w) = d_G(v, w) = 1$ and if $w \notin N_{G_1}(u) \cup N_{G_1}(v)$, then $d_G(u, w) = d_G(v, w) = 2$. Hence $n_u(e|G) = d_{G_1}(u) - t_{uv}(G_1) + 1, n_v(e|G) = d_{G_1}(v) - t_{uv}(G_1) + 1$ and $n_G(e) = n_1 + n_2 + 2t_{uv}(G_1) - \left(d_{G_1}(u) + d_{G_1}(v)\right) - 2$. Then for every edge $e = uv \in E_1$,

$$\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \left(\frac{n_1+n_2+d_{G_1}(u)-d_{G_1}(v)}{2}\right)\left(\frac{n_1+n_2+d_{G_1}(v)-d_{G_1}(u)}{2}\right)$$
$$= \frac{(n_1+n_2)^2}{4} + \frac{d_{G_1}(u)d_{G_1}(v)}{2} - \frac{d_{G_1}^2(u)+d_{G_1}^2(v)}{4}.$$

Therefore

$$\sum_{uv\epsilon E_1}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \sum_{uv\epsilon E_1}\frac{(n_1+n_2)^2}{4} + \sum_{uv\epsilon E_1}\frac{d_{G_1}(u)d_{G_1}(v)}{2} - \sum_{uv\epsilon E_1}\frac{d_{G_1}^2(u)+d_{G_1}^2(v)}{4}$$
$$= \frac{(n_1+n_2)^2}{4}m_1 + \frac{M_2(G_1)}{2} - \frac{F(G_1)}{4}. \quad (1)$$

Similarly,

$$\sum_{uv\epsilon E_2}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \frac{(n_1+n_2)^2}{4}m_2 + \frac{M_2(G_2)}{2} - \frac{F(G_2)}{4}. \quad (2)$$

Let $e = uv \in E_3$ such that $u \in V(G_1)$ and $v \in V(G_2)$. If $w \in N_{G_1}(u) \cup N_{G_2}(v)$, then $d_G(u, w) = d_G(v, w) = 1$. Hence $n_u(e|G) = n_2 - d_{G_2}(v) + 1, n_v(e|G) = n_1 - d_{G_1}(u) + 1$ and $n_G(e) = d_{G_1}(u) + d_{G_2}(v) - 2$. Then for every edge $e = uv \in E_3$,

$$\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \left(\frac{2n_2+d_{G_1}(u)-d_{G_2}(v)}{2}\right)\left(\frac{2n_1+d_{G_2}(v)-d_{G_1}(u)}{2}\right)$$
$$= n_1n_2 + \frac{n_1-n_2}{2}d_{G_1}(u) + \frac{n_2-n_1}{2}d_{G_2}(v) - \frac{d_{G_1}^2(u)}{4} - \frac{d_{G_2}^2(v)}{4} + \frac{d_{G_1}(u)d_{G_2}(v)}{2}.$$

Set $Y = \sum_{uv\epsilon E_3}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right)$. Then,

$$Y = \sum_{uv\epsilon E_3} n_1n_2 + \sum_{uv\epsilon E_3}\frac{n_1-n_2}{2}d_{G_1}(u) + \sum_{uv\epsilon E_3}\frac{n_2-n_1}{2}d_{G_2}(v) - \sum_{uv\in E_3}\frac{d_{G_1}^2(u)}{4} - \sum_{uv\epsilon E_3}\frac{d_{G_2}^2(v)}{4} + \sum_{uv\epsilon E_3}\frac{d_{G_1}(u)d_{G_2}(v)}{2}$$

$$= n_1^2 n_2^2 + m_1 n_2(n_1 - n_2) + m_2 n_1(n_2 - n_1) + 2m_1 m_2 - \frac{n_2 M_1(G_1)}{4} - \frac{n_1 M_1(G_2)}{4}. \qquad (3)$$

By Equations (1), (2) and (3), we have:

$$Sz^*(G) = \frac{(n_1+n_2)^2}{4} m_1 + \frac{M_2(G_1)}{2} - \frac{F(G_1)}{4} + \frac{(n_1+n_2)^2}{4} m_2 + \frac{M_2(G_2)}{2} - \frac{F(G_2)}{4} + n_1^2 n_2^2 + 2m_1m_2$$
$$- \frac{n_1 M_1(G_2)}{4} + m_1 n_2(n_1 - n_2) + m_2 n_1(n_2 - n_1) - \frac{n_2 M_1(G_1)}{4}$$
$$= \frac{2M_2(G_1) + 2M_2(G_2) - n_2 M_1(G_1) - n_1 M_1(G_2) - F(G_1) - F(G_2)}{4}$$
$$+ \frac{4n_1^2 n_2^2 + 8m_1 m_2 + m_1(n_1^2 + 6n_1 n_2 - 3n_2^2) + m_2(n_2^2 + 6n_1 n_2 - 3n_1^2)}{4}.$$

Let P_n, $n \geq 2$ and C_n, $n \geq 3$ denote the path and the cycle on n vertices, respectively.

Corollary 2. The following equalities are hold:

1. $Sz^*(P_n + P_m) = \frac{4n^2m^2 + n^3 + m^3 + 3nm^2 + 3mn^2 + 2n^2 + 2m^2 - 2n - 2m - 12nm + 4}{4}$.
2. $Sz^*(P_n + C_m) = \frac{4n^2m^2 + n^3 + m^3 + 3nm^2 + 3mn^2 - n^2 + 3m^2 - 2m - 6nm - 2}{4}$.
3. $Sz^*(C_n + C_m) = \frac{4n^2m^2 + n^3 + m^3 + 3nm^2 + 3mn^2}{4}$.

2.2. THE CORONA PRODUCT OF GRAPHS

The corona product $G = G_1 o G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is as the graph obtained by taking one copy of G_1 and $|V_1|$ copies of G_2 and joining the *i*-th vertex of G_1 to every vertex in *i*-th copy of G_2. Obviously, $|V(G)| = |V_1| + |V_1||V_2|$ and $|E(G)| = |E_1| + |V_1||E_2| + |V_1||V_2|$.

Theorem 3. Let G_1 be a graph of order n_1 and of size m_1 and let G_2 be a graph of order n_2 and of size m_2. If $G = G_1 o G_2$, then

$$Sz^*(G) = \frac{(n_1 n_2 + n_1)^2}{4}(m_1 + m_2) + n_1 n_2(n_1 n_2 + n_1 - 1)$$
$$+ n_1 m_2(n_1 n_2 + n_1 - 2) - \frac{(n_2+1)^2}{4} \sum_{uv \in E_1} (n_u^2(e|G_1) + n_v^2(e|G_1))$$
$$+ \frac{2(n_2+1)^2 \mathrm{Sz}(G_1) + 2M_2(G_2) - n_1 M_1(G_2) - F(G_2)}{4}.$$

Proof. By definition, $Sz^*(G) = \sum_{uv \epsilon E(G)} (n_u(e|G) + \frac{n_G(e)}{2})(n_v(e|G) + \frac{n_G(e)}{2})$. We partition the edges of G in to three subsets E_1, E_2 and E_3, as $E_1 = \{e = uv \mid u, v \in V(G_1)\}$, $E_2 = \{e = uv \mid u, v \in V(G_2)\}$ and $E_3 = \{e = uv \mid u \in V(G_1), v \in V(G_2)\}$. Let $e = uv \in E_1$. Then for each vertex w closer to u than v, the vertices of the copy of G_2 attached to w are also closer to u than v. Since each copy of G_2 has exactly n_2 vertices, then $n_u(e|G) = (n_2 + 1)n_u(e|G_1)$. Similarly $n_v(e|G) = (n_2 + 1)n_v(e|G_1)$. Then $n_G(e) = n_1n_2 + n_1 - (n_2 + 1)n_u(e|G_1) - (n_2 + 1)n_v(e|G_1)$. Hence for every edge $e = uv \in E_1$,

$$\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \frac{(n_1n_2+n_1)^2}{4} + \frac{(n_2+1)^2 n_u(e|G_1)n_v(e|G_1)}{2} - \frac{(n_2+1)^2(n_u^2(e|G_1)+n_v^2(e|G_1))}{4}.$$

Define $Z = \sum_{uv\epsilon E_1}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right)$. Then,

$$Z = \sum_{uv\epsilon E_1}\frac{(n_1n_2+n_1)^2}{4} + \sum_{uv\epsilon E_1}\frac{(n_2+1)^2 n_u(e|G_1)n_v(e|G_1)}{2} - \sum_{uv\epsilon E_1}\frac{(n_2+1)^2(n_u^2(e|G_1)+n_v^2(e|G_1))}{4}$$
$$= \frac{(n_1n_2+n_1)^2}{4}m_1 + \frac{(n_2+1)^2 Sz(G_1)}{2} - \frac{(n_2+1)^2}{4}\sum_{e=uv\epsilon E_1}(n_u^2(e|G_1) + n_v^2(e|G_1)). \quad (4)$$

Let $e = uv \in E_2$. If $w \in V(G_2)$ and $w \in N_{G_2}(u) \cap N_{G_2}(v)$, then $d_G(u,w) = d_G(v,w) = 1$ and if $w \notin N_{G_2}(u) \cup N_{G_2}(v)$, then $d_G(u,w) = d_G(v,w) = 2$. Hence $n_u(e|G) = d_{G_2}(u) - t_{uv}(G_2) + 1$, $n_v(e|G) = d_{G_2}(v) - t_{uv}(G_2) + 1$ and $n_G(e) = n_1n_2 + n_1 + 2t_{uv}(G_2) - \left(d_{G_2}(u) + d_{G_2}(v)\right) - 2$. Hence for every edge $e = uv \in E_2$,

$$\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \left(\frac{n_1n_2+n_1+d_{G_2}(u)-d_{G_2}(v)}{2}\right)\left(\frac{n_1n_2+n_1+d_{G_2}(v)-d_{G_2}(u)}{2}\right)$$
$$= \frac{(n_1n_2+n_1)^2}{4} + \frac{d_{G_2}(u)d_{G_2}(v)}{2} - \frac{d_{G_2}^2(u)+d_{G_2}^2(v)}{4}.$$

Therefore

$$\sum_{uv\epsilon E_2}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \sum_{uv\epsilon E_2}\frac{(n_1n_2+n_1)^2}{4} + \sum_{uv\epsilon E_2}\frac{d_{G_2}(u)d_{G_2}(v)}{2} - \sum_{uv\epsilon E_2}\frac{d_{G_2}^2(u)+d_{G_2}^2(v)}{4}$$
$$= \frac{(n_1n_2+n_1)^2}{4}m_2 + \frac{M_2(G_2)}{2} - \frac{F(G_2)}{4}. \quad (5)$$

Let $e = uv \in E_3$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Hence $n_u(e|G) = n_1n_2 + n_1 - d_{G_2}(v) - 1$. Since $v \in N(v,G)$, we have $n_v(e|G) = 1$ and so $n_G(e) = d_{G_2}(v)$. Hence

$$\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \left(\frac{2(n_1n_2+n_1-1)-d_{G_2}(v)}{2}\right)\left(\frac{2+d_{G_2}(v)}{2}\right)$$
$$= (n_1n_2 + n_1 - 1) + \frac{n_1n_2+n_1-2}{2}d_{G_2}(v) - \frac{d_{G_2}^2(v)}{4}.$$

Therefore,

$$\sum_{uv\epsilon E_3}\left(n_u(e|G) + \frac{n_G(e)}{2}\right)\left(n_v(e|G) + \frac{n_G(e)}{2}\right) = \sum_{uv\epsilon E_3}(n_1n_2 + n_1 - 1) - \sum_{uv\epsilon E_3}\frac{d_{G_2}^2(v)}{4} + \sum_{e=uv\epsilon E_3}\frac{n_1n_2+n_1-2}{2}d_{G_2}(v)$$

$$= n_1n_2(n_1n_2 + n_1 - 1) - \frac{n_1M_1(G_2)}{4}$$
$$+n_1m_2(n_1n_2 + n_1 - 2). \qquad (6)$$

By Equations (4), (5) and (6), we have:

$$\begin{aligned} Sz^*(G) &= \frac{(n_1n_2+n_1)^2}{4}m_1 + \frac{(n_2+1)^2Sz(G_1)}{2} - \frac{(n_2+1)^2}{4}\sum_{uv\epsilon E_1}(n_u^2(e|G_1) + n_v^2(e|G_1)) \\ &+ \frac{(n_1n_2+n_1)^2}{4}m_2 + \frac{M_2(G_2)}{2} - \frac{F(G_2)}{4} + n_1n_2(n_1n_2 + n_1 - 1) \\ &+n_1m_2(n_1n_2 + n_1 - 2) - \frac{n_1M_1(G_2)}{4} \\ &= \frac{(n_1n_2+n_1)^2}{4}(m_1 + m_2) + n_1n_2(n_1n_2 + n_1 - 1) \\ &+n_1m_2(n_1n_2 + n_1 - 2) - \frac{(n_2+1)^2}{4}\sum_{uv\epsilon E_1}(n_u^2(e|G_1) + n_v^2(e|G_1)) \\ &+ \frac{2(n_2+1)^2\mathrm{Sz}(G_1)+2M_2(G_2)-n_1M_1(G_2)- F(G_2)}{4}. \end{aligned}$$

Corollary 4. The following equalities are hold:

1. $Sz^*(P_noP_m) = \frac{2n^3m^2+3n^2m^3+24n^2m^2+4n^3m-2nm^2+2n^3+15mn^2-15n^2+40n-52nm-6}{4}$.
2. $Sz^*(P_noC_m) = \frac{2n^3m^2+2n^2m^3+28n^2m^2+4n^3m-2nm^2+2n^3+26mn^2-2n-52nm}{4}$.
3. $Sz^*(C_noP_m) = \frac{n^3m^2+n^2m^3+9n^2m^2+2n^3m+n^3+3mn^2-5n^2-16nm-2n-2}{4}$.
4. $Sz^*(C_noC_m) = \frac{n^3m^2+n^2m^3+10n^2m^2+2n^3m+n^3+9mn^2-16nm}{4}$.

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