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Remarks on Distance–Balanced Graphs

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ABSTRACT

Distance-balanced graphs are introduced as graphs in which every edge uv has the following property: the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Basic properties of these graphs are obtained. In this paper, we study the conditions under which some graph operations produce a distance-balanced graph.

Keywords: Distance–balanced graphs, graph operation.

1. INTRODUCTION

For an edge *e* = *ab* of a graph *G*, let $n_a^G(e)$ be the number of vertices closer to *a* than to *b*. That is, $n_a^G(e)$ =|{u ∈ V(G) | d(u, a) < d(u, b)}| . In addition, let $n_0^G(e)$ be the number of vertices with equal distances to *a* and *b*; $n_0^G(e)$ = |{u ∈ V(G) | d(u, a) = d(u, b) }| .

Here is our key definition. We call a graph *G* distance-balanced, if $n_a^G(e) = n_b^G(e)$ holds for any edge *e* = *ab* of *G*. These graphs were, at least implicitly, first studied by Handa [4] who considered distance-balanced partial cubes. The term itself, however, is due to Jerebic et al. [1] who studied distance–balanced graphs in the framework of various kinds of graph products. The transmission *T(u)* of a vertex u ∈ V is defined as follows:

$$T(u) = \sum_{v \in V} d(u,v).$$

A graph *G* is said to be transmission-regular if all its vertices have the same transmission. As examples of transmission-regular graphs, we can cite the complete graph K_n on n ≥ 2 vertices, the complete bipartite graph $K_{n,n}$ on 2n ≥ 2 vertices.

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Let *G* and *H* be two graphs. The corona product G o H is obtained by taking one copy of *G* and |V(G)| copies of *H*; and by joining each vertex of the i-th copy of *H* to the i-th vertex of *G*, i = 1, 2, …, |V(G)|, see [2,3]. The join G + H of graphs *G* and *H* with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G \bigcup H$ together with all the edges joining V_1 and V_2. The symmetric difference $G \oplus H$ of two graphs *G* and *H* is the graph with vertex set V(G) × V(H) and edge set

$$E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) \mid u_1 v_1 \in E(G) \quad \text{or} \quad u_2 v_2 \in E(H) \text{ but not both}\}.$$

The cluster G{H} is obtained by taking one copy of *G* and |V(G)| copies of a rooted graph *H*, and by identifying the root of the ith copy of *H* with the i^{th} vertex of *G*, i = 1,2 ,..., |V(G)|. The composite graph G{H} was studied by Schwenk [9]. Throughout this paper our notation is standard and taken mainly from the standard book of graph theory. We encourage the reader to consult papers [5,7,8,10–12] for background material as well as basic computational techniques.

2. MAIN RESULTS

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree *k* is called a k-regular graph or regular graph of degree *k*. In this section, we study the conditions under which some graph operations produce a distance-balanced graph. We begin by the following theorem which states the relationship between distance-balanced and transmission-regular graphs:

Theorem 1. A graph *G* is distance-balanced if and only if *G* is transmission-regular.

Proof. It is well–known fact that if *G* is a connected graph and uv = e $\in$ E(G), then $n_u^G(e) = n_v^G(e)$ if and only if T(u) = T(v) [6], proving the result. ▼

Theorem 2. Let *G* and *H* be connected graphs. Then G + H is distance-balanced if and only if *G* and *H* are *r* and *k* regular graphs, respectively, and $|V(G)| - r = |V(H)| - k$.

Proof. Consider the following partition of E(G + H):

$$A = \{uv \in E(G+H) \mid u, v \in V(G)\},$$
$$B = \{uv \in E(G+H) \mid u, v \in V(H)\},$$
$$C = \{uv \in E(G+H) \mid u \in V(G) \text{ and } v \in V(H)\}.$$

We first assume that *G* and *H* are r– and k–regular graphs respectively, and $|V(G)| - r = |V(H)| - k$. Let uv = e $\in$ A and $m_0^G(e) = |\{x \in V(G) \mid d(u, x) = d(v, x) = 1\}|$. Notice that

$$d_{G+H}(x,y)=\begin{cases}0 & x=y\\ 1 & (x\in V(G) \text{ and } y\in V(H)) \text{ or } (xy\in E(H)) \text{ or } (xy\in E(G))\\ 2 & \text{otherwise}\end{cases}$$

Thus we have $n_u^{G+H}(e) = \deg_G(u) - m_0^G(e)$ and $n_v^{G+H}(e) = \deg_G(v) - m_0^G(e)$. Since *G* is regular $\deg_G(u) = \deg_G(v)$, and thus $n_u^{G+H}(e) = n_v^{G+H}(e)$. We now assume that uv = e$\in$B. In a similar way we can see that $n_u^{G+H}(e) = n_v^{G+H}(e)$. Assume that uv=e$\in$C. Then we have $n_u^{G+H}(e) = |V(H)| - \deg_H(v)$ and $n_v^{G+H}(e) = |V(G)| - \deg_G(u)$. Therefore, $n_u^{G+H}(e) = n_v^{G+H}(e)$ and thus G + H is distance-balanced. Conversely, assume that G + H is distance-balanced. By above argument for an edge e of A, we see $n_u^{G+H}(e) = n_v^{G+H}(e)$ implies that any two adjacent vertices of *G* have the same degree. Since *G* is connected, this implies that *G* is r–regular for some *r*. In a similar way we can see that *H* is k-regular, for some *k*. For an edge uv = e $\in$ C, it follows again from earlier analysis that $n_u^{G+H}(e) = |V(H)| - \deg_H(v)$ and $n_v^{G+H}(e) = |V(G)| - \deg_G(u)$. Since G + H is distance - balanced, two above equations imply that $|V(H)| - \deg_H(v) = |V(G)| - \deg_G(u)$. ▼

Corollary. Let *G* and *H* be connected graphs. G + H is transmission-regular if and only if *G* and *H* be *r* and *k* regular respectively, such that $|V(G)| - r = |V(H)| - k$.

Proof. The proof follows from Theorems 1 and 2. ▼

A graph *G* is called nontrivial if $|V(G)| > 1$.

Theorem 3. The corona product of two arbitrary, nontrivial and connected graphs is not distance–balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that uv = e $\in$ E(GoH) such that u$\in$V(G) and v$\in$V(H_i). Thus, we have :

$$n_u^{GoH}(e) = |V(G)|\,(|V(H)| + 1) - \deg_{GoH}(v) \text{ and } n_v^{GoH}(e) = 1.$$

Therefore, we have $n_u^{GoH}(e) \neq n_v^{GoH}(e)$. Thus *GoH* is not distance-balanced. ▼

Corollary. The corona product of two arbitrary, nontrivial and connected graphs is not transmission - regular.

Proof. The proof follows from Theorems 1 and 3. ▼

Let $e = (a,x)(b,y) \in E(G\oplus H)$ such that $ab \in E(G)$, and $N_{(a,x)}(e) = \{(u,v) \in V(G\oplus H) \mid d((u,v),(a,x)) < d((u,v),(b,y))\}$. Consider the following partition of $N_{(a,x)}(e)$:

$A_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\in E(G) , vx\notin E(H) , ub\in E(G) , vy\in E(H) \}$,
$B_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid (u,v) \neq (b,y), au\in E(G) , vx\notin E(H) , ub\notin E(G) , vy\notin E(H) \}$,
$C_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\notin E(G) , vx\in E(H) , ub\in E(G) , vy\in E(H) \}$,
$D_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\notin E(G) , vx\in E(H) , ub\notin E(G) , vy\notin E(H) \}$ and
$F_{(a,x)} = \{(a,x)\}$. We have:

Theorem 4. Let *G* and *H* be nontrivial and regular graphs. Then the symmetric difference $G \oplus H$ is distance-balanced.

Proof. Let $e = (a,x)(b,y) \in E(G\oplus H)$, where $ab\in E(G)$. Then $n_{(a,x)}(e) = |N_{(a,x)}(e)|$, $N_{(a,x)}(e) = A_{(a,x)} \cup B_{(a,x)} \cup C_{(a,x)} \cup D_{(a,x)} \cup F_{(a,x)}$ and $N_{(b,y)}(e) = A_{(b,y)} \cup B_{(b,y)} \cup C_{(b,y)} \cup D_{(b,y)} \cup F_{(b,y)}$. On the other hand, since *G* and *H* are regular, $|A_{(a,x)}| = |A_{(b,y)}|, \ldots, |B_{(a,x)}| = |B_{(b,y)}|$, $|C_{(a,x)}| = |C_{(b,y)}|$, $|D_{(a,x)}| = |D_{(b,y)}|$ and $|F_{(a,x)}| = |F_{(b,y)}|$. Therefore, $n_{(a,x)}(e) = n_{(b,y)}(e)$. If $e = (a,x)(b,y)$, $xy\in E(H)$, then a similar argument shows that $n_{(a,x)}(e) = n_{(b,y)}(e)$, proving the result. ▼

Theorem 5. The cluster of two arbitrary, nontrivial and connected graphs is not distance-balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that $uv = e \in E(G\{H\})$ such that *u* is the root of the i^{th} copy of *H* and $u \neq v \in V(H_i)$. Thus, we have :

$$n_u^{G\{H\}}(e) = |V(H)|\,(\,|V(G)| - 1) + n_u^H(e) \quad \text{and} \quad n_v^{G\{H\}}(e) = n_v^H(e).$$

Therefore, $n_u^{G\{H\}}(e) \neq n_v^{G\{H\}}(e)$ and so $G\{H\}$ is not distance-balanced. ▼

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