Iranian Journal of Mathematical Chemistry Vol. 4, No. 1, March 2013, pp. 27 − 40 **IJMC**

Infinite product representation of solution of indefinite Sturm-Liouville problem

S. MOSAZADEH[•]

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-51167, I. R. Iran

(*Received December 1, 2012; Accepted January 19, 2013*)

ABSTRACT

In this paper, we investigate infinite product representation of the solution of a Sturm-Liouville equation with an indefinite weight function which has two zeros and/or singularities in a finite interval. First, by using of the asymptotic estimates provided in [W. Eberhard, G. Freiling, K. Wilcken-Stoeber, Indefinite eigenvalue problems with several singular points and turning points, Math. Nachr. 229, 51-71 (2001)] for a special fundamental system of the solutions of Sturm-Liouville equation, we obtain the asymptotic behavior of it's solutions and eigenvalues, then we obtain the infinite product representation of solution of the equation.

Keywords: Singularities, turning points, Sturm-Liouville problem, non-definite problem, infinite products, Hadamard's theorem.

1 INTRODUCTION

We consider the Sturm-Liouville equation of the form

$$y'' + (\lambda\phi^2(t) - q(t))y = 0, \qquad 0 \le t \le 1, \tag{1}$$

with initial conditions $y(0,\lambda) = 1, y'(0,\lambda) = 0,$ on a finite interval $I = [0,1]$. Here $\lambda = \rho^2$ is the spectral parameter. We assume that the *weight function* ϕ^2 is real with a finite number of zeros and/or *singularities* of first order in the open interval $(0,1)$, these zeros and singular points are the so–called *turning points* of (1). Moreover, these turning points are admitted to be singularities of first order of the *potential function* $q(t)$. The Sturm–Liouville

[•] Email: *s.mosazadeh@kashanu.ac.ir*

problem is said to be non-definite if the quadratic form $\int_0^1 |y(t)|^2 \phi^2(t)dt$ associated with this equation is *indefinite* on the space of all differentiable functions y in the interval *I,* having the special property (see [9] for more details).

The representation of solutions of Sturm–Liouville equations by means of an infinite product is a direct consequence of the fact that any solution $y(t,\lambda)$ defined by a fixed set of initial conditions (as we have seen above) is necessarily an entire function of λ for each fixed $t \in I$, whose order does not exceed 1/2 (see [3]). It follows from the classical Hadamard's factorization theorem that such solutions are expressible as an infinite product, and so this gives an alternate description that has not been used as of yet for approximation purposes in the various applications.

The importance of asymptotic analysis in obtaining information on the solution of Sturm-Liouville equation (1) with multiple singularities and turning points was realized by Freiling and Yurko [5] and Eberhard, Freiling and Stoeber in [4]. Also, inverse problem for equation (1) with singularities or turning points of even order were studied in [10].

The subject Sturm-Liouville problem can also be seen inside the wider context of ordinary differential equations on multistructures, that have been a subject of increasing interest in the recent years, in relation with several problems arising in physics, engineering, chemistry, quantum chemistry and chemical engineering. Also, this autonomous equation arises in mechanics, combustion theory, and the theory of mass transfer with chemical reactions. For example, in (1), to a quantum physicist or chemist, $q(t)$ is a potential function describing a potential field, an eigenvalue λ is an *energy level* and its eigenfunction is the corresponding *wave function* of a particle, the two together describing a *bound state* (for details see [11]). Also, equation (1) in the singularity case, appears in some chemical models ([13]), and in the chemical photodissociation of methyliodide (see [12] and [2]).

The inverse problem of reconstructing the potential function $q(t)$ from the given spectral information and corresponding dual equation cannot be studied by using the asymptotic forms. In fact, in asymptotic methods one cannot generally express the exact solution in closed form. The closed form of the solution is needed in methods connected with dual equations. The representing solution of the infinite product form plays an important role in investigating the corresponding dual equations. In the previous article ([10]), first an equation with one turning point of even order was considered, and derived a formula for the asymptotic distribution of the eigenvalues and the solutions. Then, by a replacement, the equation with turning point transformed to the differential equation with a singularity of the form (1) on the interval $[0,T]$, where $q(t)$ is a real function having a singularity and its form is

$$q(t) = \frac{F}{(t-t_1)^2} + q_0(t),$$

where $0 < t_1 < T$, and $F = \mu^2 - \frac{1}{4}$, $q_0(t)(t-t_1)^{1-2\mu} \in L(0,T)$, $\mu = (4\ell+2)^{-1}$, $\ell \in \mathrm{N}$. The solution $y(t,\lambda)$ of such an equation (1) with initial conditions was found to have the infinite product form

$$y(t,\lambda) = \frac{1}{2} r_1 \{\csc \pi\mu\}^k \prod_{n\geq 1} \frac{(\omega_n(t) - \lambda)}{\varsigma_n^{\ 2}}, \qquad t \in C_k, \quad k = 0,1,$$

where $C_0 = (0,t_1)$, $C_1 = (t_1,T)$, $r_1 \in R$, $\{\varsigma_n\}_{n\geq 1}$ is the sequence of positive zeros of $J'_{1/2}$ ($J_{1/2}$ is the Bessel function of order 1/2), and the sequence $\{\omega_n(t)\}_{n\geq 1}$ represents the sequence of positive eigenvalues of corresponding boundary value problem

$$\sqrt{\omega_n(t)} = \frac{(n\pi - \frac{\pi}{2})}{t} + O(\frac{1}{n}), \qquad t \in (0,T) \setminus \{t_1\}.$$

In this paper, first, we define a fundamental system of solutions (FSS) of equation (1) for $|\rho| \to \infty$ (see section 2). Using these asymptotic solutions we derive a formula for the asymptotic distribution of the eigenvalues, further we obtain the infinite product representation of the solution, see Section 4.

2 NOTATIONS AND PRELIMINARY RESULTS

We consider the differential equation

$$y'' + (\lambda\phi^2(t) - q(t))y = 0, \qquad t \in I = [0,1], \tag{2}$$

where $\lambda = \rho^2$ is a real parameter, $q(t), \phi^2(t)$ are real functions, $q(t)$ has two singular points t_1, t_2 of first order in *I*, ($0 < t_1 < t_2 < 1$), that these points are turning points of ϕ^2 .

Definition 1. (i) We define the following intervals for fixed $\varepsilon > 0$ (ε is sufficiently small):

$$I_{1,\varepsilon} = [0, t_2 - \varepsilon], \qquad I_{2,\varepsilon} = [t_1 + \varepsilon, 1].$$

(ii) We recall that there are four different types of zeros of order ℓ_ν. For $\nu = 1,2$:

$$
T_\nu := \begin{cases} I, & \text{if } \ell_\nu \text{ is even and } \phi^2(t)(t-t_\nu)^{-\ell_\nu} < 0 \quad \text{in } I_{\nu,\varepsilon}, \\ II, & \text{if } \ell_\nu \text{ is even and } \phi^2(t)(t-t_\nu)^{-\ell_\nu} > 0 \quad \text{in } I_{\nu,\varepsilon} \\ III, & \text{if } \ell_\nu \text{ is odd and } \phi^2(t)(t-t_\nu)^{-\ell_\nu} < 0 \quad \text{in } I_{\nu,\varepsilon} \\ IV, & \text{if } \ell_\nu \text{ is odd and } \phi^2(t)(t-t_\nu)^{-\ell_\nu} < 0 \quad \text{in } I_{\nu,\varepsilon}, \end{cases}
$$

is called type of t_ν.

Assumption 1. (i) The functions

$$
\phi_{\nu,0} : I_{\nu,\varepsilon} \to R, \qquad \phi_{\nu,0}(t) := (t-t_\nu)^{-\ell_\nu}\phi^2(t), \qquad \nu = 1,2,
$$

are non–vanishing and real–analytic, where $\ell_1 = \ell_2 = 1$.

(ii) For $t \in I_{\nu,\varepsilon}$, $t \neq t_\nu$, $\nu = 1,2$, the function $q(t)$ has the form

$$
q(t) = A_\nu (t-t_\nu)^{-1},
$$

with positive constants A_ν.

So, according to Definition 1 and Assumption 1, t_1 is of type III while t_2 is of type IV. According to the type of t_1, we know from [4] that in the sector

$$
S_{-1} = \left\{ \rho \;\middle|\; \arg\rho \in [-\frac{\pi}{4}, 0] \right\},
$$

there is exists an FSS of (2) $\{\, w_{1,1}(t,\rho), w_{1,2}(t,\rho) \,\}$ and such that

$$
w_{1,1}(t,\rho) = \begin{cases} |\phi(t)|^{-\frac{1}{2}} e^{i\rho\int_{t_1}^{t}|\phi(x)|dx}[1] & 0 \le t < t_1, \\ |\phi(t)|^{-\frac{1}{2}} e^{\rho\int_{t_1}^{t}|\phi(x)|dx + i\frac{\pi}{4}}[1] & t_1 < t < t_2, \end{cases} \tag{3}
$$

$$
w_{1,2}(t,\rho) = \begin{cases} |\phi(t)|^{-\frac{1}{2}} \left\{ e^{-i\rho\int_{t_1}^{t}|\phi(x)|dx}[1] + ie^{i\rho\int_{t_1}^{t}|\phi(x)|dx}[1] \right\} & 0 \le t < t_1, \\ |\phi(t)|^{-\frac{1}{2}} e^{-\rho\int_{t_1}^{t}|\phi(x)|dx + i\frac{\pi}{4}}[1] & t_1 < t < t_2, \end{cases} \tag{4}
$$

where $[1]=1+O(\frac{1}{\rho})$, as $\rho\to\infty$, and for next uses we have

$$w'_{1,1}(t,\rho)=\begin{cases} i\rho|\phi(t)|^{\frac{1}{2}}e^{i\rho\int_{t_1}^{t}|\phi(x)|dx}[1] & 0\le t<t_1, \\ \rho|\phi(t)|^{\frac{1}{2}}e^{\rho\int_{t_1}^{t}|\phi(x)|dx+i\frac{\pi}{4}}[1] & t_1<t<t_2, \end{cases} \tag{5}$$

$$w'_{1,2}(t,\rho)=\begin{cases} -\rho|\phi(t)|^{\frac{1}{2}}\left\{ie^{-i\rho\int_{t_1}^{t}|\phi(x)|dx}[1]+e^{i\rho\int_{t_1}^{t}|\phi(x)|dx}[1]\right\} & 0\le t<t_1, \\ -\rho|\phi(t)|^{\frac{1}{2}}e^{-\rho\int_{t_1}^{t}|\phi(x)|dx+i\frac{\pi}{4}}[1] & t_1<t<t_2. \end{cases} \tag{6}$$

On the other hand, since t_2 is of type IV, we also have the following FSS $\{\,w_{2,1}(t,\rho),w_{2,2}(t,\rho)\,\}$

$$w_{2,1}(t,\rho)=\begin{cases} |\phi(t)|^{-\frac{1}{2}}e^{\rho\int_{t_2}^{t}|\phi(x)|dx}[1] & t_1<t<t_2, \\ |\phi(t)|^{-\frac{1}{2}}\left\{e^{i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1]+ie^{-i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1]\right\} & t_2<t\le 1, \end{cases} \tag{7}$$

$$w_{2,2}(t,\rho)=\begin{cases} |\phi(t)|^{-\frac{1}{2}}e^{-\rho\int_{t_2}^{t}|\phi(x)|dx}[1] & t_1<t<t_2, \\ |\phi(t)|^{-\frac{1}{2}}e^{-i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1] & t_2<t\le 1. \end{cases} \tag{8}$$

That leads to the followings :

$$w'_{2,1}(t,\rho)=\begin{cases} \rho|\phi(t)|^{\frac{1}{2}}e^{\rho\int_{t_2}^{t}|\phi(x)|dx}[1] & t_1<t<t_2, \\ \rho|\phi(t)|^{\frac{1}{2}}\left\{ie^{i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1]+e^{-i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1]\right\} & t_2<t\le 1, \end{cases} \tag{9}$$

$$w'_{2,2}(t,\rho)=\begin{cases} -\rho|\phi(t)|^{\frac{1}{2}}e^{-\rho\int_{t_2}^{t}|\phi(x)|dx}[1] & t_1<t<t_2, \\ -i\rho|\phi(t)|^{\frac{1}{2}}e^{-i\rho\int_{t_2}^{t}|\phi(x)|dx-i\frac{\pi}{4}}[1] & t_2<t<1. \end{cases} \tag{10}$$

It follows that the wronskian of FSS satisfies

$$\begin{cases} W\left(w_{1,1}(t,\rho), w_{1,2}(t,\rho) \right) = -2i\rho[1], \\ W\left(w_{2,1}(t,\rho), w_{2,2}(t,\rho) \right) = -2\rho[1], \end{cases} \tag{11}$$

as $|\rho| \to \infty$.

Notation 1. For $-2 \le k \le 1$ and $(t,\rho) \in [0,1] \times S_k$ we denote

$$E_k(t,\rho) = \sum_{n=1}^{\vartheta(t)} e^{\rho \alpha_k \beta_{kn}(t)} b_{kn}(t),$$

and $\alpha_{-2} = \alpha_1 = -1$, $\alpha_0 = -\alpha_{-1} = i$, $\beta_{kn}(t) \neq 0$, also

$$0 < \delta < \beta_{k1}(t) < \beta_{k2}(t) < \ldots \le \beta_{kn}(t) \le 2\max\{R_+(1), R_-(1)\},$$

where the integer-valued functions ϑ and b_{kn} are constant in every interval $[0, t_1 - \varepsilon]$ and $[t_1 + \varepsilon, t_2 - \varepsilon]$ for ε sufficiently small and

$$R_+(t) = \int_0^t \sqrt{\max\{0, \phi^2(x)\}}dx \quad , \quad R_-(t) = \int_0^t \sqrt{\max\{0, -\phi^2(x)\}}dx .$$

3 ASYMPTOTIC FORM OF THE SOLUTION

We consider the differential equation (2) with the following conditions

$$C(0,\lambda) = 1 \quad , \quad C'(0,\lambda) = 0. \tag{12}$$

Applying the FSS $\{ w_{1,1}(t,\rho), w_{1,2}(t,\rho) \}$ for $t \in I_{1,\varepsilon}$ we have

$$C(t,\rho) = c_1 w_{1,1}(t,\rho) + c_2 w_{1,2}(t,\rho),$$

that using of Cramer's rule leads to the equation

$$C(t,\rho) = \frac{1}{W(\rho)}\left(w'_{1,2}(0,\rho) w_{1,1}(t,\rho) - w'_{1,1}(0,\rho) w_{1,2}(t,\rho) \right)$$

where

$$W(\rho) = W\left(w_{1,1}, w_{1,2} \right) = -2i\rho[1].$$

Taking (3)-(6) in to account we derive

$$C(t,\rho)=\begin{cases}\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\left\{\ e^{i\rho\int_0^t|\phi(x)|dx}[1]+e^{-i\rho\int_0^t|\phi(x)|dx}[1]\ \right\} & 0\le t<t_1,\\ \frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\left\{\ M_1(\rho)e^{\rho\int_{t_1}^t|\phi(x)|dx}[1]+M_2(\rho)e^{-\rho\int_{t_1}^t|\phi(x)|dx}[1]\ \right\} & t_1<t<t_2,\end{cases} \tag{13}$$

where

$$\begin{cases}M_1(\rho)=-ie^{-i\rho\int_0^{t_1}|\phi(x)|dx+i\frac{\pi}{4}}+e^{i\rho\int_0^{t_1}|\phi(x)|dx+i\frac{\pi}{4}},\\ M_2(\rho)=e^{-i\rho\int_0^{t_1}|\phi(x)|dx+i\frac{\pi}{4}}.\end{cases} \tag{14}$$

In addition, differentiating (13) we calculate

$$C'(t,\rho)=\begin{cases}\frac{1}{2}i\rho|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{1}{2}}\left\{\ e^{i\rho\int_0^t|\phi(x)|dx}[1]-e^{-i\rho\int_0^t|\phi(x)|dx}[1]\ \right\} & 0\le t<t_1,\\ \frac{1}{2}\rho|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{1}{2}}\left\{\ M_1(\rho)e^{\rho\int_{t_1}^t|\phi(x)|dx}[1]-M_2(\rho)e^{-\rho\int_{t_1}^t|\phi(x)|dx}[1]\ \right\} & t_1<t<t_2.\end{cases}$$

Hence we have estimated the solution of (2) defined by the initial conditions (12) in $I_{1,\varepsilon}$. In order to find the solution in $I_{2,\varepsilon}$, we fix $t\in(t_1,t_2)$ and use (7)-(10), and Cramer's rule to determine the connection coefficients $A_1(\rho),A_2(\rho)$ with

$$\begin{cases}C(t,\rho)=A_1(\rho)w_{2,1}(t,\rho)+A_2w_{2,2}(t,\rho),\\ C'(t,\rho)=A_1(\rho)w'_{2,1}(t,\rho)+A_2w'_{2,2}(t,\rho),\end{cases}$$

for $T_2=IV$. Consequently

$$\begin{cases}A_1(\rho)=\frac{1}{2}|\phi(0)|^{\frac{1}{2}}M_1(\rho)e^{\rho\int_{t_1}^{t_2}|\phi(x)|dx}[1],\\ A_2(\rho)=\frac{1}{2}|\phi(0)|^{\frac{1}{2}}M_2(\rho)e^{-\rho\int_{t_1}^{t_2}|\phi(x)|dx}[1].\end{cases} \tag{15}$$

Substituting (15) and estimates of $w_{2,1}(t,\rho)$ and $w_{2,2}(t,\rho)$ from (7) and (8) in the case $t_2<t\le 1$ we derive the continuation of the solution to the interval $(t_2,1]$ in the form :

$$C(t,\rho)=\frac{1}{2}\left|\phi(0)\right|^{\frac{1}{2}}\left|\phi(t)\right|^{-\frac{1}{2}}\left\{\ N_1(\rho)e^{i\rho\int_{t_2}^{t}|\phi(x)|dx}[1]+N_2(\rho)e^{-i\rho\int_{t_2}^{t}|\phi(x)|dx}[1]\ \right\},$$

where

$$\begin{cases} N_1(\rho)=-ie^{-i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t_2}|\phi(x)|dx}+e^{i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t_2}|\phi(x)|dx}, \\ N_2(\rho)=e^{-i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t_2}|\phi(x)|dx}+ie^{i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t_2}|\phi(x)|dx}+e^{-i\rho\int_0^{t_1}|\phi(x)|dx-\rho\int_{t_1}^{t_2}|\phi(x)|dx}. \end{cases}$$

Thus, we deduce the following theorem.

Theorem 1. Let $C(t,\rho)$ be the solution of (2) under the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$, then the following estimates hold :

$$C(t,\rho)=\begin{cases} \frac{1}{2}\left|\phi(0)\right|^{\frac{1}{2}}\left|\phi(t)\right|^{-\frac{1}{2}}e^{i\rho\int_0^t|\phi(x)|dx}E_k(t,\rho), & 0\le t<t_1, \\ \frac{1}{2}\left|\phi(0)\right|^{\frac{1}{2}}\left|\phi(t)\right|^{-\frac{1}{2}}e^{i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t}|\phi(x)|dx+i\frac{\pi}{4}}E_k(t,\rho), & t_1<t<t_2, \\ \frac{1}{2}\left|\phi(0)\right|^{\frac{1}{2}}\left|\phi(t)\right|^{-\frac{1}{2}}e^{i\rho\int_0^{t_1}|\phi(x)|dx+\rho\int_{t_1}^{t_2}|\phi(x)|dx+i\rho\int_{t_2}^{t}|\phi(x)|dx}E_k(t,\rho), & t_2<t\le 1. \end{cases}$$

4 EIGENVALUES AND INFINITE PRODUCT REPRESENTATION OF THE SOLUTION

We consider the boundary value problem $L_1=L_1(\phi^2(t),q(t),s)$ for equation (2) with boundary conditions

$$C(0,\lambda)=1,\quad C'(0,\lambda)=0,\quad C(s,\lambda)=0.$$

The boundary value problem L_1 for $s\in(0,t_1)$ has a countable set of positive eigenvalues $\{\lambda_n{}^+(s)\}_{n\ge1}$. From (13), we have the following asymptotic distribution for each $\{\lambda_n{}^+(s)\}_{n\ge1}$ holds :

$$\rho_n^+(s)=\sqrt{\lambda_n{}^+(s)}=\frac{n\pi-\frac{\pi}{2}}{\int_0^{t_1}|\phi(x)|dx}+O(\frac{1}{n}). \tag{16}$$

The spectrum $\{\lambda_n\}$ of boundary value problem L_1 for $t_1 < s < t_2$, consist of two sequences of negative and positive eigenvalues: $\{\lambda_n(s)\} = \{\lambda_n^-(s)\} \cup \{\lambda_n^+(s)\}$, $n \in \mathrm{N}$, such that

$$\rho_n^-(s) = \sqrt{-\lambda_n^-(s)} = -\frac{n\pi - \frac{\pi}{4}}{\int_{t_1}^{t} |\phi(x)|dx} + O(\frac{1}{n}), \qquad \rho_n^+(s) = \sqrt{\lambda_n^+(s)} = \frac{n\pi - \frac{\pi}{4}}{\int_{0}^{t_1} |\phi(x)|dx} + O(\frac{1}{n}).$$

(17)

Similarly for $t_2 < s < 1$, from the estimates of $C(t,\rho)$ we see that :

$$\rho_n^-(s) = \sqrt{-\lambda_n^-(s)} = -\frac{n\pi - \frac{\pi}{4}}{\int_{t_1}^{t_2} |\phi(x)|dx} + O(\frac{1}{n}), \quad \rho_n^+(s) = \sqrt{\lambda_n^+(s)} = \frac{n\pi - \frac{\pi}{4}}{\int_{t_2}^{t} |\phi(x)|dx} + O(\frac{1}{n}). \qquad (18)$$

Since the solution $C(s,\rho)$ of Sturm-Liouville equation defined by a fixed set of initial conditions is an entire function of ρ for each fixed $s \in [0,1]$, thus it follows from the classical Hadamard's factorization theorem (see [8, p. 24]) that such solution is expressible as an infinite product. For fixed $s \in (0,t_1)$ by Halvorsen's result [6], $C(s,\rho)$ is an entire function of order $\frac{1}{2}$. Therefore we can use Hadamard's theorem to represent the solution in the form

$$C(s,\lambda) = h(s)\prod_{n\geq 1}\left(1 - \frac{\lambda}{\lambda_n(s)}\right),$$

where $h(s)$ is a function independent of λ but may depend on s and the infinite number of negative eigenvalues, $\{\lambda_n(s)\}_{n=1}^{\infty}$ form the zero set of $C(s,\lambda)$ for each s.

Since $C(s,\lambda_n(s)) = 0$, these $\lambda_n(s)$ correspond to eigenvalues of the boundary value problem L_1 on the closed interval $[0,s]$, $0 < s < t_1$. We rewrite the infinite product as

$$C(s,\lambda)=h(s)\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n(s)}\right)=h_1(s)\prod_{n\geq 1}\left(\frac{\lambda_n(s)-\lambda}{\zeta_n^2}\right), \tag{19}$$

with

$$h_1(s):=h(s)\prod_{n\geq 1}\frac{\zeta_n^2}{\lambda_n(s)}, \tag{20}$$

where $\zeta_n=\dfrac{n\pi-\frac{\pi}{2}}{R_+(s)}$, and $R_+(s)$ is defined in Notation 1.

Now (16) implies that $\dfrac{\zeta_n^2}{\lambda_n(s)}=1+O\left(\dfrac{1}{n^2}\right)$. It follows from the results of [7] that the infinite product $\prod\dfrac{\zeta_n^2}{\lambda_n(s)}$ is absolutely convergent on any compact subinterval of $(0,t_1)$. The function $\dfrac{\zeta_n^2}{\lambda_n(s)}$ is continuous and so the O-term is uniformly bounded in s.

Theorem 2. Let $C(t,\lambda)$ be the solution of (2) satisfying the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. Then for $0<t<t_1$,

$$C(t,\lambda)=\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2},$$

where $R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(x)\}}dx$, ζ_n, $n\geq 1$, is the sequence of positive zeros of J_1', derivative of the Bessel function of order one, the sequence $\lambda_n(t), n\geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t]$.

Proof. According to [6] the infinite product

$$\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2},$$

is an entire function of λ, whose roots are precisely $\lambda_n(t)$, $n\geq 1$. From [1, p. 370] we have

$$J_\nu'(\tau)=\sqrt{\frac{2\pi}{\tau}}\left\{-R(\nu,\tau)\sin\theta-S(\nu,\tau)\cos\theta\right\},$$

where ν is fixed and

$$R(\nu,\tau) \approx \sum_{k=0}^{\infty}(-1)^k \frac{4\nu^2+16k^2-1}{4\nu^2-(4k+1)^2}\left\{\frac{(\nu,2k)}{(2\tau)^{2k}}\right\}$$
$$= 1-\frac{(\mu-1)(\mu+15)}{2!(8\tau)^2}+\ldots$$

$$S(\nu,\tau) \approx \sum_{k=0}^{\infty}(-1)^k \frac{4\nu^2+4(2k+1)^2-1}{4\nu^2-(4k+1)^2}\left\{\frac{(\nu,2k)}{(2\tau)^{2k+1}}\right\}$$
$$= \frac{\mu+3}{8\tau}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8\tau)^3}+\ldots$$
$$\theta = \tau-\left(\frac{\nu}{2}+\frac{1}{4}\right)\pi,$$

as $|\tau| \to \infty$, where $\mu = 4\nu^2$. Now, by inserting $\tau = R_+(t)\sqrt{\lambda}$, and from [1, p. 370], we get

$$\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2} = 2\cos(\sqrt{\lambda}R_+(t))[1].$$

Thus from (13) and (19), we obtain

$$h_1(t) = \frac{C(t,\lambda)}{\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2}} = \frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}. \qquad \blacksquare$$

Similarly, for $s = t$, $t_1 < t < t_2$, the boundary value problem L_1 on $[0,t]$ has a infinite number of negative and positive eigenvalues with are denoted (17). By Hadamard's theorem, the solution on $[0,t]$, $t_1 < t < t_2$, has the form

$$C(t,\lambda) = g(t)\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n^-(t)}\right)\left(1-\frac{\lambda}{\lambda_n^+(t)}\right).$$

Let $\tilde{j}_n$, $n = 1,2,3,\ldots$, be the positive zeros of $J_1'(\tau)$. Then (see [1])

$$\frac{-\tilde{j}_n^{\,2}}{R_-^2(t)\lambda_n^-(t)} = 1+O\left(\frac{1}{n^2}\right), \qquad \frac{\tilde{j}_n^{\,2}}{R_+^2(t)\lambda_n^+(t)} = 1+O\left(\frac{1}{n^2}\right).$$

Consequently, the infinite products $\prod \frac{-\tilde{j}_n^{\,2}}{R_-^2(t)\lambda_n^-(t)}$, $\prod \frac{\tilde{j}_n^{\,2}}{R_+^2(t)\lambda_n^+(t)}$ are absolutely convergent for each $t \in (t_1, t_2)$. Therefore we may write

$$C(t,\lambda) = g_1(t)\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t_1)}{\tilde{j}_n^2}, \tag{21}$$

with

$$g_1(t) = g(t)\prod \frac{-\tilde{j}_n^{\,2}}{R_-^2(t)\lambda_n^-(t)}\prod \frac{\tilde{j}_n^{\,2}}{R_+^2(t)\lambda_n^+(t)}.$$

Theorem 3. For $t_1 < t < t_2$,

$$C(t,\lambda) = \frac{\pi}{8}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\left(R_-(t)R_+(t)\right)^{\frac{1}{2}}\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t_1)}{\tilde{j}_n^2}.$$

Proof. According From Lemmas 2 and 3 of [7] the infinite products

$$\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t)}{\tilde{j}_n^2} \quad , \quad \prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t_1)}{\tilde{j}_n^2}$$

are entire functions of λ for fixed t, those roots are precisely $\{\lambda_n^-(t)\}$ and $\{\lambda_n^+(t)\}$, $n \geq 1$, respectively. Moreover

$$\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t_1)}{\tilde{j}_n^2} = \frac{4e^{R_-(t)\sqrt{\lambda}}}{\pi\left(R_-(t)R_+(t)\right)^{\frac{1}{2}}\sqrt{\lambda}}\left\{\cos(R_+(t)\sqrt{\lambda}-\frac{\pi}{4})+O(\frac{1}{\sqrt{\lambda}})\right\}$$

as $\lambda \to \infty$. Thus by (21) and using of the asymptotic expansion of $C(t,\lambda)$ in (13) we get

$$g_1(t) = \frac{C(t,\lambda)}{\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t_1)}{\tilde{j}_n^2}} = \frac{\pi}{8}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\left(R_-(t)R_+(t)\right)^{\frac{1}{2}}. \qquad \blacksquare$$

We can proceed similarly, for $s = t$, $t_2 < t < 1$, to obtain

$$C(t,\lambda) = f_1(t)\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t_2)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t)}{\tilde{j}_n^2}.$$

Thus, we have the following theorem.

Theorem 4. For $t_2 < t < 1$,

$$C(t,\lambda) = \frac{\pi}{8}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\left(R_-(t)R_+(t)\right)^{\frac{1}{2}}\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t_2)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t)}{\tilde{j}_n^2}.$$

REFERENCES

1. M. Abramowitz, J. A. Stegun, *Handbook of mathematical functions with formalas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, 55, United States Department of Commerce, Washington, DC, 1964.
2. W. O. Amrein, A. M. Hinz, D. B. Pearson, *Sturm–Liouville theory: Past and Present*, Birkhäuser Verlag, Basel, 2005.
3. F. V. Atkinson, A. B. Mingarelli, Asymptotics of the number of zeros and of the eigenvalues of general weighted Sturm–Liouville problems, *J. Reine Angew. Math.* **375**, 380–393 (1987).
4. W. Eberhard, G. Freiling, K. Wilcken–Stoeber, Indefinite eigenvalue problems with several singular points and turning points, *Math. Nachr.* **229**, 51–71 (2001).
5. G. Freiling, V. Yurko, On the determination of differential equations with singularities and turning points, *Results Math.* **41**, 275–290 (2002).
6. S. G. Halvorsen, A function theoretic property of solutions of the equation $x'' + (\lambda w - q)x = 0$, *Quart. J. Math. Oxford Ser.* **38**, 73–76 (1987).
7. A. Jodayree Akbarfam, A. B. Mingarelli, The canonical product of the solution of the Sturm–Liouville equation in one turning point case, *Canad. Appl. Math. Quart.* **8**, 305–320 (2000).
8. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, American Mathematical Society, Providence, RI, 1964.
9. A. B. Mingarelli, A survey of the regular weighted Sturm–Liouville problem, The non–definite case, *Proceeding of the Workshop on Differential Equations,*

Tsinghua University, Beijing, The People's Republic of China, 3–7 June, 1985, World Scientific, Singapore, 1986.

10. A. Neamaty, S. Mosazadeh, On the canonical solution of the Sturm–Liouville problem with singularity and turning point of even order, *Canad. Math. Bull.* **54**, 506–518 (2011).
11. E. Schrodinger, An undulatory theory of the mechanics of atoms and molecules, *Phys. Rev.* **28**, 1049–1070 (1926).
12. J. K. Shaw, A. P. Baronavski, H. D. Ladouceur,W. O. Amrein, Applications of the Walker method in Spectral Theory and Computational Methods of Sturm−Liouville problems, *Lecture Notes in Pure. Appl. Math.* **191**, 377–395 (1997).
13. G. Vanden Berghe, V. Fack, H. De Meyer, Numerical methods for solving radial Schrodinger equations, *J. Comp. Appl. Math.* **28**, 391–401 (1989).