

## ***The Uniqueness Theorem for Inverse Nodal Problems with a Chemical Potential***

SEYFOLLAH MOSAZADEH\*

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317–53153, I R Iran

---

### ARTICLE INFO

---

#### Article History:

Received 3 February 2016

Accepted 16 April 2016

Published online 5 November 2016

Academic Editor: Ali Reza Ashrafi

---

#### Keywords:

Boundary Value Problem

Inverse Nodal Problem

Eigenvalues

Nodal Points

---



---

### ABSTRACT

---

In this paper, an inverse nodal problem for a second-order differential equation having a chemical potential on a finite interval is investigated. First, we estimate the nodal points and nodal lengths of differential operator. Then, we show that the potential can be uniquely determined by a dense set of nodes of the eigenfunctions.

© 2017 University of Kashan Press. All rights reserved

---

## 1 INTRODUCTION

There are many problems in mathematics, chemistry, physics and some engineering sciences which are connected to the second-order differential equations. For example, in the process of the formation of *methyliodide* ( $CH_3I$ ) by the biological and photochemical production mechanisms in a biogeochemical module, the following equation appears:

$$\frac{dc}{dt} = P - S + F_{air-Sea} + \frac{\partial}{\partial z} \left( A_v \frac{dc}{dz} \right), \quad (1)$$

which describes the evolution of methyl iodide concentration ( $c$  [ $\text{mmolm}^{-3}$ ]) over time under production ( $P$ ), degradation ( $S$ ), air–sea exchange ( $F$ ), as well as turbulent vertical diffusion ( $A_v$ –diffusion coefficient) (see [26]). Using the separation of variables technique

---

\* Corresponding Author (Email address: s.mosazadeh@kashanu.ac.ir)  
DOI: 10.22052/ijmc.2016.39228

one equation (1), we can transform the equation (1) to the following second-order differential equation:

$$y'' + \left( \lambda - \frac{A}{x^2} + q(x) \right) y = 0, \quad (2)$$

where  $\lambda$  is the spectral parameter,  $A$  is a real number, the potential  $q(x)$  is real-valued. Equation (2) has a singularity at the endpoint  $x = 0$ . For other examples, in quantum chemistry or quantum mechanics, we refer to the quantum modeling of the *hydrogen atom*, or the *Hellman equation* to finding an approximation for the simplified description of complex systems, which can be transformed to (2) (see also [3, 4, 6, 13, 15, 17, 24]).

Inverse problems associated with the equation (2) with  $A=0$  have various versions. The first version was studied by Borg and Levinson, and it is shown that the potential  $q(x)$  can be uniquely determined from the given boundary condition and one possible reduced spectrum [5, 18]. For the second version, using two spectra  $\lambda_n$  and  $\lambda'_n$ , Marchenko uniquely determined the potential  $q(x)$  and the corresponding boundary conditions [20]. Finally, Gelfand and Levitan proved that  $q(x)$  uniquely determined by the spectral function [12].

Some inverse problems having singularities or turning points, and/or discontinuity conditions were studied by the above methods in many works (see [1, 2, 8-11, 16, 19, 23, 27]). Note that, in [22], we considered a second-order differential equation of Sturm-Liouville type having two turning points and singularities in a finite interval. Then, its asymptotic form of the solutions was studied, and obtained the infinite representation of the solutions of differential equation which plays an important role in investigating the corresponding inverse problem.

In later years, in some interesting works but without singularity, inverse problems were investigated using a new spectral data which are so-called *nodal points*, and their corresponding inverse problems are so-called *inverse nodal problems*. McLaughlin seems to have been the first to consider this method for the one-dimensional Schrödinger equations [21]. For other works, see also [7, 14, 25].

In this work, we consider the inverse nodal problem associated with the singular differential equation (2) and the Dirichlet boundary condition

$$y(0) = 0 = y(1), \quad (3)$$

on the interval  $(0,1)$ . We also assume that

$$q(x) \in x^{2-2k_0} \in L^1(0,1), \quad (4)$$

where  $k_0$  is a member of  $\{2,3,4,\dots\}$ . The problem (2)-(3) has infinitely many nontrivial solutions. The values of  $\lambda$  for which there exist nontrivial solutions are so-called *eigenvalues*, and their corresponding nontrivial solutions  $y(x,\lambda)$  are so-called *eigenfunctions*. All the eigenvalues are real and the set of the eigenvalues is countably infinite, and also the eigenvalues can be arranged in increasing order as follows

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots,$$

such that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . In the present paper, first, we obtain the asymptotic formula for the eigenvalues, the nodes of the eigenfunctions and the nodal lengths (the section 2). Then, we prove that the set of the nodal points of the boundary value problem (2)-(3) is dense in the interval (0,1) and the potential  $q(x)$  can be uniquely determined from this new kind of spectral data (see the Section 3).

## 2 ASYMPTOTIC FORMULA FOR NODAL POINTS

We consider the boundary value problem  $L=L(q(x))$  defined by (2)-(3). Assume that in (2),

$$A = v^2 - \frac{1}{4}, \quad v = k_0 - \frac{1}{2}, \quad k_0 \in \{2,3,4,\dots\}. \tag{5}$$

From [11], we know that the equation (2) has two solutions  $y_1(x,\lambda)$  and  $y_2(x,\lambda)$ , which are linearly independent with respect to  $x$ , and also have the following asymptotic form as  $\lambda \rightarrow \infty$  :

$$y_1(x, \lambda) = \lambda^{(k_0-1)/2} \left\{ (-1)^{k_0-1} e^{i\sqrt{\lambda}x} [1]_0 + e^{-i\sqrt{\lambda}x} [1]_0 \right\}, \tag{6}$$

$$y_2(x, \lambda) = \frac{1}{4} i \lambda^{-k_0/2} \left\{ -e^{i\sqrt{\lambda}x} [1]_0 + (-1)^{k_0-1} e^{-i\sqrt{\lambda}x} [1]_0 \right\}, \tag{7}$$

where  $[1]_0 = 1 + O((\sqrt{\lambda}x)^{-1})$ . Therefore, the solution  $y(x,\lambda)$  of the equation (2) under the condition  $y(0)=0$  can be written as a linear combination of  $y_1$  and  $y_2$ . Also, since the boundary value problem  $L$  is self-adjoint and  $y_1, y_2$  are entire in  $\lambda$ , thus all of the eigenvalues of  $L$  are real and simple. In the case when  $k_0$  is odd, it follows from (3), (7) that  $y(x,\lambda) = y_2(x,\lambda)$  and the asymptotic form of the eigenvalues as follows

$$\sqrt{\lambda_n(q)} = n\pi + O\left(\frac{1}{n}\right). \tag{8}$$

Similarly, in the case when  $k_0$  is even, we derive from (3), (6) that  $y(x,\lambda) = y_1(x,\lambda)$  and also the eigenvalues of  $L$  may be calculated as (8).

For the boundary value problem  $L$  an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunctions  $y_n(x) = y(x,\lambda_n)$  has exactly  $n-1$  (simple) zeros inside the interval (0,1), namely:

$$0 < x_n^{(1)} < x_n^{(2)} < \dots < x_n^{(n-1)} < 1.$$

The set

$$X_L := \left\{ x_n^{(j)} \right\}, \quad n \geq 1, \quad j = \overline{1, n-1}, \tag{9}$$

is called the set of nodal points of the problem  $L$ . Also, let

$$I_n^{(j)} := [x_n^{(j)}, x_n^{(j+1)}]$$

be the  $j^{th}$  nodal domain of the  $n^{th}$  eigenfunction  $y_n$ , and let

$$\ell_n^{(j)} := |I_n^{(j)}| = x_n^{(j+1)} - x_n^{(j)}$$

be the associated *nodal length*. Inverse nodal problems consist in recovering the potential  $q(x)$  from the given set  $X_L$  of nodal points or from a certain its part.

Now, in the following theorem, we develop asymptotic expressions for nodal points  $x_n^{(j)}$  and the nodal lengths  $\ell_n^{(j)}$  ( $n=1,2,3,\dots, j=1,2,\dots,n-1$ ) at which  $y_n$ , the eigenfunction corresponding to the eigenvalue  $\lambda_n$  of the problem  $L$ , vanishes.

**Theorem 1.** We consider the equation (2) under Dirichlet boundary condition (3). Let  $q(x)$  satisfies (4), then the nodal points of the problem  $L$  defined by (2)-(3) are

$$\begin{cases} x_n^{(j)} = \frac{j}{n} + O\left(\frac{1}{n}\right), \\ n = 1, 2, 3, \dots, \quad j = 1, 2, 3, \dots, n-1, \end{cases} \quad (10)$$

and the nodal lengths are

$$\ell_n^{(j)} = \frac{1}{n} + O\left(\frac{1}{n}\right).$$

**Proof.** Suppose  $\nu = k_0 - 1/2$  and  $k_0$  is odd. Then, by (7)-(8) and solving  $y_2(x, \lambda_n) = 0$ , we approximate the nodal points of the form (10). Similarly, in the when  $k_0$  is even, using (6), (8) and from  $y_1(x, \lambda_n) = 0$  we arrive at (10). Moreover,

$$\begin{aligned} \ell_n^{(j)} &= x_n^{(j+1)} - x_n^{(j)} \\ &= \left( \frac{j+1}{n} + O\left(\frac{1}{n}\right) \right) - \left( \frac{j}{n} + O\left(\frac{1}{n}\right) \right) \\ &= \frac{1}{n} + O\left(\frac{1}{n}\right). \quad \square \end{aligned}$$

Theorem 1, specially the relation (10), provide the sufficient conditions for the uniqueness theorem in the next section.

### 3 THE UNIQUENESS THEOREM

In this section, we show that the set of the nodal points  $x_n^{(j)}$  of the form (10) is dense in  $(0,1)$ . Then, we prove a uniqueness theorem for the solution of the inverse nodal problem associated with the boundary value problem  $L$ .

First, we consider the equation

$$w''(x, \lambda) + \lambda w(x, \lambda) = 0, \quad 0 \leq x \leq 1, \quad (11)$$

with the boundary conditions

$$w(0, \lambda) = 0 = w(1, \lambda). \quad (12)$$

It is easily shown that the solution of the problem (11)-(12) is  $w(x, \lambda) = \sin(\sqrt{\lambda}x)$ . Furthermore, the exact eigenvalues of the problem  $L_0$  defined by (11)-(12) are

$$\xi_n = n^2\pi^2, \tag{13}$$

and their corresponding eigenfunctions are

$$w_n(x) = w(x, \xi_n) = \sin(n\pi x). \tag{14}$$

Since for each  $n \in \{2,3,4,\dots\}$  there exist  $k \in \{0,1,2,\dots\}$  and  $m \in \{1,2,\dots,2^k\}$  such that  $n=2^{k+1}-m+1$ , so according to (13)-(14), the set

$$\left\{ (2^{k+1} - m + 1)^2\pi^2 \mid k = 0,1,2,\dots, m = 1,2,\dots,2^k \right\},$$

consists of all eigenvalues of (11)-(12) except  $\xi_1=\pi^2$ . Moreover, the eigenfunction corresponding to the eigenvalue  $\xi_n=(2^{k+1}-m+1)\pi^2$  is

$$w(x, \xi_n) = \sin((2^{k+1} - m + 1)\pi x),$$

so that  $m/(2^{k+1}-m+1)$  is a zero of the eigenfunction  $w_n(x)$ . Therefore, the set of the nodal points of  $L_0$  is

$$\begin{aligned} X_{L_0} &:= \left\{ \xi_n^j \right\}_{n \geq j, j=1, \overline{n-1}} \\ &= \left\{ \frac{m}{2^{k+1} - m + 1} \mid k = 0,1,2,\dots, m = 1,2,\dots,2^k \right\} \cup \{0\}. \end{aligned} \tag{15}$$

**Lemma 1.** The set  $X_{L_0}$ , defined by (15), is dense in  $[0,1]$ .

**Proof.** For each fixed  $k \in \{0,1,2,\dots\}$ , we have

$$= \left\{ \frac{m}{2^{k+1} - m + 1} \mid m = 1,2,\dots,2^k \right\} = \left\{ \frac{1}{2^{k+1}}, \frac{2}{2^{k+1} - 1}, \frac{3}{2^{k+1} - 2}, \dots, \frac{2^k}{2^k + 1} \right\}.$$

Moreover,

$$\frac{1}{2^{k+1}} - 0 = \frac{1}{2^{k+1}}, \quad 1 - \frac{2^k}{2^k + 1} = \frac{1}{2^k + 1}, \tag{16}$$

and for  $m=1,2,\dots,2^k-1$ ,

$$\begin{aligned} \bar{\ell}_{m,k} &:= \frac{m+1}{2^{k+1} - (m+1) + 1} - \frac{m}{2^{k+1} - m + 1} \\ &= \frac{2^{k+1} + 1}{(2^{k+1} - m) - (2^{k+1} - m + 1)}. \end{aligned}$$

Hence, there exists a sufficiently large number  $\bar{k}$  such that for each  $k > \bar{k}$  we have

$$\bar{\ell}_{m,k} < \frac{1}{k+1}. \tag{17}$$

Now, let  $\bar{x}_{m,k} := m/(2^{k+1}-m+1)$ . Then, for each  $x \in [0,1]$ , there exists  $m \in \{1,2,\dots, 2^k-1\}$  such that

$$x \in [0, \bar{x}_{1,k}] \vee x \in [\bar{x}_{m,k}, \bar{x}_{m+1,k}] \vee x \in [\bar{x}_{2^k,k}, 1]. \tag{18}$$

On the other hand, the right sides of equations (16) and (17) tend to zero as  $k \rightarrow \infty$ . This together with equation (18) completes the proof.  $\square$

**Theorem 2.** The set of the nodal points of the boundary value problem  $L, X_L$ , is dense in the interval  $(0,1)$ .

**Proof.** It follows from (15) that the nodal points  $\xi_n^{(j)}$  of  $L_0$  have the form

$$\xi_n^{(j)} = \frac{j}{n}, \quad n \geq 2, \quad j = 1, 2, 3, \dots, n-1.$$

Thus, using (10) we obtain

$$x_n^{(j)} = \xi_n^{(j)} + O\left(\frac{1}{n}\right). \tag{19}$$

By (19) and Lemma 1, we conclude that  $X_L$  is dense in  $(0,1)$ .  $\square$

Now, we prove the main result of this section.

**Theorem 3.** Consider the boundary value problems defined by

$$y'' + \left( \lambda - \frac{A}{x^2} + q_i(x) \right) y = 0, \quad i = 1, 2, \quad x \in (0,1), \tag{20}$$

and Dirichlet condition

$$y(0) = 0 = y(1). \tag{21}$$

Let  $q_1, q_2$ , satisfy the condition (4) and  $x_n^{(j)}(q_1) = x_n^{(j)}(q_2)$ . Then  $q_1 = q_2$  (a.e.).

**Proof.** First, we consider the case when  $k_0$  is odd, in (5). Let  $x$  be an arbitrary, fixed number in the interval  $[0,1]$ . Since the set of the nodal points  $X_L$ , defined in (9), is dense in the interval  $(0,1)$  by Theorem 2, it follows that there exists a subsequence  $\{n_k\}$ ,  $k=1,2,3,\dots$ , such that

$$\lim_{k \rightarrow \infty} x_{n_k}^{(j)} = x. \tag{22}$$

Let  $\tilde{y}_i(x) = y_2(x, \lambda_{n_k}(q_i))$  be the solution of (20)-(21) with the potential  $q_i(x)$ . Then, using (20) we derive

$$\frac{d}{dx}(\tilde{y}_2\tilde{y}_1' - \tilde{y}_1\tilde{y}_2')(x) = \left\{ q_1(x) - q_2(x) + \lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) \right\} \tilde{y}_1(x)\tilde{y}_2(x). \quad (23)$$

Integrating (23) from 0 to  $x(j, n_k) = x_{n_k}^{(j)} := x_{n_k}^{(j)}(q_1) = x_{n_k}^{(j)}(q_2)$ , we get

$$(\tilde{y}_2\tilde{y}_1' - \tilde{y}_1\tilde{y}_2')(x)|_0^{x(j, n_k)} = \int_0^{x(j, n_k)} \left\{ q_1(t) - q_2(t) + \lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) \right\} \tilde{y}_1(t)\tilde{y}_2(t)dt. \quad (24)$$

Since  $\tilde{y}_1(x(j, n_k)) = \tilde{y}_2(x(j, n_k)) = 0$ , the left side of (24) is equal to zero for each  $k \in \{1, 2, 3, \dots\}$ . Thus,

$$\int_0^{x(j, n_k)} \left\{ q_1(t) - q_2(t) + \lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) \right\} \tilde{y}_1(t)\tilde{y}_2(t)dt = 0,$$

for  $k = 1, 2, 3, \dots$ . We are done if we can show

$$\int_0^x (q_1(t) - q_2(t))dt = 0.$$

For this goal, by (8) we have

$$\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence, together with (22) and (24) these results imply

$$\lim_{k \rightarrow \infty} n_k^2 \pi^2 \int_0^x (q_1(t) - q_2(t))\tilde{y}_1(t)\tilde{y}_2(t)dt = 0. \quad (25)$$

Moreover, it follows from (7) that there exists a constant  $C$  such that for sufficiently large  $k$ , we have

$$\left| \tilde{y}_1(x)\tilde{y}_2(x) - (n_k\pi)^{-2} \sin^2(n_k\pi x) \right| < C(n_k\pi)^{-3}.$$

So,

$$n_k^2 \pi^2 \tilde{y}_1(x)\tilde{y}_2(x) \approx \sin^2(n_k\pi x), \quad k \rightarrow \infty, \quad (26)$$

Therefore, by (25)–(26) we get

$$\int_0^x (q_1(t) - q_2(t))dt = 0. \quad (27)$$

Finally, since  $x$  was chosen arbitrary in the interval  $[0, 1]$ , together with (27) this yields  $q_1 = q_2$  (a.e.). In the case when  $k_0$  is even, Theorem 3 can be proved similarly, by (6) and the same way as above. □

Theorem 3 shown that the solution of the inverse nodal problem associated with (2)–(3), the potential function  $q(x)$ , can be uniquely determined by a dense set of nodes of the eigenfunctions.

**ACKNOWLEDGMENT** This research is partially supported by the University of Kashan under grant number 464151/1.

## REFERENCES

1. R. Kh. Amirov, On system of Dirac differential equations with discontinuity conditions inside an interval, *Ukrainian Math. J.* **57** (2005) 712–727.
2. R. Amirov and N. Topsakal, On inverse problem for singular Sturm-Liouville operator with discontinuity conditions, *Bull. Iranian Math. Soc.* **40** (2014) 585–607.
3. W. O. Amrein, A. M. Hinz and D. B. Pearson, *Sturm–Liouville Theory: Past and Present*, Birkhäuser Verlag, Basel, 2005.
4. R. P. Bell, *The Tunnel Effect in Chemistry*, Springer-Science, University Press, Cambridge, 1980.
5. G. Borg, Eine umkehrung der Sturm–Liouville sehen eigenwertaufgabe, *Acta Math.* **78** (1945) 1–96.
6. J. P. Boyd, Sturm-Liouville eigenvalue problems with an interior pole, *J. Math. Physics* **22** (1981) 1575–1590.
7. Y. H. Cheng, C. K. Law and J. Tsay, Remarks on a new inverse nodal problem, *J. Math. Anal. Appl.* **248** (2000) 145–155.
8. W. Eberhard, G. Freiling and K. Wilcken-Stoeber, Indefinite eigenvalue problems with several singular points and turning points, *Math. Nachr.* **229** (2001) 51–71.
9. G. Freiling and V. Yurko, On constructing differential equation with singularities from incomplete spectral information, *Inv. Prob.* **14** (1998) 1131–1150.
10. G. Freiling and V. Yurko, On the determination of differential equations with singularities and turning points, *Results Math.* **41** (2002) 275–290.
11. G. Freiling and V. Yurko, Inverse problems for differential operators with singular boundary conditions, *Math. Nachr.* **278** (2005) 1561–1578.
12. I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function, *Amer. Math. Soc. Trans.* **1** (1951) 253–304.
13. D. M. Haaland and R. T. Meyer, Reaction of excited iodine atoms with methyl iodide rate constant determinations, *Int. J. Chem. Kinetics* **6** (1974) 297–308.
14. O. Hald and J. R. McLaughlin, Solutions of inverse nodal problems, *Inv. Prob.* **5** (1989) 307–347.



15. H. Hellmann, A new approximation method in the problem of many electrons, *J. Chem. Phys.* **3** (1935) p. 61.
16. A. Jodayree Akbarfam and A. B. Mingarelli, The canonical product of the solution of the Sturm-Liouville equation in one turning point case, *Canad. Appl. Math. Quart.* **8** (2000) 305–320.
17. K. Jörgens, Spectral theory of second-order ordinary differential operators, *Lecture Notes: Series no.2, Matematisk Institut, Aarhus Universitet*, 1962/63.
18. N. Levinson, The inverse Sturm–Liouville problem, *Mat. Tidsskr. B.* (1949) 25–30.
19. H. R. Marasi and A. Jodeyree Akbarfam, On the canonical solution of indefinite problem with  $m$  turning points of even order, *J. Math. Anal. Appl.* **332** (2007) 1071–1086.
20. V. A. Marchenko, Concerning the theory of a differential operator of the second order, *Dokl. Akad. Nauk. SSSR (N.S.)* **72** (1950) 457–460.
21. J. R. McLaughlin, Inverse spectral theory using nodal points as data—a uniqueness result, *J. Diff. Eqs.* **73** (1988) 354–362.
22. S. Mosazadeh, Infinite product representation of solution of indefinite Sturm-Liouville Problem, *Iranian J. Math. Chem.* **4** (2013) 27–40.
23. S. Mosazadeh, The stability of the solution of an inverse spectral problem with a singularity, *Bull. Iranian Math. Soc.* **41** (2015) 1061–1070.
24. J. K. Shaw, A. P. Baronavski, H. D. Ladouceur and W. O. Amrein, Applications of the Walker method in Spectral Theory and Computational Methods of Sturm-Liouville problems, *Lecture Notes in Pure and Appl. Math.* **191** (1997) 377–395.
25. C. T. Shieh and V. A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, *J. Math. Anal. Appl.* **347** (2008) 266–272.
26. I. Stemmler, M. Rothe, I. Hense and H. Hepach, Numerical modeling of methyl iodide in the eastern tropical Atlantic, *Biogeosciences* **10** (2013) 4211–422.
27. V. Yurko, Recovering singular differential operators on noncompact star-type graphs from Weyl functions, *Tamkang J. Math.* **42** (2011) 223–236.