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Computing Some Topological Indices of Tensor Product of Graphs

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ABSTRACT

A topological index of a molecular graph G is a numeric quantity related to G which is invariant under symmetry properties of G. In this paper we obtain the Randić, geometric-arithmetic, first and second Zagreb indices , first and second Zagreb coindices of tensor product of two graphs and then the Harary, Schultz and modified Schultz indices of tensor product of a graph G with complete graph of order n are obtained.

Keywords: Topological index, tensor product.

1. INTRODUCTION

A topological index of a molecular graph G is a numeric quantity related to G which is invariant under symmetry properties of G. The first and second Zagreb indices were originally defined as $M_1(G)=\sum_{a\in V(G)}\delta_G^2 a$ and $M_2(G)=\sum_{ab\in E(G)}\delta_G a\ \delta_G b$, respectively. The first Zagreb index can be also expressed as a sum over edges of G, $M_1(G)=\sum_{ab\in E(G)}[\delta_G a+\delta_G b]$, see [1, 2]. The first and second Zagreb coindices are defined as $\overline{M}_1(G)=\sum_{ab\notin E(G)}[\delta_G a+\delta_G b]$ and $\overline{M}_2(G)=\sum_{ab\notin E(G)}\delta_G a\ \delta_G b$, see [3]. In 1975, the chemist Milan Randić proposed a topological index based on the degrees of the end vertices of an edge in studying the properties of alkane [4]. The Randić index of a graph G is defined as $R(G)=\sum_{ab\in E(G)}(1/\sqrt{\delta_G a\ \delta_G b})$. The geometric–arithmetic index (GA) was conceived, $GA(G)=\sum_{ab\in E(G)}(\sqrt{\delta_G a\,\delta_G b}/\frac{1}{2}(\delta_G a\,\delta_G b))$. Other topological indices

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that will be used in this paper are the Schultz and modified Schultz indices and they are defined as follows:

$$W_+(G) = \sum_{\{a,b\}\subseteq V(G)} (\delta_G a + \delta_G b)\, d_G(a,b),$$

$$W_*(G) = \sum_{\{a,b\}\subseteq V(G)} \delta_G a\, \delta_G b\, d_G(a,b),$$

respectively, see [5, 6] for details. The Harary index *H*(*G*) is defined as $H(G) = \sum_{\{a,b\}\subseteq V(G)} (1/\sqrt{\delta_G a\, \delta_G b})$ [7]. For any two simple graphs *G* and *H*, the tensor product $G\otimes H$ of G and H has vertex set $V(G\otimes H) = V(G)\times V(H)$ and edge set $E(G\otimes H) = \{(a,b)(c,d) \mid \quad ac\in E(G) \ \ and \ \ bd\in E(H)\}$. It is easy to prove that $|E(G\otimes H)| = 2|E(G)||E(H)|$ [8]. In [9], the vertex PI index was proposed and the Wiener and vertex PI indices of this graph operation were computed in [10]. In this paper we study on some topological indices of tensor product of graph. At the beginning the Randić, GA, first and second Zagreb indices and first and second Zagreb coindices are computed. For obtaining Zagreb coindices of tensor product of graphs, we need another graph operations that recall them in the next stage.

The disjunction $G\vee H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which (a,b) is adjacent with (c,d) whenever *a* is adjacent with *c* in *G* or b is adjacent with *d* in *H*.

The symmetric difference $G\oplus H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which (a,b) is adjacent with (c,d) whenever *a* is adjacent with *c* in *G* or b is adjacent with d in H, but not both.

For computing topological indices which related to distance in graphs, we use the useful and simple definitions and result in [11] for distance of vertices in tensor product of graphs.

Definition 1.1. Let *G* be a graph. We define $d'_G(x,y)$ for $x,y\in V(G)$ as follows:

i. If $d_G(x,y)$ is odd then $d'_G(x,y)$ is defined as the length of the shortest even walk joining *x* and *y* in *G*, and if there is no shortest even walk then $d'_G(x,y) = +\infty$.

ii. If $d_G(x,y)$ is even then $d'_G(x,y)$ is defined as the length of the shortest odd walk joining *x* and *y* in *G*, and if there is no shortest odd walk then $d'_G(x,y) = +\infty$.

iii. If $d_G(x,y) = +\infty$, then $d'_G(x,y) = +\infty$.

Definition 1.2. Let G and H be two graphs and $(a,b),(c,d)\in V(G\otimes H)$. The relation *R* on the vertices of $G\otimes H$ is defined as follows:
$(a,b)\ R\ (c,d)$ if and only if $d_G(a,c), d_H(b,d) < +\infty$ and $d_G(a,c) + d_H(b,d)$ is even.

Theorem 1.3. Let *G* and *H* be graphs and $(a,b),(c,d)\in V(G\otimes H)$.

i. If $(a,b)\ R\ (c,d)$, then $d_{G\otimes H}((a,b),(c,d)) = Max\ \{d_G(a,c), d_H(b,d)\}$.

ii. If $(a,b) \not R\ (c,d)$ then,

$$d_{G\otimes H}((a,b),(c,d)) = Min\{\ Max\{d_G(a,c), d'_H(b,d)\},\ Max\{d'_G(a,c), d_H(b,d)\}\}.$$

We use the above definitions and Theorem for computing Schultz, modified Schultz and Harary indices of tensor product of complete graph K_n and a graph G.

2. MAIN RESULTS

In this section, the Zagreb indices and coindices are computed for tensor product of graphs.

Theorem 2.1. Let G and H be graphs. The first and second Zagreb indices of tensor product of G and H are given by:

$$M_1(G\otimes H) = M_1(G)M_1(H),$$

$$M_2(G\otimes H) = 2M_2(G)M_2(H).$$

Proof. By definition of Zagreb indices,

$$\begin{aligned} M_1(G\otimes H) &= \sum_{(a,b)\in V(G\otimes H)} (\delta_{G\otimes H}(a,b))^2 \\ &= \sum_{(a,b)\in V(G\otimes H)} (\delta_G a\ \delta_H b)^2 \\ &= \sum_{a\in V(G)} \sum_{b\in V(H)} (\delta_G a)^2\ (\delta_H b)^2 \\ &= \sum_{a\in V(G)} (\delta_G a)^2 \sum_{b\in V(H)} (\delta_H b)^2 \\ &= M_1(G)M_1(H). \end{aligned}$$

Also,

$$\begin{aligned} M_2(G\otimes H) &= \textstyle\sum_{(a,b)(c,d)\in E(G\otimes H)} \delta_{G\otimes H}(a,b)\delta_{G\otimes H}(c,d) \\ &= 2\textstyle\sum_{ac\in E(G), bd\in E(H)} (\delta_G a\quad \delta_H b)\ (\delta_G c\ \delta_H d) \\ &= 2\textstyle\sum_{ac\in E(G)} \delta_G a\ \delta_G c \sum_{bd\in E(H)} \delta_H b\ \delta_H d \\ &= 2M_2(G)M_2(H), \end{aligned}$$

which completes the proof. □

Theorem 2.2. Let G and H be graphs. The first and second Zagreb coindices of tensor product of G and H are computed as follows:

$$\begin{aligned} \overline{M}_1(G\otimes H) = {} & 2\,|E(G)|(\overline{M}_1(H) + M_1(H)) + 2\,|E(H)|(\overline{M}_1(G) + M_1(G)) \\ & + M_1(G\oplus H) + \overline{M}_1(G\vee H), \end{aligned}$$

$$\begin{aligned}\overline{M}_2(G\otimes H) =&\, 2M_1(G)(\overline{M}_2(H)+M_2(H))+2M_1(H)(\overline{M}_2(G)M_2(G))\\ &+M_2(G\oplus H)+\overline{M}_2(G\vee H).\end{aligned}$$

Proof. By definition

$$\begin{aligned}\overline{M}_1(G\otimes H) &= \sum\nolimits_{(a,b)(c,d)\notin E(G\otimes H)}[\delta_{G\otimes H}(a,b)+\delta_{G\otimes H}(c,d)]\\
&= \sum\nolimits_{(a,b)(c,d)\notin E(G\otimes H)}[\delta_G a\ \delta_H b+\delta_G c\ \delta_H d]\\
&+\sum\nolimits_{(a,b)(c,d)\notin E(G\otimes H)\ ,\ b=d}[\delta_G a\ \delta_H b+\delta_G c\ \delta_H d]\\
&+\sum\nolimits_{ac\in E(G),bd\notin E(H)}[\delta_G a\,\delta_H b+\delta_G c\ \delta_H d]\\
&+\sum\nolimits_{ac\notin E(G),bd\in E(H)}[\delta_G a\ \delta_H b+\delta_G c\ \delta_H d]\\
&+\sum\nolimits_{ac\notin E(G),bd\notin E(H)}[\delta_G a\ \delta_H b+\delta_G c\ \delta_H d]\\
&= \sum\nolimits_{bd\notin E(H)}\delta_G a\ (\delta_H b+\ \delta_H d)\\
&+\sum\nolimits_{bd\in E(H)}\delta_G a\ (\delta_H b+\ \delta_H d)\\
&+\sum\nolimits_{ac\notin E(G)}\delta_H b(\delta_G a+\ \delta_G c)\\
&+\sum\nolimits_{ac\in E(G)}\delta_H b(\delta_G a+\ \delta_G c)\\
&+M_1(G\oplus H)+\overline{M}_1(G\vee H)\\
&= 2\,|\,E(G)\,|(\overline{M}_1(H)+M_1(H))+2\,|\,E(H)\,|(\overline{M}_1(G)+M_1(G))\\
&+M_1(G\oplus H)+\overline{M}_1(G\vee H).\end{aligned}$$

By similar method the second Zagreb coindex are obtained. □

Theorem 2.3. Let *G* and *H* be graphs. The Randić index of tensor product of *G* and *H* is computed as follows:

$$R(G\otimes H)=2R(G)R(H).$$

Proof. By definition

$$\begin{aligned}R(G\otimes H) &= \sum_{(a,b)(c,d)\in E(G\otimes H)}\frac{1}{\sqrt{\delta_{G\otimes H}(a,b)\delta_{G\otimes H}(c,d)}}\\
&= 2\sum_{ac\in E(G)}\sum_{bd\in E(H)}\frac{1}{\sqrt{\delta_G a\,\delta_G c}}\frac{1}{\sqrt{\delta_H b\,\delta_H d}}\\
&= 2\sum_{ac\in E(G)}\sum_{bd\in E(H)}\frac{1}{\sqrt{\delta_G a\,\delta_G c}}\frac{1}{\sqrt{\delta_H b\,\delta_H d}}\\
&= 2R(G)R(H).\end{aligned}$$

□

Theorem 2.4. Let G and H be graphs and G be k-regular. The GA index of tensor product of G and H is computed as follows:

$$GA(G\otimes H)=2|E(G)|\,GA(H).$$

Proof. By definition

$$\begin{aligned}GA(G\otimes H)&=\sum_{(a,b)(c,d)\in E(G\otimes H)}\frac{\sqrt{\delta_{G\otimes H}(a,b)\,\delta_{G\otimes H}(c,d)}}{\frac{1}{2}(\delta_{G\otimes H}(a,b)+\delta_{G\otimes H}(c,d))}\\&=\sum_{(a,b)(c,d)\in E(G\otimes H)}\frac{\sqrt{\delta_G a\,\delta_H b}\,\sqrt{\delta_G c\,\delta_H d}}{\frac{1}{2}(\delta_G a\,\delta_H b+\delta_G c\,\delta_H d)}\\&=\sum_{(a,b)(c,d)\in E(G\otimes H)}\frac{k\sqrt{\delta_H b\,\delta_H d}}{\frac{1}{2}k(\delta_H b+\delta_H d)}=2|E(G)|\,GA(H).\end{aligned}$$

□

Suppose G is a graph. Define the set $T_G\subseteq E(G)$ as follows:

$$T_G=\{ab\in E(G)\,|\,ab\text{ is contained in a triangle}\}.$$

Theorem 2.5. Let G be a graph and K_n be a complete graph of order n. The Harary index of tensor product of K_n and G is computed as follows:

$$H(K_n\otimes G)=n^2H(G)+\tfrac{1}{2}\binom{n}{2}|V(G)|-\tfrac{2}{3}n|E(G)|+\tfrac{1}{6}n|T_G|.$$

Proof. By definition of Harary index,

$$H(K_n\otimes G)=\sum_{\{(a,b)(c,d)\}\subseteq V(K_n\otimes G)}\frac{1}{d((a,b),(c,d))}.$$

For each $(a,b),(c,d)\in V(K_n\otimes G)$ exactly one of the following cases hold:

$$A_1=\{\{(a,b),(c,d)\}\,|\,a\neq c,b\neq d,\,(a,b)R(c,d)\},$$

$$A_2=\{\{(a,b),(c,d)\}\,|\,a\neq c,b\neq d,\,(a,b)\not R(c,d)\},$$

$$A_3=\{\{(a,b),(c,d)\}\,|\,a\neq c,b=d\},$$

$$A_4=\{\{(a,b),(c,d)\}\,|\,a=c,b\neq d,\,(a,b)R(c,d)\},$$

$$A_5=\{\{(a,b),(c,d)\}\,|\,a=c,\,b\neq d,\,(a,b)\not R(c,d)\}.$$

Therefore,

$$\begin{aligned} H(K_n \otimes G) &= \sum_{\{(a,b)(c,d)\}\in A_1} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \\ &+ \sum_{\{(a,b)(c,d)\}\in A_3} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \\ &+ \sum_{\{(a,b)(c,d)\}\in A_5} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \end{aligned}$$

We evaluate each sums separately. It is obvious, if $\{a,c\} \subseteq V(K_n)$, then $d_{K_n}(a,c) = 1$ and $d'_{K_n}(a,c) = 2$. By using notation of Definitions 1.1 and 1.2, one can see that, if $(a,b)\not R(c,d)$ and $a \neq c, b \neq d$ then,

$$Min\{\ Max\{d_{K_n}(a,c), d'_G(b,d)\},\ Max\{d'_{K_n}(a,c), d_G(b,d)\}\} = d_G(b,d),$$

$$Max\ \{d_{K_n}(a,c), d_G(b,d)\} = d_G(b,d).$$

Hence

$$\begin{aligned} \sum_{\{(a,b)(c,d)\}\in A_1\cup A_2} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} &= \sum_{\{(a,b)(c,d)\}\in A_1} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \\ &+ \sum_{\{(a,b)(c,d)\}\in A_2} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \\ &+ \sum_{\{(a,b)(c,d)\}\in A_2} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} \\ &= \sum_{\{a,c\}\subseteq V(K_n)} \sum_{\{b,d\}\subseteq V(G)} \frac{1}{d_G(b,d)} \\ &= 2\binom{n}{2} H(G) \end{aligned}$$

By attention to the set A_3, we have:

$$\sum_{\{(a,b)(c,d)\}\in A_3} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} = \sum_{\substack{\{a,c\}\subseteq V(K_n) \\ b\in V(G)}} \frac{1}{2} = \tfrac{1}{2}|V(G)|\binom{n}{2}.$$

For computing the 4–th summation, we know that,

$$A_4 = \{(a,b)(a,d) \mid a \in V(K_n), \{b,d\} \subseteq V(G)\ and\ 2 \mid d_G(b,d)\}.$$

Hence,

$$\sum_{\{(a,b)(c,d)\}\in A_4} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))}$$

$$= \sum_{\substack{a\in V(K_n)\\ \{b,d\}\subseteq V(G)\\ 2|d_G(b,d)}} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))}$$

$$= \sum_{\substack{a\in V(K_n)\\ \{b,d\}\subseteq V(G)\\ 2|d_G(b,d)}} \frac{1}{d_G(b,d)}.$$

Now we can compute the 5−th summation,

$$\sum_{\{(a,b)(c,d)\}\in A_5} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))} = \sum_{\substack{a\in V(K_n)\\ \{b,d\}\subseteq V(G)\\ 2\nmid d_G(b,d)}} \frac{1}{d_{K_n\otimes G}((a,b),(c,d))}$$

If $d_G(b,d)$ is odd then by Theorem 1.3,

$$d_{K_n\otimes G}((a,b),(a,d)) = Min\{\ Max\{d_{K_n}(a,a), d'_G(b,d)\},\ Max\{d'_{K_n}(a,a), d_G(b,d)\}\}$$
$$= Min\{Max\ \{3, d_G(b,d)\}, d'_G(b,d)\}$$

By attention to different cases for $d_G(b,d)$ and $d'_G(b,d)$, we can see:

$$Min\{Max\ \{3, d_G(b,d)\}, d'_G(b,d)\} = \begin{cases} d_G(b,d) & d_G(b,d)\geq 3 \\ 2 & d_G(b,d)\ \&\ d'_G(b,d)=2 \\ 3 & d_G(b,d)=1\ \&\ d'_G(b,d)\geq 4 \end{cases}$$

Hence, the following sets are defined:

$A'_5 = \{\{(a,b),(a,d)\} \mid a\in V(K_n),\ d_G(b,d)\geq 3 \text{ and } d_G(a,b) \text{ is odd}\}$,

$A''_5 = \{\{(a,b),(a,d)\} \mid a\in V(K_n),\ d_G(b,d)=1 \text{ and } d'_G(a,b)=2\}$,

$A'''_5 = \{\{(a,b),(a,d)\} \mid a\in V(G),\ d_G(b,d)=1 \text{ and } d'_G(a,b)\geq 4\}$.

Thus,

$$\begin{aligned}\sum_{\{(a,b)(c,d)\}\in A_5}\frac{1}{d_{K_n\otimes G}((a,b),(c,d))} &= \sum_{\{(a,b)(c,d)\}\in A_5}\frac{1}{d_{K_n\otimes G}((a,b),(c,d))}\\ &+\sum_{\{(a,b)(c,d)\}\in A_5}\frac{1}{d_{K_n\otimes G}((a,b),(c,d))}\\ &+\sum_{\{(a,b)(c,d)\}\in A_5}\frac{1}{d_{K_n\otimes G}((a,b),(c,d))}\\ &+\sum_{\{(a,b)(c,d)\}\in A_5}\frac{1}{d_{K_n\otimes G}((a,b),(c,d))}\\ &=\sum_{\{(a,b),(a,d)\}\in A_5'}\frac{1}{d_G(b,d)}\\ &+\sum_{\{(a,b),(a,d)\}\in A_5''}\frac{1}{2}+\sum_{\{(a,b),(a,d)\}\in A_5'''}\frac{1}{3}\\ &=n\left(\sum_{2\nmid d_G(b,d)}\frac{1}{d_G(b,d)}-|E(G)|\right)+\tfrac{1}{2}n\,|T_G|\\ &+\tfrac{1}{3}n(|E(G)|-|T_G|).\end{aligned}$$

By above calculations,

$$H(K_n\otimes G)=n^2H(G)+\tfrac{1}{2}\binom{n}{2}|V(G)|-\tfrac{2}{3}n|E(G)|+\tfrac{1}{6}n|T_G|.$$

Theorem 2.6. Let G be a graph and K_n be a complete graph of order n. The Schultz and modified Schultz indices of tensor product of K_n and G are given by:

$$W_+(K_n\otimes G)=\binom{n}{2}[2nW_+(G)+8n(n-1)|E(G)|+2M_1(G)+2\textstyle\sum_{e=bd\notin T_G}(\delta_G b+\delta_G d)],$$

$$W_*(K_n\otimes G)=(n-1)^2[n^2W_*(G)+4n(n-1)M_1(G)+(2n-1)M_2(G)+n\textstyle\sum_{e=bd\notin T_G}d\delta_G b\,\delta_G d]..$$

Proof. We just prove the Schultz index of $K_n\otimes G$, modified Schultz index is obtained similarly. By using the proof of Theorem 2.5 and definition of Schultz index, we have:

$$\begin{aligned}W_+(K_n\otimes G)&=\sum_{\{(a,b),(c,d)\}\subseteq V(K_n\otimes G)}[\delta_{K_n\otimes G}(a,b)+\delta_{K_n\otimes G}(c,d)]d_{K_n\otimes G}((a,b),(c,d))\\ &=\sum_{i-1}^{5}\sum_{\{(a,b),(c,d)\}\in A_i}[\delta_{K_n\otimes G}(a,b)+\delta_{K_n\otimes G}(c,d)]d_{K_n\otimes G}((a,b),(c,d))\end{aligned}$$

$$=(n-1)\sum_{i-1}^{5}\sum_{\{(a,b),(c,d)\}\in A_i}[\delta_G b+\delta_G d]\,d_{K_n\otimes G}((a,b),(c,d))$$

$$=2(n-1)\binom{n}{2}W_+(G)+8(n-1)\binom{n}{2}|E(G)|$$

$$+\,n(n-1)\sum_{\substack{\{b,d\}\subseteq V(G)\\ 2|d_G(b,d)}}d_G(b,d)[\delta_G b+\delta_G d]$$

$$+\,n(n-1)\sum_{\substack{\{b,d\}\subseteq V(G)\\ 2\nmid d_G(b,d)}}d_G\ (b,d)[\delta_G b+\delta_G d]$$

$$-\,n(n-1)\sum_{\substack{\{b,d\}\subseteq V(G)\\ e=bd\in E(G)}}d_G(b,d)[\delta_G b+\delta_G d]$$

$$+\,2n(n-1)\sum_{\substack{\{b,d\}\subseteq V(G)\\ e=bd\in T_G}}[\delta_G b+\delta_G d]$$

$$+\,3n(n-1)\sum_{\substack{\{b,d\}\subseteq V(G)\\ e=bd\notin T_G}}[\delta_G b+\delta_G d].$$

By above calculations, we conclude that:

$$W_+(K_n\otimes G)=\binom{n}{2}[2nW_+(G)+8(n-1)|E(G)|+2M_1(G)+2\textstyle\sum_{e=bd\notin T_G}(\delta_G b+\delta_G d)].$$

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