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Optimal Control of Switched Systems by a Modified Pseudo Spectral Method

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ABSTRACT

In the present paper, we develop a modified pseudo spectral scheme for solving an optimal control problem which is governed by a switched dynamical system. Many real–world processes such as chemical processes, automotive systems and manufacturing processes can be modeled as such systems. For this purpose, we replace the problem with an alternative optimal control problem in which the switching times appear as unknown parameters. Using the Legendre–Gauss–Lobatto quadrature and the corresponding differentiation matrix, the alternative problem is discretized to a nonlinear programming problem. At last, we examine three examples in order to illustrate the efficiency of the proposed method.

1. INTRODUCTION

It is well known that pseudospectral (PS) methods are powerful methods for the numerical solution of differential equations. In fact, they arose from spectral methods which were traditionally used to solve fluid dynamics problems [1, 2]. They can often achieve ten digits of accuracy where a finite difference scheme or a finite element method would get two or three [3]. The key point in PS methods is that they avoid the poor behavior of the classical polynomial interpolation methods by removing the restriction to equally spaced interpolation points.

The variational method of optimal control theory, which typically consists of the calculus of variations and Pontryagin's methods, can be used to derive a set of necessary conditions that must be satisfied by an optimal control law and its associated state–control equations [4, 5]. These necessary conditions of optimality lead to a generally nonlinear

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two–point boundary value problem that must be solved to determine the explicit expression for the optimal control. Except in some special cases, the solution of this two–point boundary value problem is difficult and not practical to obtain.

Various alternative computational techniques for optimal control problems have been developed in the literature. The techniques are basically of three types: parameterization on both state and control [6, 7, 8], parameterization on control only [9, 10] and nonparameterization [11, 12, 13]. As a technique of the first type, PS methods can be interpreted as direct transcription methods for discretizing a continuous optimal control problem into a nonlinear programming (NLP) problem [14, 15, 16, 17, 18, 19]. The resulting NLP problem can be solved numerically by the well developed algorithms [20, 21].

Although PS methods enjoy many nice properties, but their use in solving problems with nonsmooth solutions or problems with switches may cause major difficulties. The reason lies in the famous Gibbs phenomenon which happens when a nonsmooth function is approximated by means of a finite number of smooth functions [2]. In [22], the authors developed the method of PS knotting in order to address this issue. In fact, they introduced the concepts of hard and soft knots to eliminate the mentioned difficulties.

The switched systems are a particular class of hybrid systems. The hybrid systems arise in varied contexts in chemical processes, automotive engine control, traffic control, and manufacturing processes, etc. The abundance of hybrid phenomena in many engineering systems in general, and in the chemical process industries in particular has fostered a large and growing body of research work in this area [23, 24, 25, 26, 27, 28, 29, 30]. In [31], the authors discussed important hybrid aspects of chemical processing plants. Recently, optimal control of switched systems arising in fermentation processes has been studied in [32]. A hybrid system consists of several subsystems and a switching law, where the switching law is determined by a switching sequence and a set of switching times. At each time instant, only one subsystem is active. A hybrid system can be described by a differential inclusion of the form

$$x(t) \in \left\{ f_v(t, x(t), u(t)) : \quad v \in \{1,2,\ldots,M\} \right\}, \tag{1}$$

where $t \in [t_0, t_f]$, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ and for each $v \in \{1,2,\ldots,M\}$, $f_v : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is continuously differentiable with respect to its arguments. A switching law σ for system (1) is defined as $\sigma = ((t_0, i_0), (t_1, i_1), \ldots, (t_{K-1}, i_{K-1}))$, where $1 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_{K-1} \le t_f$, and $i_k \in \{1,2,\ldots,M\}$ for $k = 0,1,\ldots,K-1$. Note here $t_1,\ldots,t_{K-1}$ are the switching instants. An optimal control of such a system involves finding a control $u(t)$, and a switching law σ such that the corresponding state trajectory subject to the dynamical system (1) departs from a given initial state and minimize a given cost functional. In [33], a method which is based on parameterization of the switching instants is proposed for this kind of optimal control problems.

In this paper, we investigate a modified Legendre PS scheme in order to explore accurate and efficient solutions of optimal control problems for switched systems. Here, we consider the optimal control problems in which a prespecified sequence of active subsystems is given. In order to explore numerical solutions of such problems, we need to seek the solutions of both the optimal switching instants and the optimal piecewise input. The rest of this paper is organized as follows. The problem statement is given in Section 2. In Section 3, we describe the preliminaries for subsequent development. The present method is proposed in Section 4. Then, three examples are provided in Section 5 to illustrate the efficiency of the proposed method. Conclusions are presented in Section 6.

2. PROBLEM STATEMENT

We consider switched systems defined on the fixed time interval $[t_0, t_f]$ with $K-1$ switches, consisting of the subsystems

$$x(t) = f_k(x(t), u(t)), \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, K, \tag{2}$$

with initial conditions

$$x(t_0) = x_0, \tag{3}$$

where $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ is the state function and $u(t) = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m$ is the corresponding control function. Also, $f_k: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, 2, \ldots, K$, are given functions. We assume that the switching sequence is preassigned, such that

$$t_0 \le t_1 \le \cdots \le t_{K-1} \le t_K = t_f, \tag{4}$$

where the switching times $t_1, \ldots, t_{K-1}$ are decision variables. Our objective is to find a piecewise continuous function $u(t)$ and switching instants $t_1, \ldots, t_{K-1}$ subject to the condition (4) for the switched system (2) and (3) such that the cost functional

$$J = \phi(x(t_f)) + \int_{t_0}^{t_f} g(x(t), u(t)) dt \tag{5}$$

is minimized. It is noted that the considered problem is an optimal control problem in Bolza form. Also, the vector functions $f_k: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, 2, \ldots, K$, and the scalar functions $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\phi: \mathbb{R}^n \to \mathbb{R}$, are assumed to be smooth with respect to all their arguments.

3. PRELIMINARIES

Let $\tau_0 < \tau_1 < \cdots < \tau_N$ be the Legendre–Gauss–Lobatto (LGL) nodes where $\tau_0 = -1$, $\tau_N = 1$ and $\tau_1, \ldots, \tau_{N-1}$ are the roots of $\dot{P}_N(\tau)$. Here $\dot{P}_N(\tau)$ is the derivative of the N–th order Legendre polynomial $P_N(\tau)$. In other words, the LGL points $\tau_0, \tau_1, \ldots, \tau_N$ are the $N+1$ roots of $(1-\tau^2)\dot{P}_N(\tau)$. The reader is referred to [1, 34] for details.

Let $h(t)$ be a continuous real function which is defined on $[-1,1]$. The Lagrange interpolating polynomial of degree N interpolates the function $h(t)$ at the points $\tau_0, \tau_1, \ldots, \tau_N$, as

$$h(\tau) = \sum_{j=0}^{N} h(\tau_j) L_j(\tau). \tag{6}$$

Here for $j = 0,1,\ldots,N$, $L_j(\tau)$ denotes the Lagrange polynomial of degree N corresponding to the point τ_j, defined by

$$L_j(\tau) = \prod_{i=0, i\neq j}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}.$$

Note that the Lagrange polynomials satisfy in the Kronecker property

$$L_j(\tau_i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}.$$

In order to approximate the derivative of $h(t)$ at the points τ_i, $i = 0,1,\ldots,N$, the interpolation formula (6) is differentiated yielding

$$\dot{h}(\tau_i) = \sum_{j=0}^{N} d_{ij} h(\tau_j), \tag{7}$$

where $d_{ij} = \dot{L}_j(\tau_i)$. The $(N+1)\times(N+1)$ matrix $D = [d_{ij}]$ is the so–called derivative matrix. According to [1]

$$d_{ij} = \begin{cases} \dfrac{P_N(\tau_i)}{P_N(\tau_j)} \cdot \dfrac{1}{\tau_i - \tau_j}, & i \neq j \\ \dfrac{-N(N+1)}{4}, & i = j = 0 \\ \dfrac{N(N+1)}{4}, & i = j = N \\ 0, & otherwise \end{cases}.$$

Furthermore, for approximating the definite integral of $h(t)$ on $[-1,1]$, the LGL quadrature rule is used. According to this quadrature rule, the definite integral is replaced by a summation, in which the values of $h(t)$ at the LGL points are utilized, as

$$\int_{-1}^{1} h(\tau)d\tau = \sum_{j=0}^{N} w_j h(\tau_j), \tag{8}$$

where w_j, $j = 0,1,\ldots,N$, are the LGL weights, corresponding to the LGL points τ_j, $j = 0,1,\ldots,N$, given by

$$w_j = \frac{2}{N(N+1)} \cdot \frac{1}{\left[P_N(\tau_j)\right]^2}, \qquad j = 0,1,\ldots,N.$$

4. PROPOSED METHOD

We suppose that in the problem stated in the Eqs. (2)–(5), the switching sequence is preassigned and $t_1,\ldots,t_{K-1}$ are the corresponding unknown switching times for which the condition (4) holds.

We denote the restriction of vector functions $x(t)$ and $u(t)$ to the k–th subinterval $[t_{k-1},t_k)$ by $x^k(t)$ and $u^k(t)$, respectively. According to these notations, the dynamic subsystems in Eq. (2) are expressed as

$$x^k(t) = f_k(x^k(t),u^k(t)), \quad t_{k-1} \le t < t_k, \quad k = 1,\ldots,K, \tag{9}$$

$$x^k(t_{k-1}) = \lim_{t \to t_{k-1}^-} x^{k-1}(t), \qquad k = 2,\ldots,K. \tag{10}$$

Note that in Eq. (10), the continuity constraints are added in order to guarantee the continuity of state functions. Accordingly, the cost functional (5) reformulated as

$$J = \phi(x^K(t_f)) + \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} g(x^k(t),u^k(t))dt, \tag{11}$$

and the initial conditions (3) restated as

$$x^1(t_0) = x_0. \tag{12}$$

To apply the approximations described in the previous section, we must transfer each subinterval to the interval $[-1,1]$. For this purpose, we use the transformation formula $\tau = \dfrac{2t-(t_{k-1}+t_k)}{t_k - t_{k-1}}$ in the k–th subinterval $[t_{k-1},t_k)$. In this respect, the problem is restated in the following alternative form:

$$\min J = \phi(x^K(1)) + \sum_{k=1}^{K} \left(\frac{t_k - t_{k-1}}{2}\right) \int_{-1}^{1} g(x^k(\tau),u^k(\tau))d\tau \tag{13}$$

$$s.t. x^k(\tau) = \left(\frac{t_k - t_{k-1}}{2}\right) f_k(x^k(\tau),u^k(\tau)), \qquad k = 1,\ldots,K, \tag{14}$$

$$x^k(-1) = x^{k-1}(1), \qquad k = 2,\ldots,K, \tag{15}$$

$$x^1(-1) = x_0. \tag{16}$$

The alternative problem (13)–(16) provides us with some advantage, namely that it no longer has varying switching instants. In fact, the switching instants are considered as parameters in the alternative problem.

It has to be noted that for $k = 1,\dots,K$, the components of vector functions $x^k(\tau)$ and $u^k(\tau)$ are smooth on $[-1,1]$ and then can be expanded in terms of Lagrange basis functions according to Eq. (6). Therefore, using the formula (8), the performance index J in Eq. (13) is approximated as

$$J = \phi(X_N^{(K)}) + \sum_{k=1}^{K}\left(\frac{t_k - t_{k-1}}{2}\right)\sum_{j=0}^{N} g(X_j^{(k)}, U_j^{(k)}) w_j, \tag{17}$$

where $X_j^{(k)}$ and $U_j^{(k)}$ are vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and defined by

$$X_j^{(k)} = x^k(\tau_j), U_j^{(k)} = u^k(\tau_j), \quad j = 0,1,\dots,N, \quad k = 1,\dots,K.$$

Also, using the formula (7), the alternative dynamical systems (14) are approximated by

$$DX^{(k)} - \left(\frac{t_k - t_{k-1}}{2}\right)F^{(k)} = 0, \quad k = 1,\dots,K, \tag{18}$$

where $X^{(k)}$ and $F^{(k)}$ are $(N+1)\times n$ matrices, respectively, defined by

$$X^{(k)} = \begin{bmatrix} X_0^{(k)} \\ X_1^{(k)} \\ \vdots \\ X_N^{(k)} \end{bmatrix}, \quad F^{(k)} = \begin{bmatrix} f_k(X_0^{(k)}, U_0^{(k)}) \\ f_k(X_1^{(k)}, U_1^{(k)}) \\ \vdots \\ f_k(X_N^{(k)}, U_N^{(k)}) \end{bmatrix}.$$

Furthermore, the continuity constraints (15) and the initial conditions (16), respectively, are stated as

$$X_0^{(k)} - X_N^{(k-1)} = 0, \quad k = 2,\dots,K, \tag{19}$$

and

$$X_0^{(1)} = x_0. \tag{20}$$

We also assume that no two endpoints of subintervals coincide. Then, for a small given $\varepsilon > 0$, we add the extra constraints

$$t_k - t_{k-1} > \varepsilon, \quad k = 1,\dots,K. \tag{21}$$

In summary, the alternative optimal control (13)–(16) is discretized to the following NLP problem: Find vectors $X_j^{(k)}$, $U_j^{(k)}$, $j = 0,1,\dots,N$, $k = 1,\dots,K$ and the parameters t_k, $k = 1,\dots,K-1$ to minimize the expression (17) subject to the constraints (18)–(21).

The relations between the solutions of obtained NLP problem and the solutions of alternative problem (13)–(16) are given by

$$x^{(k)}(\tau)=\sum_{j=0}^{N}X_j^{(k)}L_j(\tau),\qquad k=1,\dots,K,$$

and

$$u^{(k)}(\tau)=\sum_{j=0}^{N}U_j^{(k)}L_j(\tau),\qquad k=1,\dots,K.$$

5. ILLUSTRATIVE EXAMPLES

In this section, we consider three examples to illustrate the efficiency of proposed method. Here, we consider the numerical examples given in [33]. According to the present method, each example in modeled using the mathematical software package Maple 17 and the resulting NLP problems are solved by the command NLPSolve.

Example 1. Consider a switched system consisting of nonlinear subsystems

$$subsystem1:\ \begin{cases}\dot{x}_1(t) &= x_1(t)+u(t)\sin x_1(t)\\ \dot{x}_2(t) &= -x_2(t)-u(t)\cos x_2(t)\end{cases},$$
$$subsystem2:\ \begin{cases}\dot{x}_1(t) &= x_2(t)+u(t)\sin x_2(t)\\ \dot{x}_2(t) &= -x_1(t)-u(t)\cos x_1(t)\end{cases},$$
$$subsystem3:\ \begin{cases}\dot{x}_1(t) &= -x_1(t)-u(t)\sin x_1(t)\\ \dot{x}_2(t) &= x_2(t)+u(t)\cos x_2(t)\end{cases}.$$

Assume that $t_0=0$, $t_f=3$ and the system switches at $t=t_1$ from subsystem 1 to 2 and at $t=t_2$ from subsystem 2 to 3 ($0\le t_1\le t_2\le 3$). The initial conditions are $x_1(0)=2$ and $x_2(0)=3$. We want to find optimal switching instants t_1, t_2 and an optimal input $u(t)$ such that the cost functional

$$J=\frac{1}{2}(x_1(3)-1)^2+\frac{1}{2}(x_2(3)+1)^2+\frac{1}{2}\int_0^3[(x_1(t)-1)^2+(x_2(t)+1)^2+u^2(t)]dt$$

is minimized.

In Table 1, we listed the results of optimal switching instants t_1, t_2 and optimal cost J obtained by the present method with $K=3$ and different values of N. In the last of Table 1, we reported the CPU time (seconds) for the computations of the corresponding results. Also, in Figure 1, we plot the graphs of optimal control and the corresponding state trajectory obtained by the present method with $K=3$ and $N=9$.column

Table 1. The results of optimal switching instants t_1, t_2 and optimal cost J obtained by the present method with $K = 3$ and different values of N, for Example 1.

	t_1	t_2	J	CPU time (seconds)
$K = 3$ $N = 6$	0.22451889	1.01940266	5.44119735	4.04
$N = 8$	0.22452199	1.02006802	5.44100709	5.03
$N = 10$	0.22451866	1.02002342	5.44097522	6.20
$N = 12$	0.22451838	1.02002491	5.44097350	8.15
$N = 14$	0.22451835	1.02002485	5.44097350	9.96

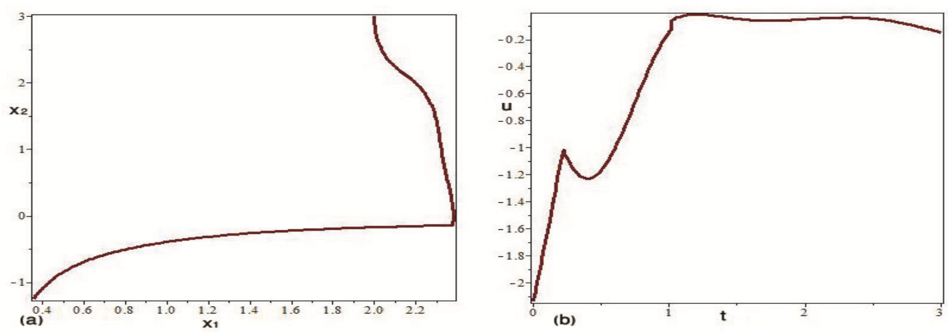

Figure 1: The graphs of (a) state trajectory and (b) optimal control obtained by the present method with $K = 3$ and $N = 9$, for Example 1.

Example 2. Consider a switched system consisting

$$subsystem 1: \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

$$subsystem 2: \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u(t).$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ $(0 \le t_1 \le 2)$ from subsystem 1 to 2. The initial conditions are $x_1(0) = 0$ and $x_2(0) = 2$. We want to find an optimal switching instant t_1 and an optimal input $u(t)$ such that the cost functional

$$J = \frac{1}{2}(x_1(2) - 4)^2 + \frac{1}{2}(x_2(2) - 2)^2 + \frac{1}{2}\int_0^2 [(x_2(t) - 2)^2 + u^2(t)]dt$$

is minimized.

We applied the proposed method to solve this example. In Table 2, we reported the results of t_1 and J obtained by the present method with $K = 2$ and different values of N. Also in Figure 2, we plot the graphs of optimal control and the corresponding state trajectory with $K = 2$ and $N = 9$.

Table 2. The results of optimal switching instant t_1 and optimal cost J obtained by the present method with $K = 2$ and different values of N, for Example 2.

	t_1	J	CPU time (seconds)
$K = 2$ $N = 6$	0.19007133	9.78402619	2.62
$N = 8$	0.18967215	9.76657993	3.04
$N = 10$	0.18967109	9.76654884	3.46
$N = 12$	0.18967110	9.76654882	4.42
$N = 14$	0.18967107	9.76654882	5.60

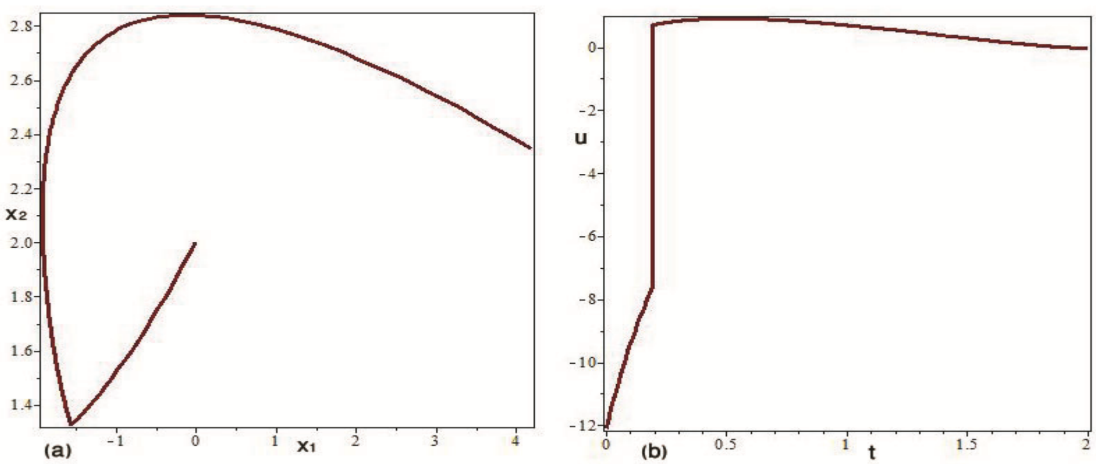

Figure 2: The graphs of (a) state trajectory and (b) optimal control obtained by the present method with $K = 2$ and $N = 9$, for Example 2.

Example 3. Consider a switched system with internally forced switching only consisting of

$$subsystem\,1: \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$
$$subsystem\,2: \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

Assume that $t_0 = 0$, $t_f = 2$ and the system state starts at $x_1(0) = 1$, $x_2(0) = 1$, following subsystem 1 (subsystem 1 is active for $c(x_1(t), x_2(t)) = x_1(t) + x_2(t) - 7 \le 0$ and subsystem 2 is active for $c(x_1(t), x_2(t)) \ge 0$). Assume that upon intersecting the hyper surface $c(x_1, x_2) = 0$, the system switches from subsystem 1 to 2. Also, assume there is only one switching which takes place at time t_1 ($0 \le t_1 \le 2$). We want to find an optimal input $u(t)$ such that the cost functional

$$J = \frac{1}{2}(x_1(2) - 10)^2 + \frac{1}{2}(x_2(2) - 6)^2 + \frac{1}{2}\int_0^2 u^2(t)dt$$

is minimized.

Note that we have not considered state constraints in the subsystems of our problem modeled by Eqs. (2)–(5). For this reason, we state our technique in order to approximate state constraints. By setting $K = 2$, according to the proposed method, we have two sets of state functions values: $X_j^{(1)}$, $j = 0,1,\ldots,N$, are the values of state functions in subsystem 1, and $X_j^{(2)}$, $j = 0,1,\ldots,N$, are the values of state functions in subsystem 2. According to this, we obtain the constraints $c(X_j^{(1)}) \le 0$, $j = 0,1,\ldots,N$, in subsystem 1, and $-c(X_j^{(2)}) \le 0$, $j = 0,1,\ldots,N$, in subsystem 2. These new inequality constraints must be added to Eqs. (18)–(21).

In Table 3, we listed the results of t_1 and J obtained by the present method with $K = 2$ and different values of N. Also in Figure 3, we plot the graphs of optimal control and the corresponding state trajectory with $K = 2$ and $N = 9$.

Table 3. The results of optimal switching instant t_1 and optimal cost J obtained by the present method with $K = 2$ and different values of N, for Example 3.

	t_1	J	CPU time (seconds)
$K = 2$ $N = 6$	1.16328653	0.11315919	5.78
$N = 8$	1.16293205	0.11309541	6.00
$N = 10$	1.16278027	0.11306590	6.46
$N = 12$	1.16270441	0.11305120	6.84
$N = 14$	1.16266144	0.11304283	7.42

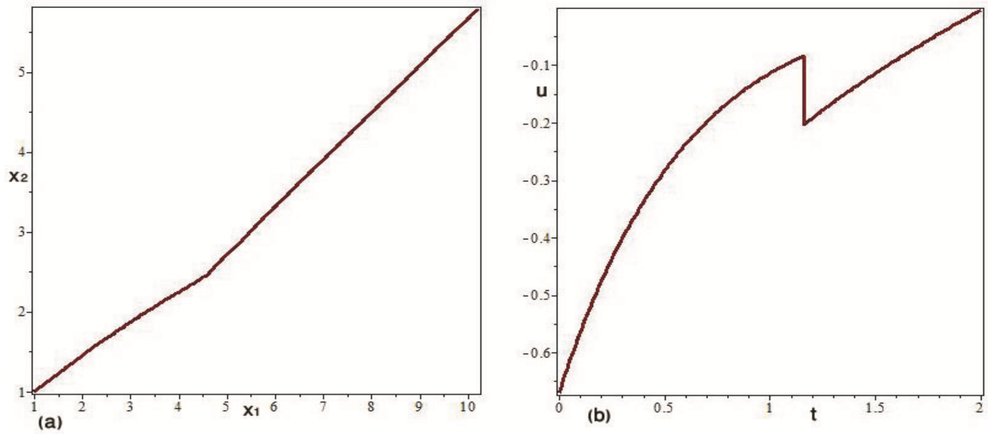

Figure 3: The graphs of (a) state trajectory and (b) optimal control obtained by the present method with $K = 2$ and $N = 9$, for Example 3.

6. CONCLUSION

In this paper, we have considered a class of optimal control problems governed by switched systems. Such systems arise in varied contexts in chemical processes, automotive engine control, traffic control, and manufacturing processes, etc. We have proposed a modified Legendre pseudospectral scheme in order to explore accurate solutions. For this purpose, we have restated the problem in form of an alternative problem in which the switching instants are considered as parameters. Then, we can solve the obtained NLP problem using existing subroutines. Three numerical examples considered in order to show the validity and applicability of the proposed method.

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