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Computing Wiener and hyper–Wiener indices of unitary Cayley graphs

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ABSTRACT

The unitary Cayley graph X_n has vertex set Z_n = {0, 1,…, n-1} and vertices u and v are adjacent, if gcd(u–v, n) = 1. In [A. Ilić, The energy of unitary Cayley graphs, Linear Algebra Appl. 431 (2009) 1881–1889], the energy of unitary Cayley graphs is computed. In this paper the Wiener and hyper–Wiener index of X_n is computed.

Keywords: Unitary Cayley graphs, Wiener index, hyper–Wiener index.

1. INTRODUCTION

Let H be a connected graph with vertex and edge sets V(H) and E(H), respectively. As usual, the distance between the vertices u and v of H is denoted by d(u,v) and it is defined as the number of edges in a minimal path connecting the vertices u and v.

A topological index is a real number related to a graph. It must be a structural invariant, i.e., it preserves by every graph automorphisms. There are several topological indices have been defined and many of them have found applications as means to model chemical, pharmaceutical and other properties of molecules. The Wiener index W is one of the most studied topological index, see for details [4,5]. It is equal to the sum of distances between all pairs of vertices of the respective graph,[11].

The hyper-Wiener index was proposed by Klein et al. [9], as a generalization of the Wiener index of graph. It is defined as WW(G)=1/2W(G)+1/2 $\sum_{\{u,v\}\subseteq V(G)} d(u,v)^2$. We encourage the reader to consult [2,3,6,8] for mathematical properties of hyper–Wiener index and its applications in chemistry.

Let G be a multiplicative group with identity 1. For $S\subseteq G$; $1\notin S$ and $S^{-1}=\{s^{-1} \mid s\in S\}=S$ the Cayley Graph X=Cay(G;S) is the undirected graph having vertex set V(X)=G and edge set

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$E(X)=\{\{a,b\} \mid ab^{-1}\in S\}$. The Cayley graph X is regular of degree |S|. Its connected components are the right cosets of the subgroup generated by S. So X is connected, if S generates G. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [1].

For a positive integer n>1 the unitary Cayley graph $X_n = Cay(Z_n;U_n)$ is defined by the additive group of the ring Z_n of integers modulo n and the multiplicative group U_n of its units. If we represent the elements of Z_n by the integers 0, 1,…, n-1, then it is well known that $U_n=\{a\in Z_n \mid gcd(a,n)=1\}$. So X_n has vertex set $V(X_n)=Z_n=\{0, 1, \dots, n-1\}$ and $ab\in E(X_n)$ if and only if gcd(a–b,n)=1. The graph X_n is regular of degree $|U_n|=\varphi(n)$, where $\varphi(n)$ denotes the Euler function. If n=p is a prime number, then $X_n=K_p$ is the complete graph on p vertices. If $n = p^t$ is a prime power then X_n is a complete p-partite graph. In this paper we give formulas for the calculation of the hyper−Wiener index of unitary Cayley graphs.

In the following lemma, well-known properties of unitary Cayley graphs are introduced, [7,10], for more details.

Lemma 1. The number of common neighbors of distinct vertices a, b in the unitary Cayley graph X_n is given by $F_n(a-b)$, where for integers n, s we have:

$$F_n(s) = n\prod_{p|n}\left(1-\frac{g(p)}{p}\right)$$

and p is a prime number and $g(p) = \begin{cases} 1 & p \mid s \\ 2 & \text{otherwise} \end{cases}$.

Lemma 2. The unitary Cayley graph X_n, n≥2, is bipartite if and only if n is even.

2. HYPER−WIENER INDEX OF UNITARY CAYLEY GRAPHS

In this section, hyper−Wiener index of the graph X_n were computed. We assume that $d_G(v)$ is the sum of distances between v and all other vertices of G. Then

$$W(G) = \sum_{\{v,u\}\subseteq V(G)} d(v,u) = 1/2\sum_{v\in V(G)} d_G(v)$$

and

$$WW(G) = 1/4\sum_{v\in V(G)}\left(\sum_{u\in V(G)} d\ (v,u) + d\ (v,u)^2\right).$$

Theorem 1. The Wiener index of unitary Cayley graph X_n is as follows:

$$W(X_n)=\begin{cases} 1/2n(n-1) & \text{if } n \text{ is a prime number,}\\ 3/4n^2-n & \text{if } n=2^{\alpha} \text{ and } \alpha>1,\\ n^2-1/2n\varphi(n)-n & \text{if } n \text{ is odd, but not a prime number,}\\ 5/4n^2-n\varphi(n)-n & \text{if } n \text{ is even and has an odd prime divisor.}\end{cases}$$

Proof. If n is a prime number, then $X_n=K_n$ is the complete graph and d(u,v)=1, for every u,v. then by definition Wiener index we have $W(X_n)=|E(X_n)|=1/2n(n-1)$. If $n=2^{\alpha}$ and $\alpha>1$, then X_n is the complete bipartite graph with vertex partition $V(X_n)=\{0,2, \dots, n-2\} \cup \{1,3, \dots, n-1\}$. Thus $W(X_n)= W(K_{2^{\alpha-1},2^{\alpha-1}}) =3/4n^2-n$.

Suppose n is odd, but not a prime number. let $p_1, p_2, \dots, p_t$ be the different prime divisors of n, $n=p_1^{r_1}\times p_2^{r_2}\times\dots\times p_t^{r_t}$ and $p_i\neq 2$, $1\le i\le t$. By Lemma 1, the number of common neighbors of vertices $a\neq b$ is $F_n(a-b)$ and by definition $F_n(s)$ all factors in the expansion of $F_n(a-b)$ are positive. In this case there is a common neighbor to every pair of distinct vertices, which implies d(a,b)=1 or d(a,b)=2. Thus

$$\begin{aligned} W(X_n) &= 1/2\sum_{v\in V(X_n)} d_G(v) = 1/2\sum_{v\in V(X_n)}\Big(\sum_{u\in V(X_n)} d(v,u)\Big) = 1/2\sum_{v\in V(G)}\Big(\sum_{uv\in E(X_n)} 1+\sum_{uv\notin E(X_n)} 2\Big)\\ &= 1/2\sum_{v\in V(X_n)}(\varphi(n)+2(n-\varphi(n)-1)) = 1/2n(-\varphi(n)+2n-2)\\ &= n^2-1/2n\varphi(n)-n \end{aligned}$$

Finally, we consider the case where n is even and has an odd prime divisor p, $n\neq 2^{\alpha}$. By Lemma 2, X_n is the bipartite graph with vertex partition $V(X_n)=A\cup B$, where $A = \{0,2, \dots, n-2\}$ and $B = \{1,3, \dots, n-1\}$. Therefore, for computing the wiener index, it is enough to calculate $d_G(u)$, for every $u\in V(X_n)=A\cup B$. To calculate $d_G(u)$, we consider two cases that $u\in A$ or $u\in B$. If $u,v\in A$ then vertices u and v of X_n are not adjacent and by Lemma 1, they have common neighbor and so d(u,v)=2. If $u\in A$ and $v\in B$ then all $\varphi(n)$ neighbors of u are in B. Let $B=B_1\cup B_2$, where $B_1=\{v\in B \mid uv\in E(X_n)\}$ and $B_2=\{v\in B \mid uv\notin E(X_n)\}$. We show that for $u\in A$ and $v\in B_2$, d(u,v)=3. To do this, we assume that $w\in B_1$, $uw\in E(X_n)$. Now w and v are both odd and therefore have a common neighbor $z\in A$, which implies d(u,v)=3 and we have

$$\begin{aligned} d_G(u) &= \sum_{v\in V(X_n)} d(u,v) = \sum_{v\in A} d(u,v) + \sum_{v\in B} d(u,v) \\ &= \sum_{v\in A} 2 + \sum_{v\in B_1\cup B_2} d(u,v) = 2(n/2-1) + \sum_{v\in B_1} 1 + \sum_{v\in B_2} 3 \\ &= 2(n/2-1) + \varphi(n) + 3(n/2-\varphi(n)) \\ &= 5/2n - 2\varphi(n) - 2 \end{aligned}$$

Similarly, if $u\in B$ then $d_G(u) = 5/2n - 2\varphi(n) - 2$ and we have:

$$\begin{aligned} W(X_n) &= 1/2\sum_{u\in V(X_n)} d_G(u) = 1/2\sum_{u\in A\cup B} d_G(u) \\ &= 1/2\sum_{u\in A} 5/2n - 2\varphi(n) - 2 + 1/2\sum_{u\in B} 5/2n - 2\varphi(n) - 2 \\ &= 1/2n(5/2n - 2\varphi(n) - 2) = 5/2n^2 - n\varphi(n) - n \end{aligned}$$

Which completes the proof. ■

Corollary. The hyper–Wiener index of unitary Cayley graph X_n is as follows:

$$WW(X_n) = \begin{cases} 1/4n^2(n-1) & \text{if } n \text{ is a prime number,} \\ n^2 - 3/2n & \text{if } n = 2^{\alpha} \text{ and } \alpha > 1, \\ n^2 - 1/2n\varphi(n) - n & \text{if } n \text{ is odd, but not a prime number,} \\ 1/4n^2 - 5/2n\varphi(n) - 3/2n & \text{if } n \text{ is even and has an odd prime divisor.} \end{cases}$$

Proof. The proof is straightforward by Theorem 1. ■

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