Iranian Journal of Mathematical Chemistry, Vol. 4, No.1, March 2013, pp. 1 – 20 **IJMC**

A nonlinear second order field equation – similarity solutions and relation to a Bellmann-type equation - Applications to Maxwellian Molecules

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(*Received Jan 23, 2013; Accepted January 30, 2013*)

ABSTRACT

In this paper Lie's formalism is applied to deduce classes of solutions of a nonlinear partial differential equation (nPDE) of second order with quadratic nonlinearity. The equation has the meaning of a field equation appearing in the formulation of kinetic models. Similarity solutions and transformations are given in a most general form derived to the first time in terms of reciprocal Jacobian elliptic functions.

By using a special transformation the first derivative of the equation can be transformed off leading to a further nPDE. The latter equation is also studied as well as algebraic properties and group invariant solutions could be derived. This new classes of solutions obtained are closely related to solutions of the kinetic model and so far, expressions for a generating function considering normalized moments are also deduced.

Finally, the connection to Painlevé's first equation is shown whereby these classes of solutions are solutions due to the invariant properties too. For practical use in numerical calculations some series representations are given explicitly. In view of the point of novelty it is further shown how to derive a Bellman-type equation to the first time and asymptotic classes of solutions result by appropriate transformations. The importance of the present paper is the relation to the Boltzmann Equation which describes the one particle distribution function in a gas of particles interacting only through binary collisions. Since transformations remain an equation invariant, solutions of the new transformed equation also generates solutions of physical relevance. Normalized moments are discussed finally.

Keywords: Classical Lie group formalism, classes of similarity solutions, nonlinear partial differential equations (nPDEs), Maxwellian molecules, Bellmann's typeequation**.**

1. INTRODUCTION

Applied to nPDG, Lie's classical method leads to group invariant solutions. Physically significant solutions arising from symmetry methods allow one to investigate the physical

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behavior of general classes of solutions. Group invariant solutions obtained via Lie's approach provide insight into the physical models themselves.

Explicit solutions also serve as benchmarks in the design, accuracy, testing and, comparison of numerical algorithms. In general, one can say that a solution of an nPDE in two independent variables can be constructed by two invariants of the group. One of these two invariants becomes the new independent variable $\zeta = \zeta(x, t)$, the so-called similarity variable and the other invariant plays the role of a dependent variable $S(\zeta)$.

Basic elements and explanations of the algorithm are given in the references, e.g. [1–4].

1.1 Some short notes to the Boltzmann equation (BE)

The BE [5] governs the temporal evolution of the one particle distribution function in a gas of particles interacting only through binary collisions. The main difficulties in solving the BE are largely due to the complex mathematical structure of the collision term.
The detailed form of this term depends upon the precise nature of the intermolecular forces. The exact solution for general intermolecular forces and arbitrary initial conditions are not known. It is therefore of great importance to study simplified models for which one can obtain special solutions.

In constructing model-BE one looks for special intermolecular forces between the particles in the gas such that the differential scattering cross-section or collision rate has a simple dependence on the energies of the colliding particles or on the scattering angle.
The best-known example in this category is the so-called Maxwell molecules or pseudo-Maxwell molecules [6]. The most important stimulus undoubtedly came from the discovery of an exact (similarity) solution of the BE for Maxwell molecules, found by Bobylev [7] and independently by Krook and Wu [8], known as the BKW-mode. Numerically solving of a non-linear BE-model and showing relaxation processes was presented by Tjon [9]. An important development was the Laguerre series solution firstly obtained by Ernst [10] whereby the distribution function is given in an expansion in terms of generalized Laguerre polynomials.

A further development of interest was the introduction of the so-called very hard particle model (VHPM) in which the distribution function has non-trivial energy dependence. A version of this model with discrete energies was solved by Rouse and Simons [11] who gave the solution in form of a set of algebraic recursion relations. The given remarks are far from being complete; an extensive collection of relevant papers can be found in [12].

1.2. Some notes about similarity solutions relating to the BE

Bobylev [7] presents a group theoretical method for obtaining similarity solutions. Another analysis was given by Tenti and Hui [13], both showed that other class of exact solutions exist for the BE in a system containing sources and sinks of particles.

The group method yields, apart from the BKW-mode, class of similarity solutions each associated with non-linear eigenvalues. This mode has been studied by many authors, e.g. [14, 15].

In their papers, Euler et. al. [16] presented a similarity analysis by using the Krook/Wu model and discussed the resulting ODEs in connection to Painlevé-analysis. However, explicit similarity solutions are missed. Some similarity solutions can be found by Vijayakumar and Bhutani [17].

2. Outline the Problem - Derivation and Analysis

In the formation of Maxwellian tails the following nPDG is considered:

$$\frac{\partial^2 u}{\partial x \partial \tau} + \frac{\partial u}{\partial x} + u^2 = 0, \tag{1}$$

with $u = u(x,\tau)$, $u \in C^2(-\infty,\infty)$, $(x,\tau) \in \Omega$, $\Omega \subset \nabla^n$, $\tau \in \nabla^+ \setminus \{0\}$, x acts as a local coordinate and τ means a time coordinate. A short derivation of physical point of view is given [5]: The state of a gas is described by a distribution function $n f(\upsilon,\tau)$ where n is a constant density, υ is a velocity variable and further $\upsilon = |v|$. Now, the BE in a simplified form by using collision dynamics can be written as [6]:

$$\frac{\partial f(\upsilon,\tau)}{\partial \tau} = -f(\upsilon,\tau) + \frac{1}{4\pi} \int d^3 w \int_0^{\pi} d\chi \sin\chi \int_0^{2\pi} d\varepsilon\, f(\upsilon',\tau)\, f(w',\tau) \tag{2}$$

with $$\upsilon'^2 = \frac{1}{2}\left(\upsilon^2 + w^2\right) - \frac{1}{2}\left(\upsilon^2 - w^2\right)\cos\chi + |v \times w| \sin\chi \cos\varepsilon$$

$$w'^2 = \frac{1}{2}\left(\upsilon^2 + w^2\right) - \frac{1}{2}\left(\upsilon^2 - w^2\right)\cos\chi - |v \times w| \sin\chi \cos\varepsilon\ . \tag{3}$$

By definition, normalized moments $p_k(\tau)$ of the function $f(\upsilon,k)$ are introduced by

$$p_k(\tau) = \frac{\sqrt{\pi}}{2\,(2\beta^2)^k\, \Gamma(k+3/2)} \int \upsilon^{2k} f(\upsilon,k) d^3\upsilon\,,\quad p_0(\tau) = p_1(\tau) = 1\,,\quad p_k(\infty) = 1\,,\ k = 0,\,1. \tag{4}$$

Next, a generating function $G(\xi,\tau)$ for the normalized moments is considered by

$$G(\xi,\tau)=\sum_{k=0}^{\infty}\xi^{k}\,p_k(\tau)\ , \tag{5}$$

to derive the desired equation:

$$\frac{\partial}{\partial\xi}\left(\xi\frac{dG}{d\tau}+\xi G\right)=G^2. \tag{5a}$$

Hence, the transformation $x=\frac{1-\xi}{\xi}$, $u(x,\tau)=\xi G(\xi,\tau)$ just generates the nPDE (1). Otherwise, a generalized (p+1)th BE in (1+1) dimensions can be formulated by [12]:

$$(\partial_t+1)(-\partial_x)^p\ u(x,t)=\frac{\Gamma(2p)}{\Gamma(p)}\ u^2(x,t)\,, \tag{6}$$

in which p is a positive integer and Γ means the gamma function (t is equivalent to τ used above).

For $p=1$ we end up by eq. (1) and moreover, for $p>1$ higher order BE can be obtained. To use symmetry groups in any application [18, 19] we first need to find the symmetries of eq. (1). The result is a system of eight linear homogeneous PDEs for the infinitesimals $\xi_i=\xi_i(x,u)$ and $\phi_i=\phi_i(x,u)$, in which $\vec{x}$ and $\vec{u}$ are vectors of the independent and dependent variables. Here, ξ_1 stands for the first independent variable x and ξ_2 for the second variable τ. In our case only one dependent variable u is present; therefore the related infinitesimal is ϕ.

These quantities constitute the determining equations generated by Fréchet's derivative:

$$\frac{\partial\xi_1}{\partial u}=\frac{\partial\xi_2}{\partial u}=\frac{\partial^2\phi}{\partial u^2}=\frac{\partial\xi_1}{\partial\tau}=\frac{\partial\xi_2}{\partial x}=\frac{\partial^2\phi}{\partial u\,\partial x}=0\,,\quad \frac{\partial\xi_2}{\partial\tau}+\frac{\partial^2\phi}{\partial u\,\partial\tau}=0\,,$$

$$2u\phi+u^2\frac{\partial\xi_1}{\partial x}-u^2\frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial x}-u^2\frac{\partial^2\phi}{\partial u\,\partial\tau}+\frac{\partial^2\phi}{\partial x\,\partial\tau}=0\,. \tag{7}$$

Solving this system we end up by an explicit representation for the infinitesimals ξ_1,ξ_2 and ϕ.

The derived symmetry group constitutes a four-dimensional finite point group allowing translations and scaling symmetries with the group parameters denoted by k_1 to k_4. Curiously, the variable $u(x,\tau)$ appears linear in the infinitesimal ϕ (similar as in the case of the KdV equation):

$$\begin{aligned}\xi_1 &= k_1 + k_2 x \\ \xi_2 &= -k_3 e^{-\tau} + k_4 \\ \phi &= \left(k_3 e^{\tau} - k_2\right) u(x,\tau)\end{aligned} \tag{8}$$

Eq.(1) admits a four-dimensional Lie algebra V with the generating vector fields:

$$V_1 = \partial_x\,,\quad V_2 = x\,\partial_x - u\,\partial_u\,,\quad V_3 = e^{\tau}\,u\,\partial_u - e^{\tau}\,\partial_\tau\,,\quad V_4 = \partial_\tau\quad . \tag{9}$$

These vector fields further constitute a Lie algebra by:

$$[V_1,V_4] = [V_1,V_3] = [V_2,V_4] = [V_2,V_3] = 0\,,\ [V_4,V_3] = V_3 \text{ and } [V_1,V_2] = -V_1\,. \tag{10}$$

For this four-dimensional Lie algebra, the commutator Table 1 for V_i will be a 4×4 table whose (i,j)th entry expresses the Lie bracket $\lfloor V_i, V_j \rfloor$ given in (9). The table is skew-symmetric and the diagonal elements are all zero. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i,j)th entry and the related structure constants can be easily read of Tab.1:

$$C_{1,2,1} = C_{4,3,3} = -1 \text{ and } C_{2,1,1} = C_{3,4,3} = 1 \tag{11}$$

Theorem: The Lie algebra of eq.(1) is solvable.

Proof*:* A Lie algebra L is called solvable if $V^{(n)} = 0$ for some $n > 0$. $V^{(1)}$ represents an ideal $\{V_1, V_2, V_3\}$, $V^{(2)}$ an ideal with $\{V_1, V_2\}$; this can be reduced to $V^{(4)} = 0$ for $n = 4$. □.

Notes: Other useful algebraic group properties are mentioned: Eq.(1) has no Casimir operator, the group order is four containing 15 subgroups and the metric (or Cartanian tensor) is given by:

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ with } \det(g) = 0. \tag{11a}$$

Since the condition $\det(g) = 0$ holds the given algebra is degenerated. Alternatively, the metric tensor in eq. (211a) is written as $g_{im} = \sum_{i,k=1}^{n} c_{lk}^{i} c_{mi}^{k}$ in a compact form.

Let us now discuss special class of similarity solutions relating to special subgroups. Some abbreviations are useful: The similarity variable is denoted by ζ, the similarity function itself by $S(\zeta)$, briefly we set S instead of $S(\zeta)$. Consequently, a prime denotes differentiation w.r.t. ζ, that is $d/d\zeta$. Results are summarized in Tab.3 by setting different $k_i = 0$.

Table 1. Commutator of the nonlinear Field Equation (1).

	V_1	V_2	V_3	V_4
V_1	0	$-V_1$	0	0
V_2	V_1	0	0	0
V_3	0	0	0	V_3
V_4	0	0	$-V_3$	0

In order to derive classes solutions of eq.(1) the following nODEs of second order are of interest:

For ***Case A, Case B*** and ***Case C***:

$$S'' - S^2 + S = 0,\ S : R \times R \to R, \zeta \in R, -\infty \le \zeta \le \infty, \tag{12}$$

for ***Case D*** and ***Case E***:

$$S'' + 2\,S' - S^2 + S = 0, S : R \times R \to R,\ \zeta \in R, -\infty \le \zeta \le \infty, \tag{13}$$

and finally for ***Case F*** and ***Case G***:

$$\zeta^2 S'' + 3\zeta S' + S^2 + S = 0,\ S : R \times R \to R,\ \zeta \in R \setminus \{0\}, -\infty \le \zeta \le \infty. \tag{14}$$

Note: $\zeta = 0$ and $\zeta = \infty$ are the regular points of the Eq.(14).

Eq. (12) and eq.(13) are solved explicitly by use of elliptic functions as shown in [19, 20, 21].

$$\int \frac{dS}{\sqrt{\frac{2}{3}S^3 - S^2 + 2C_1}} = \zeta - \zeta_0 \quad \text{and} \quad \frac{1}{\sqrt{3}} \int \frac{dS}{\sqrt{\frac{2}{3}S^3 - S^2 + 2C_1}} = \zeta - \zeta_0\ , \tag{15}$$

where ζ_0 and C_1 means arbitrary constant of integrations. Due to the equivalence of the polynomials P(S), the solution function of (13) differs only by a constant factor in form of $\sqrt{3}$. Therefore we restrict he analysis in view of eq.(12). By considering the polynomial of third order, abbreviated by P(S) we have to treat the following cases:

(i) if $C_1 = 0$, (15) can be analyzed without introducing elliptic functions leading to logarithmic solutions (num. factors suppressed) by an implicit representation: $(\zeta - \zeta_0) \sim 1/\mathrm{S} - \ln\mathrm{S} + \ln(2\mathrm{S}-3)$,

(ii) if $C_1 > 0$, the polynomial P(S) has one real zero and two complex zeros,

(iii) if $C_1 < 0$, we observe the same situation as in (ii) and can therefore be analyzed equal to (ii).

For (ii) we end up by an explicit general expression for the similarity function S in the form

$$\mathrm{S} = -\mathrm{P} + \sqrt{\mathrm{A}} \left[\left(1 - \mathrm{cn} \left(\frac{\mathrm{A}^{1/4}(\zeta - \zeta_0)}{\sqrt{2}} \right) \right)^{-1} - \frac{1}{2} \right], \tag{16}$$

with some real numbers $\mathrm{A} = (\mathrm{B}(\mathrm{B}-1)+1)^2/\mathrm{B}^2$, $\mathrm{P} = (\mathrm{B}^2 - \mathrm{B} + 1)/2\mathrm{B}$, $\mathrm{B} = (\,11 - 2\sqrt{30}\,)^{1/3}$, that is numerically $B \equiv 0{,}3571$, (also for $A = 0{,}4133$ and $P = -0{,}2952$). From the graphical behaviour it is seen that the domain of the cosine amplitude claims $\mathrm{cn}(\zeta, \mathrm{k}) \neq -1 \Rightarrow \zeta \neq 2\mathrm{k}, 6\mathrm{k}, 10\mathrm{k}, \ldots$, e.g. for $\zeta_0 = 0$.

Tab.2 gives an overview of all similarity transformations for the different cases under consideration.

Now, by using Tab.2, general expressions for the solution function $u(x,\tau)$ could derived.

For ***Case A:***

$$\mathrm{u}(\mathrm{x},\tau) = \frac{1}{\mathrm{x}} \left\{ -\mathrm{P} + \sqrt{\mathrm{A}} \left[\left(1 - \mathrm{cn} \left(\frac{\mathrm{A}^{1/4}(\tau - \tau_0)}{\sqrt{2}} \right) \right)^{-1} - \frac{1}{2} \right] \right\}, \quad \mathrm{x} \neq 0, \tag{17}$$

for ***Case B:***

$$u(x,\tau) = -\mathrm{P} + \sqrt{\mathrm{A}} \left[\left(1 - \mathrm{cn} \left(\frac{\mathrm{A}^{1/4}(\mathrm{x} - \mathrm{x}_0 - \tau + \tau_0)}{\sqrt{2}} \right) \right)^{-1} - \frac{1}{2} \right], \tag{18}$$

for ***Case C:***

$$u(x,\tau)=\frac{1}{1+x}\left\{-P+\sqrt{A}\left[\left(1-\mathrm{cn}\left(\frac{A^{1/4}(\tau-\tau_0)}{\sqrt{2}}\right)\right)^{-1}-\frac{1}{2}\right]\right\} \quad , \quad x\neq -1, \quad (19)$$

Table 2. Expressions for the similarity variable ζ and related transformations for the similarity function $S(\zeta)$ depending upon the choice of the group parameters k_i.

Case	Similarity variable	Transformation	$k_i = 1$, **all other** $k = 0$
A	$\zeta=\tau$	$S = u\,x$	$k_2 = 1$
B	$\zeta = x-\tau$	$S = u$	$k_1 = k_4 = 1$
C	$\zeta=\tau$	$S = u(1+x)$	$k_1 = k_2 = 1$
D	$\zeta = \ln(x-\tau)$	$S = u$	$k_2 = k_4 = 1$
E	$\zeta = \tau - \ln(1+x)$	$S = u(1+x)$	$k_1 = k_2 = k_4 = 1$
F	$\zeta = \dfrac{1}{x - x(\cosh\tau - \sinh\tau)}$	$S = u\,x\,(e^{\tau}-1)$	$k_2 = k_3 = k_4 = 1$
G	$\zeta = \dfrac{1}{1+x-e^{\tau}(1+x)}$	$S = u(e^{\tau}-1)(1+x)$	$k_1 = k_2 = k_3 = k_4 = 1$

for ***Case D:***

$$u(x,\tau) = -P+\sqrt{A}\left[\left(1-\mathrm{cn}\left(\sqrt{\frac{3}{2}}A^{1/4}(\tau-\tau_0+\ln\left(\frac{x}{x_0}\right)\right)\right)^{-1}-\frac{1}{2}\right], \quad (20)$$

and finally, considering the last case ***Case E:***

$$u(x,\tau)=\frac{1}{(1+x)}\left\{-P+\sqrt{A}\left[\left(1-\mathrm{cn}\left(\sqrt{\frac{3}{2}}A^{1/4}(\tau-\tau_0+\ln\left(\frac{1+x}{1-x_0}\right)\right)\right)^{-1}-\frac{1}{2}\right]\right\} \quad (21)$$

with the necessary conditions $x\neq\{0,-1\}$, $x_0\neq\{0,-1\}$, and further τ_0 is an arbitrary constants (Note that by setting $x = x_0 = 0,-1$ the logarithm term vanishes and/or remains undefined).

To derive classes of solutions for Case F and Case G one has to solve eq.(14) performing the following steps: Changing the variable $S(\zeta)$ to $v(t)$, using the transformation $\zeta = e^t$, eq.(14) converts to $v'' + 2v + v^2 + v = 0$ the prime denotes d/dt. By using the transformations considered from the Tab.2 we find for ***Case F:***

$$u(x,\tau) = \frac{1}{6(e^{\tau}-1)}\left\{(3N+M-3) - \sqrt{B}\left[1 + \frac{2}{\mathrm{cn}(G/2\sqrt{3})-1}\right]\right\}, \tag{22}$$

and equivalently for ***Case G:***

$$u(x,\tau) = \frac{1}{6(e^{\tau}-1)(1+x)}\left\{(3N+M-3) - \sqrt{B}\left[1 + \frac{2}{\mathrm{cn}(K/2\sqrt{3})-1}\right]\right\}, \tag{23}$$

by introducing the quantities (to consider the angular dependence we claim $x_0 \neq 0$, $\tau_0 \neq 0$):

$$G = B^{1/4}\left(\frac{1}{x(e^{-\tau}-1)} + \frac{1}{x_0(1-\cosh\tau_0 + \sinh\tau_0)}\right), \quad x_0 \neq 0 \tag{24}$$

$$K = B^{1/4}\left(\frac{e^{\tau_0}}{e^{\tau_0} + x_0(1+x_0)} - \frac{e^{\tau}}{e^{\tau} + x(1+x)}\right), \; x_0 \neq 0 \tag{25}$$

$$B = (M^2 - 3M(N+2) + 9(4+(N-2)N)^{1/4} \tag{26}$$

and the real numbers

$$M = (297 - 54\sqrt{30})^{1/3} \quad \text{and} \quad N = (11 + 2\sqrt{30})^{1/3}, \tag{27}$$

with the numerical values $M = 1{,}071$ and $N = 2{,}800$ respectively. These derived classes of solutions can be seen as a new contribution to known results. Let us point out an interesting aspect: Using a special transformation applied to eq.(1) Painlevé's first equation results:
The first order derivation term of eq.(1) can be transformed off using the transformations:

$$\tilde{X}(x,\tau,u(x,\tau)) = x\,, \quad \tilde{T}(x,\tau,u(x,\tau)) = -e^{-\tau}\,, \tag{28}$$

$$\tilde{U}\left(\tilde{X}(x,\tau,u(x,\tau))\,\tilde{T}(x,\tau,u(x,\tau))\right) = e^{\tau}\,u(x,\tau) \tag{28a}$$

and therefore, eq.(1) is simplified to

$$\frac{\partial^2 \tilde{U}}{\partial\tilde{X}\,\partial\tilde{T}} + \tilde{U}^2 = 0\,. \tag{29}$$

Applying a Lie procedure once again, the following system of determining equations arise:

$$\frac{\partial \xi_1}{\partial \tilde{U}} = \frac{\partial \xi_2}{\partial \tilde{U}} = \frac{\partial^2 \phi}{\partial \tilde{U}^2} = \frac{\partial \xi_1}{\partial \tilde{T}} = \frac{\partial \xi_2}{\partial \tilde{X}} = \frac{\partial^2 \phi}{\partial \tilde{U} \partial \tilde{X}} = 0,$$

$$2\tilde{U}\phi + \tilde{U}^2 \frac{\partial \xi_1}{\partial \tilde{X}} - \tilde{U}^2 \frac{\partial \xi_2}{\partial \tilde{U}} - \tilde{U}^2 \frac{\partial \phi}{\partial \tilde{U}} + \frac{\partial^2 \phi}{\partial \tilde{X} \partial \tilde{T}} = 0, \tag{30}$$

where the four-dimensional Lie algebra is given by

$$\tilde{V}_1 = \partial_{\tilde{X}}, \quad \tilde{V}_2 = \tilde{X}\partial_{\tilde{X}} - \tilde{U}\partial_{\tilde{U}}, \quad \tilde{V}_3 = \tilde{T}\,\partial_{\tilde{T}} - \tilde{U}\partial_{\tilde{U}}, \quad \tilde{V}_4 = \partial_{\tilde{T}}. \tag{31}$$

The commutator relations are shown in Tab.3; hence, the related structure constants are

$$C_{1,2,1} = C_{3,4,3} = -1 \quad \text{and} \quad C_{2,1,1} = C_{4,3,3} = 1. \tag{32}$$

Table 3. Commutator of the nonlinear Field Equation (29).

	$\tilde{V}_1$	$\tilde{V}_2$	$\tilde{V}_3$	$\tilde{V}_4$
$\tilde{V}_1$	0	$-\tilde{V}_1$	0	0
$\tilde{V}_2$	$\tilde{V}_1$	0	0	0
$\tilde{V}_3$	0	0	0	$-\tilde{V}_3$
$\tilde{V}_4$	0	0	$\tilde{V}_3$	0

The solution of eq.(30) just generates the symmetry of the transformed eq.(29):

$$\begin{aligned} \xi_1 &= k_3 + k_4\tilde{X} \\ \xi_2 &= k_1 + k_2\tilde{T} \\ \phi &= -(k_2 + k_4)\,\tilde{U}(\tilde{X},\tilde{T}) \end{aligned} \tag{33}$$

Now, our intention is to choose a special subgroup, say $k_1 = k_3 = 1$ all other $k_i = 0$. The similarity variable is calculated to $\zeta = \tilde{T} + \tilde{X}$ or equivalently by using a combination of the vector fields

$$\tilde{U}(\tilde{X},\tilde{T}) = S = S\left(\tilde{V}_1 + \tilde{V}_4\right) = S\left(C_1\partial_{\tilde{X}} + C_2\partial_{\tilde{T}}\right), \tag{34}$$

with arbitrary constants C_1 and C_2. Using the transformation $\hat{\zeta} = \sqrt{6C_1C_2}\,\zeta$, Painlevé's first equation is derived [21, 22, 23]; which can also be seen as a complete reduction of eq.(29):

$$S''(\hat{\zeta}) = 6S^2(\hat{\zeta}). \tag{35}$$

The latter equation is usually solved by a reciprocal sine amplitude function in the form:

$$S(\hat{\zeta}) = C_3\left[\frac{-k^2}{1+k^2} + \frac{1}{\mathrm{sn}^2(C_3(\hat{\zeta}-\hat{\zeta}_0))}\right], \tag{36}$$

and, due to the invariant transformation properties also a solution of eq.(1) is therefore:

$$u(x,\tau) = C_3\left[\frac{-k^2}{1+k^2} + \left(\mathrm{sn}^2\left(\frac{1}{\sqrt{C_3(\hat{\zeta}-\hat{\zeta}_0)}}\right)\left(C_0\,x - C_1e^{-\tau} - C_2\right)\right)^{-1}\right], \tag{37}$$

where k^2 in eq.(36) and eq.(37) means the root of the equation $k^4 - k^2 + 1 = 0$. As a last aspect we show how to derive classes of solutions of eq.(29) for the case if the group parameters are chosen to be $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ and then, again by considering the choice $k_2 = k_3 = k_4 = 1$ and $k_1 = 0$, respectively. In the first case the transformation is given:

$$e^{-\tilde{X}}(1+\tilde{T}) = \zeta \quad \text{and} \quad e^{\tilde{X}}\,\tilde{U} = S. \tag{38}$$

We end up by another complete similarity reduction of eq.(29) related to the following nODE:

$$\zeta S'' - 2S' + S^2 = 0. \tag{39}$$

In the second case we deduce the transformation

$$\frac{\tilde{T}}{1+\tilde{X}} = \zeta \quad \text{and} \quad \tilde{U}(1+\tilde{X})^2 = S, \tag{40}$$

to proceed the corresponding nODE as

$$\zeta S'' - 3\,S' + S^2 = 0,\ \ S: R\times R \to R,\ \zeta \in R\setminus\{0\}, -\infty \le \zeta \le \infty\text{-} \tag{41}$$

Solutions can be found by taking the following steps: First, let us rewrite eq.(39) by setting a new variable $S(\zeta) \to y(x)$, introducing $(1 + y(x)) = z(x)$ to obtain the following nODE of second order:

$$x\, z'' z + 2\, z\, z' - 2\, x\, z'^2 + z = 0\,, \tag{42}$$

with the prime denoting derivation w.r.t. the independent variable x. Changing the dependent and the independent variables together by $x = 1/\xi$ and $z = 1/\eta$, inserting into eq.(42) the equation of Bellmann [24, 25, 26] results:

$$\eta'' - \eta^2\, \xi^{-3} = 0\,. \tag{43}$$

The prime means differentiation $d/d\xi$. A general case of eq. (43) can be written as $\eta'' \pm \xi^{\sigma}\, \eta^{n} = 0$. In our case we have $\sigma = -3$ and $n = 2$. An extensive discussion can be found in [25] for all positive proper solutions expressed in asymptotic terms.

In our analysis the relations for the exponents $\sigma + 2 < 0$ and $\sigma + n + 1 = 0$ holds and three cases as $\xi \to \infty$ are of interest: **(i)** $\eta' \to 0$, **(ii)** $\eta' \to \alpha \neq 0$ and **(iii)** $\eta' \to \infty$ whereby α in Case (ii) means an arbitrary constant. Case (ii) and Case (iii) are impossible for a positive proper solution following the proofs in [25]. Therefore an asymptotic expression [25, 26] for Case (i) reads:

$$\eta \sim \alpha + \frac{\alpha^{n}\, \xi^{\sigma+2}}{(\sigma+1)(\sigma+2)}(1 + O[1])\,. \tag{44}$$

Hence, by transforming back we derive an asymptotic expression for the these similarity function S:

$$S \sim \frac{\alpha^2 \zeta}{2}(1 + O[1])\,, \quad \alpha \neq 0\,. \tag{45}$$

Here a linear dependence is seen and furthermore we performed a numerical simulation plotted in figure 1 whereby the graphical representation is in agreement with the asymptotic expression eq.(45). Finally, considering both the eq.(39) and eq.(45), we derive a solution of eq.(29) in the form:

$$\tilde{U}(\tilde{X}, \tilde{T}) \sim \frac{1}{2}\alpha^2 e^{-2\tilde{X}}(1 + \tilde{T})\,. \tag{46}$$

Note: Eq.(41) is of similar structure and can therefore be handled in the same sense.

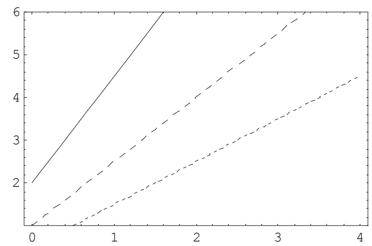


Figure 1. Some solution curves of the nonlinear equation (39) generated by different initial values: Solid line: S(0) = 2, S'(0) = 2.5, dotted line: S(0) = 1, S'(0) = 1.5, short dot line: S(0) = 0.5, S'(0) = 1, in the considered domain a linear connection can be seen according to the linear asymptotic model relation eq.(45).

3. DISCUSSION

Now we are interested in drawing some conclusions of physical relevance. As we have seen that the nonlinear eq.(1) is closely related with the BE (Maxwell distribution) and therefore admits new classes of solutions, especially eq.(21), eq.(22) and eq.(23), respectively.

Here we take full concentration since the remaining solutions are of similar structure. Using eq.(24), eq.(26) and eq.(27) one can write out eq.(22) , Case F, with some simplifications:

$$u_F(x,\tau) = \frac{X^{1/8}}{6(e^{\tau}-1)}\left\{1+2\left(\mathrm{cn}\left[\frac{1}{2}\sqrt{3}\,B^{1/4}\,\varphi\right]-1\right)^{-1}\right\}, \quad -1 \le \varphi \le 1 \ . \tag{47}$$

Briefly, $X^{1/8}$ is used for the numerical factors N and M and their algebraic connections. The argument φ stands for $\left\{(1-e)^{-1}+e^{\tau}\left(x - x\,e^{\tau}\right)^{-1}\right\}$ and for the constants we set $x_0 = \tau_0 = 1$ without loss of generality. In this case one has to claim the restriction $x \neq 0$.

The function appears in a complicate way connecting the elliptic cosine amplitude with exponential functions in their argument and an exponential co-factor acting as a damping term. Importantly, one can assume a series representation in terms of the independent variables (τ, φ) up to order two about the regular point (0,0). Such formulas are useful for numerical calculations in practice justifying their representation (for convenience, the local coordinate x is replaced by the angle):

$$u_F(\varphi,\tau) \approx \frac{1}{\varphi^2}\left\{\frac{X^{1/8}}{9\sqrt{B}}\left(\frac{8}{\tau}-\frac{2}{3}+4\right)+O[\tau]^3\right\}$$
$$+\left\{\frac{(1-2k)X^{1/8}}{9\tau}+\frac{1}{18}(2k-1)X^{1/8}+\frac{1}{208}(1-2k)X^{1/8}\,\tau+O[\tau]^3\right\} \qquad (48)$$
$$+\left\{-\frac{T\,X^{1/8}}{480\,\tau}\frac{T\,X^{1/8}}{960}-\frac{T\,X^{1/8}}{5760}+O[\tau]^3\right\}\varphi^2+O[\varphi]^3$$

by taking $\mathrm{T}=\sqrt{\mathrm{B}}\,\mathrm{X}^{1/8}\left(1+16\mathrm{k}\left(\mathrm{k}-1\right)\right)$ and usually k is the modulus. A limiting view (the approach allows to segue from elliptic to circular/hyperbolic functions) for $\mathrm{k}\to 0$ and $\mathrm{k}\to 1$ respectively, leading to

$$\lim_{\mathrm{k}\to 0}\mathrm{u}_{\mathrm{F}}(\mathrm{x},\tau)=\frac{\mathrm{X}^{1/8}}{6(\mathrm{e}^{\tau}-1)}\mathrm{ctg}^2\left[\frac{\sqrt{3}}{4}\,\mathrm{B}^{1/16}\,\varphi\right], \qquad (49)$$

$$\lim_{\mathrm{k}\to 1}\mathrm{u}_{\mathrm{F}}(\mathrm{x},\tau)=\frac{\mathrm{X}^{1/8}}{6(\mathrm{e}^{\tau}-1)}\mathrm{ctgh}^2\left[\frac{\sqrt{3}}{4}\,\mathrm{B}^{1/16}\varphi\right]. \qquad (49a)$$

For small values of τ, especially for $\tau\to 0$ the function takes singular; the first and the second derivatives are positive functions $\forall\,\varphi\in\nabla\setminus\{0\}$ and for the independent variable τ the same is true.

In addition it is also seen that the relation $\lim_{\tau\to\infty}\mathrm{u}_{\mathrm{F}}(\mathrm{x},\tau)=0$ holds. Using the same arguments one can discuss eq.(23) which differs by a rational term:

$$u_G(x,\tau)=\frac{X^{1/8}}{6(e^{\tau}-1)(1+x)}\left\{1+2\left(cn\left[\frac{1}{2}\sqrt{3}\,B^{1/4}\,\varphi\right]-1\right)^{-1}\right\},\quad -1\le\varphi\le 1 \qquad (50)$$

with the argument of the cosine amplitude explicitly $\varphi=\left(\frac{\mathrm{e}}{2+\mathrm{e}}+\frac{\mathrm{e}^{\tau}}{\mathrm{e}^{\tau}+\mathrm{x}(1+\mathrm{x})}\right)$. This function admits a useful series representation about the regular point (0,0) up to order three:

$$u_G(x,\tau) \approx \left\{-\frac{1}{\tau}\left(\frac{8W}{9\varphi^2}+\frac{V}{9}-U\varphi^2+O[\varphi]^3\right)+\left(\frac{4W}{9\varphi^2}+\frac{V}{18}+\frac{Y\varphi^2}{960}+O[\varphi]^3\right)+\right.$$
$$\left.+\left(-\frac{2W}{27\varphi^2}+\frac{V}{108}-\frac{Y\varphi^2}{5760}+O[\varphi]^3\right)\tau+O[\tau]^3\right\}+\left\{\frac{1}{\tau}\left(\frac{8W}{9\varphi^2}+\frac{V}{9}+U\varphi^2+O[\varphi]^3\right)+\right.$$
$$\left.+\left(-\frac{4W}{9\varphi^2}+\frac{V}{18}-\frac{Y\varphi^2}{960}+O[\varphi]^3\right)+\left(\frac{2W}{27\varphi^2}+\frac{V}{108}+\frac{Y\varphi^2}{5760}+O[\varphi]^3\right)\tau+O[\tau]^3\right\}x+$$
$$+\left\{-\frac{1}{\tau}\left(\frac{8W}{9\varphi^2}+\frac{V}{9}-U\varphi^2+O[\varphi]^3\right)+\left(\frac{4W}{9\varphi^2}+\frac{V}{18}+\frac{Y\varphi^2}{960}+O[\varphi]^3\right)+\right.$$
$$\left.+\left(-\frac{2W}{27\varphi^2}+\frac{V}{108}-\frac{Y\varphi^2}{5760}+O[\varphi]^3\right)\tau+O[\tau]^3\right\}x^2+O[x]^3 \qquad (51a)$$

To simplify, the following abbreviations in eq.(51a) are appropriate:

$$\begin{aligned}&\frac{1}{480}\left(\sqrt{\mathrm{B}}\left(1+16\mathrm{k}\left(\mathrm{k}-1\right)\right)\right)\mathrm{X}^{1/8}=\mathrm{U}\\&(2\mathrm{k}-1)\,\mathrm{X}^{1/8}=\mathrm{V}\\&\mathrm{X}^{1/8}/\sqrt{\mathrm{B}}=\mathrm{W}\\&\sqrt{\mathrm{B}}\;\mathrm{X}^{1/8}\left(1+16\mathrm{k}\left(\mathrm{k}-1\right)\right)=\mathrm{Y}\end{aligned} \qquad (51b)$$

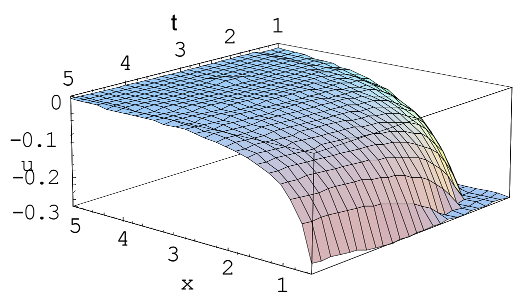
As to be expected a limiting analysis leads to a cotangent and a hyperbolic cotangent function as well as in the latter case. For both functions a graphical overview is shown in figure 2 and figure 3 respectively. Hence, one can calculate generating functions for the normalized moments in a most general form by applying the transformation given on p.5, $x=\frac{1-\xi}{\xi}$, $u(x,\tau)=\xi G(\xi,\tau)$ to derive:

$$G_F(\xi,\tau)=\frac{X^{1/8}}{6\xi(e^{\tau}-1)}\left\{1+2\left(\mathrm{cn}\left[\frac{1}{2}\sqrt{3}\,B^{1/4}\,\varphi\right]-1\right)^{-1}\right\}\;, \qquad (52)$$

$$\text{with } \varphi=\frac{\sqrt{3}\;B^{1/16}(\xi-1+e^{\tau}(1+(e-2)\xi)}{2(\xi-1)(e^{\tau}-1)(e-1)}\;. \qquad (52a)$$

$$G_G(\xi,\tau) = \frac{\xi\, X^{1/8}}{6\,(e^{\tau}-1)}\left\{1+2\left(\mathrm{cn}\left[\frac{1}{2}\sqrt{3}\,B^{1/4}\,\varphi\right]-1\right)^{-1}\right\}, \tag{53}$$

with
$$\varphi = \left(\frac{e}{2+e}+\frac{\xi^2 e^{\tau}}{1-\xi+\xi^2 e^{\tau}}\right). \tag{53a}$$



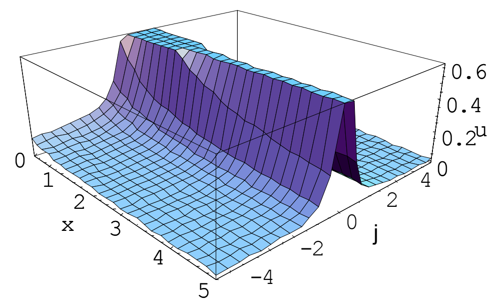


Figure 2. A surface sketch of the function $u_F(x,\tau)$, eq.(22) and eq.(47) respectively. In this animation the modulus of the elliptic cosine amplitude is chosen to $k=1$. The contribution of the exponential part acts as a damping term preventing a further slow rise for $x \geq 0$, $\tau \geq 0$ in the domain of saturation (left).

Figure 3. A surface sketch of the function $u_G(x,\varphi)$, eq.(23) and eq.(50) respectively. Similar to figure 2, the modulus is given by $k=1$. The exponential and rational parts common influences the behaviour in that sense, so that a maximum value could be observed (right).

As a surprising point we show properties of the solution function eq.(21), Case E, which differs completely from the above given class of solutions. A surface plot can be seen in figure 4. Taking into account eq.16a) and eq.(21) a compact written form becomes:

$$u_E(x,\tau) = -\frac{(1+B(B-1))^3}{2B^3(1+x)}\left\{\frac{2-\mathrm{cn}[\varphi,k]}{2\,\mathrm{cn}[\varphi,k]}\right\}, \tag{54}$$

with the argument $\varphi = \sqrt{\frac{3B(B-1)+1}{2B}}\left(\ln(1+x)+\tau-1\right)$ and the real number $B = \left(11-2\sqrt{30}\right)^{1/3}$.

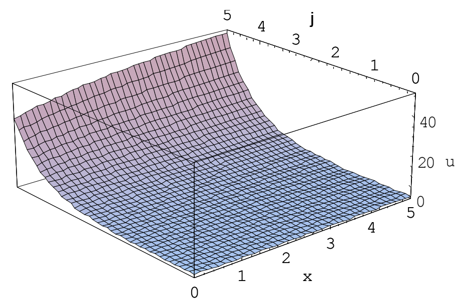


Figure 4. A surface sketch of the function $u_E(x,\varphi)$, eq.(21) and eq.(54) respectively. A typical slow rising behaviour promoted by the logarithm part in the argument of $u_E(x,\varphi)$ is seen.

Here we have to claim the restriction $x \in \nabla^+ \setminus \{0,1\}$. A limiting analysis further shows that

$$\lim_{k\to 0} u_E(x,\tau) = \frac{P(2\sec\varphi - 1)}{4B^3(1+x)} \quad \text{and} \quad \lim_{k\to 1} u_E(x,\tau) = \frac{P(2\cosh\varphi - 1)}{4B^3(1+x)}, \tag{54a}$$

holds and the a series representation up to order three in this case leads to a more convenient form:

$$u_E(x,\tau) \approx \left\{S - Sx + Sx^2 + O[x]^3\right\} + \left\{S - Sx + Sx^2 + O[x]^3\right\}\varphi^2 + O[\varphi]^3. \tag{54b}$$

The expression for the generating function for eq.(54) reads as

$$G_E(\xi,\tau) = -\frac{(1+B(B-1))^3}{2B^3\xi(1+\xi)}\left\{\frac{2-\mathrm{cn}[\varphi,k]}{2\,\mathrm{cn}[\varphi,k]}\right\}, \tag{54c}$$

and the argument of the elliptic cosine function rewritten in terms of ξ leads to

$$\varphi = R\left(\tau + \ln[1+\xi] - 1\right) \text{ with the numerical factor } R = \sqrt{\frac{3}{2}}\left(\frac{(1+B(B-1))}{B}\right)^{1/2}. \tag{54d}$$

Comparing (52) and (54c) one can see the significant difference in the structure of both generating functions.

We show the analytical structure of these new classes of solutions leading to the following interpretation: $u_F(x,\tau)$ and $u_G(x,\tau)$ respectively, contains inverse elliptic functions which are proper class of solutions like a bell-shaped behavior whereby the exponential term acts as a damping term. The rational term as well as the logarithm part influences just so that saturation is observed; especially by taking into account figure 2 and figure 3.The increasing trend of $u_E(x,\tau)$ provides a slow rise of the distribution function. The stability of the given new class of solutions is a further important (physical) question. To decide whether a solution is stable the second derivative is of interest. Since all functions result from a similarity transformation it is sufficient to consider eq.(16). Performing some analysis the following cases are found (for the numerical calculation the modulus is assumed to be $k = 0{,}5$):

$$\frac{d^2S}{d\zeta^2} = \begin{cases} \infty & \zeta = 0 \\ \text{positive} & \zeta > 0 \\ \text{positive} & \zeta < 0 \end{cases} \tag{55}$$

The solutions are stable either for the cases $\zeta \in \nabla^+$ and $\zeta \in \nabla^-$ apart from the origin, here the function remains indeterminate.

Note: Since the relation to the Bellmann Equation is new the present study will be extended in future work. Hence the results obtained for the generating functions will also play an important contribution leading to new physical insights of the behavior of Maxwellian molecules. It is expected that such considerations will also change the meaning of normalized moments and therefore the distribution function will be suitable modified.

4. CONCLUSION

The similarity reduction of two model equations, especially eq.(1) and eq.(29) were studied by Lie's classical method whereby the physical point of view is the relation to a Maxwell distribution. The model equations have been studied for group invariant solutions and a complete analysis is given. As an interesting fact new transformations and similarity solutions could be found.

Both classes of solutions contain exponential- and elliptic functions in additional form. Applying a special transformation to the model eq.(1), the relation to Painlevé's first equation is shown. An explicit analysis performed on the model eq.(29) leads to new transformations and therefore to new classes of similarity solutions.

It is shown to the first time that the determining equations eq.(39) and eq.(41) for the similarity function are related to an equation of Bellmann`s type. Asymptotic solutions are given as well as a numerical analysis to show the agreement in a graphical mode. Due to the complexity of the structure suitable series expressions are given and formulas for the generating function relating to normalized moments are also derived.

5. ACKNOWLEDGEMENTS

The author likes to dedicate the present paper to the much honoured Univ.-Prof. Dipl.-Ing. Dr. techn. Helmut Jäger, Professor Emeritus at the Institute of Experimental Physics at the Technical University Graz, Austria. The author wishes to thank Professor Jäger for all valuables contributions and guidance during the authors academic considerations.

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