A Note on Vertex–Edge Wiener Indices of Graphs

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ABSTRACT The vertex-edge Wiener index of a simple connected graph $G$ is defined as the sum of distances between vertices and edges of $G$. Two possible distances $D_1(u,e|G)$ and $D_2(u,e|G)$ between a vertex $u$ and an edge $e$ of $G$ were considered in the literature and according to them, the corresponding vertex-edge Wiener indices $W_{ve1}(G)$ and $W_{ve2}(G)$ were introduced. In this paper, we present exact formulas for computing the vertex-edge Wiener indices of two composite graphs named splice and link.

KEYWORDS Distance in graph • vertex–edge Wiener index • Splice • Link.

1. INTRODUCTION

The graphs considered in this paper are undirected, finite and simple. A topological index (also known as graph invariant) is any function on a graph that does not depend on a labeling of its vertices. The oldest topological index is the one put forward in 1947 by Harold Wiener [1,2] nowadays referred to as the Wiener index. Wiener used his index for the calculation of the boiling points of alkanes. The Wiener index $W(G)$ of a connected graph $G$ is defined as the sum of distances between all pairs of vertices of $G$:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v|G),$$

where $d(u,v|G)$ denotes the distance between the vertices $u$ and $v$ of $G$ which is defined as the length of any shortest path in $G$ connecting them. Details on the mathematical properties of the Wiener index and its applications in chemistry can be found in [1–8].

In analogy with definition of the Wiener index, the vertex-edge Wiener indices are defined based on distance between vertices and edges of a graph [9,10]. Two possible distances between a vertex $u$ and an edge $e=ab$ of a connected graph $G$ can be considered.

The first distance is denoted by $D_1(u,e|G)$ and defined as [9]:

$$D_1(u,e|G) = \min \{d(u,a|G), d(u,b|G)\},$$

and the second one is denoted by $D_2(u,e|G)$ and defined as [10]:

$$D_2(u,e|G) = \max \{d(u,a|G), d(u,b|G)\}.$$
Based on these two distances, two vertex-edge versions of the Wiener index can be introduced. The first and second vertex–edge Wiener indices of \( G \) are denoted by \( W_{ve1}(G) \) and \( W_{ve2}(G) \), respectively, and defined as 
\[
W_{ve1}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_i(u, e|G),
\]
where \( i \in \{1, 2\} \). It should be explained that, the vertex-edge Wiener index introduced in [9] is half of the first vertex-edge Wiener index \( W_{ve1} \). However, in the above summation, for every vertex \( u \) and edge \( e \) of \( G \), the distance \( D_i(u, e|G) \) is taken exactly one time into account, so the summation does not need to be multiplied by a half. The first and second vertex-edge Wiener indices are also known as minimum and maximum indices, and denoted by \( \text{Min}(G) \) and \( \text{Max}(G) \), respectively. Since these indices are considered as the vertex-edge versions of the Wiener index, their present names and notations seem to be more appropriate.

In [10,11], the vertex–edge Wiener indices of some chemical graphs were computed and in [12,13], the behavior of these indices under some graph operations were investigated. In this paper, we present exact formulas for the first and second vertex-edge Wiener indices of two composite graphs named splice and link. Readers interested in more information on computing topological indices of splice and link of graphs, can be referred to [12,14–20].

2. RESULTS AND DISCUSSION

In this section, we compute the first and second vertex–edge Wiener indices of splice and link of graphs. We start by introducing some notations.

Let \( G \) be a connected graph. For \( u \in V(G) \), we define:
\[
d(u|G) = \sum_{v \in V(G)} d(u, v|G),
\]
\[
D_i(u|G) = \sum_{e \in E(G)} D_i(u, e|G), \quad i \in \{1, 2\}.
\]

With the above definitions,
\[
W(G) = \frac{1}{2} \sum_{u \in V(G)} d(u|G),
\]
\[
W_{ve_i}(G) = \sum_{u \in V(G)} D_i(u|G), \quad i \in \{1, 2\}.
\]

2.1 SPlice

Let \( G_1 \) and \( G_2 \) be two connected graphs with disjoint vertex sets \( V(G_1) \) and \( V(G_2) \) and edge sets \( E(G_1) \) and \( E(G_2) \), respectively. For given vertices \( a_1 \in V(G_1) \) and \( a_2 \in V(G_2) \), a splice [17] of \( G_1 \) and \( G_2 \) by vertices \( a_1 \) and \( a_2 \) is denoted by \( (G_1, G_2)(a_1, a_2) \) and defined by identifying the vertices \( a_1 \) and \( a_2 \) in the union of \( G_1 \) and \( G_2 \). We denote by \( n_1 \) and \( m_1 \) the order and size of the graph \( G_1 \), respectively. It is easy to see that, 
\[
|V((G_1, G_2)(a_1, a_2))| = n_1 + n_2 - 1 \text{ and } |E((G_1, G_2)(a_1, a_2))| = m_1 + m_2.
\]
In the following lemma, the distance between two arbitrary vertices of 
\( (G_1,G_2)(a_1,a_2) \) is computed. The result follows easily from the definition of the splice of graphs, so its proof is omitted.

**Lemma 2.1** Let \( u,v \in V((G_1,G_2)(a_1,a_2)) \). Then

\[
d(u,v|(G_1,G_2)(a_1,a_2)) = \begin{cases} 
    d(u,v|G_i) & u,v \in V(G_i) \\
    d(u,v|G_j) & u,v \in V(G_j) \\
    d(u,a_i|G_1) + d(a_2,v|G_2) & u \in V(G_1), v \in V(G_2)
  \end{cases}
\]

In the following lemma, the distances \( D_1 \) and \( D_2 \) between vertices and edges of 
\( (G_1,G_2)(a_1,a_2) \) are computed.

**Lemma 2.2** Let \( u \in V((G_1,G_2)(a_1,a_2)) \) and \( e \in E((G_1,G_2)(a_1,a_2)) \). Then

\[
D_i(u,e|(G_1,G_2)(a_1,a_2)) = \begin{cases} 
    D_i(u,e|G_i) & u \in V(G_i), e \in E(G_i) \\
    D_i(u,e|G_j) & u \in V(G_j), e \in E(G_j) \\
    d(u,a_i|G_1) + D_i(a_2,e|G_2) & u \in V(G_1), e \in E(G_2) \\
    d(u,a_2|G_2) + D_i(a_1,e|G_i) & u \in V(G_2), e \in E(G_i)
  \end{cases}
\]

where \( i \in \{1,2\} \).

**Proof.** Using Lemma 2.1, the proof is obvious. \( \blacksquare \)

In the following theorem, the first and second vertex-edge Wiener indices of 
\( (G_1,G_2)(a_1,a_2) \) are computed.

**Theorem 2.3** The first and second vertex-edge Wiener indices of 
\( G = (G_1,G_2)(a_1,a_2) \) are given by:

\[
W_{ve}(G) = W_{ve}(G_1) + W_{ve}(G_2) + m_1d(a_1|G_1) + m_2d(a_2|G_2)
+ (n_2 - 1)D_i(a_1|G_1) + (n_1 - 1)D_i(a_2|G_2),
\]

where \( i \in \{1,2\} \).

**Proof.** By definition of the vertex-edge Wiener indices,

\[
W_{ve}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_i(u,e|G), \quad i \in \{1,2\}.
\]

Now, we partition the above sum into four sums as follows:

The first sum \( S_1 \) consists of contributions to \( W_{ve}(G) \) of vertices from \( V(G_i) \) and edges from \( E(G_i) \). Using Lemma 2.2, we obtain:

\[
S_1 = \sum_{u \in V(G_i)} \sum_{e \in E(G_i)} D_i(u,e|G) = \sum_{u \in V(G_i)} \sum_{e \in E(G_i)} D_i(u,e|G_i) = W_{ve}(G_i).
\]

The second sum \( S_2 \) consists of contributions to \( W_{ve}(G) \) of vertices from \( V(G_i) \) and edges from \( E(G_i) \). Similar to the previous case, we obtain:
\[ S_2 = \sum_{u \in \varphi(G_1)} \sum_{e \in E(G_2)} D_1(u, e|G_2) = W_{\mathcal{W}_1}(G_2). \]

The third sum \( S_3 \) consists of contributions to \( W_{\mathcal{W}_1}(G) \) of vertices from \( V(G_1) \setminus \{a_i\} \) and edges from \( E(G_2) \). Using Lemma 2.2, we obtain:

\[
S_3 = \sum_{u \in \varphi(G_1) \setminus \{a_i\}} \sum_{e \in E(G_2)} D_1(u, e|G) = \sum_{u \in \varphi(G_1) \setminus \{a_i\}} \sum_{e \in E(G_2)} [d(u, a_i|G_1) + D_1(a_2, e|G_2)]
\]
\[
= m_1 d(a_i|G_1) + (n_1 - 1) D_1(a_2|G_2).
\]

The last sum \( S_4 \) consists of contributions to \( W_{\mathcal{W}_1}(G) \) of vertices from \( V(G_2) \setminus \{a_i\} \) and edges from \( E(G_i) \). Similar to the previous case, we obtain:

\[
S_4 = \sum_{u \in \varphi(G_2) \setminus \{a_i\}} \sum_{e \in E(G_1)} [d(u, a_i|G_2) + D_1(a_1, e|G_1)]
\]
\[
= m_1 d(a_1|G_2) + (n_2 - 1) D_1(a_1|G_1).
\]

Now the formula of \( W_{\mathcal{W}_i}(G), i \in \{1, 2\} \), is obtained by adding the quantities \( S_1, S_2, S_3 \) and \( S_4 \).

\[ \square \]

### 2.2 Link

Let \( G_1 \) and \( G_2 \) be two connected graphs with disjoint vertex sets \( V(G_1) \) and \( V(G_2) \) and edge sets \( E(G_1) \) and \( E(G_2) \), respectively. For vertices \( a_i \in V(G_i) \) and \( a_2 \in V(G_2) \), a *link* [17] of \( G_1 \) and \( G_2 \) by vertices \( a_i \) and \( a_2 \) is denoted by \( (G_1 \sim G_2)(a_i, a_2) \) and obtained by joining \( a_i \) and \( a_2 \) by an edge in the union of these graphs. We denote by \( n_i \) and \( m_i \) the order and size of the graph \( G_i \), respectively. It is easy to see that, \( |V((G_1 \sim G_2)(a_i, a_2))| = n_1 + n_2 \) and \( |E((G_1 \sim G_2)(a_i, a_2))| = m_1 + m_2 + 1 \).

In the following lemma, the distance between two arbitrary vertices of \( (G_1 \sim G_2)(a_i, a_2) \) is computed. The result follows easily from the definition of the link of graphs, so its proof is omitted.

**Lemma 2.4** Let \( u, v \in V((G_1 \sim G_2)(a_i, a_2)) \). Then

\[
d(u, v|G_1 \sim G_2)(a_i, a_2)) = \begin{cases} d(u, v|G_1) & u, v \in V(G_1) \\ d(u, v|G_2) & u, v \in V(G_2) \\ d(u, a_i|G_1) + d(a_2, v|G_2) + 1 & u \in V(G_1), \ v \in V(G_2) \end{cases}.
\]

In the following lemma, the distances \( D_1 \) and \( D_2 \) between vertices and edges of \( (G_1 \sim G_2)(a_i, a_2) \) are computed.

**Lemma 2.5** Let \( u \in V((G_1 \sim G_2)(a_i, a_2)) \) and \( e \in E((G_1 \sim G_2)(a_i, a_2)) \). Then

\[
D_1(u, v|G_1 \sim G_2)(a_i, a_2)) = \begin{cases} d(u, v|G_1) & u, v \in V(G_1) \\ d(u, v|G_2) & u, v \in V(G_2) \\ d(u, a_i|G_1) + d(a_2, v|G_2) + 1 & u \in V(G_1), \ v \in V(G_2) \end{cases}.
\]
where \( i \in \{1, 2\} \).

**Proof.** Using Lemma 2.4, the proof is obvious. 

In the following theorem, the first and second vertex-edge Wiener indices of \((G_1 \sim G_2)(a_1, a_2)\) are computed.

**Theorem 2.6** The first and second vertex-edge Wiener indices of \(G = (G_1 \sim G_2)(a_1, a_2)\) are given by:

\[
W_{ve}(G) = W_{ve}(G_1) + W_{ve}(G_2) + \sum_{u \in V(G_1)} d(u, a_1) + \sum_{u \in V(G_2)} d(u, a_2) + n_i d_i(a_1, G_1) + n_j d_j(a_2, G_2) + n_i m_i + n_j m_j + (n_i + n_j)(i - 1),
\]

where \( i \in \{1, 2\} \).

**Proof.** By definition of the vertex-edge Wiener indices,

\[
W_{ve}(G) = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} D_i(u, e|G_1) = \sum_{u \in V(G_1) \cup E(G_1)} D_i(u, e|G_1) = W_{ve}(G_1).
\]

The second sum \( S_2 \) consists of contributions to \( W_{ve}(G) \) of vertices from \( V(G_2) \) and edges from \( E(G_2) \). Similar to the previous case, we obtain:

\[
S_2 = \sum_{u \in V(G_2) \cup E(G_2)} \sum_{e \in E(G_2)} D_i(u, e|G_2) = W_{ve}(G_2).
\]

The third sum \( S_3 \) consists of contributions to \( W_{ve}(G) \) of vertices from \( V(G_i) \) and edges from \( E(G_2) \). Using Lemma 2.5, we obtain:

\[
S_3 = \sum_{u \in V(G_1) \cup E(G_1)} \sum_{e \in E(G_2)} D_i(u, e|G_2) = \sum_{u \in V(G_i) \cup E(G_1)} [d(u, a_1) + D_i(a_2, e|G_2)] + 1
\]

\[
= m_i d(a_1, G_1) + n_i d_i(a_2, G_2) + n_i m_i.
\]

The fourth sum \( S_4 \) consists of contributions to \( W_{ve}(G) \) of vertices from \( V(G_2) \) and edges from \( E(G_i) \). Similar to the previous case, we obtain:
\[ S_4 = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} [d(u, a_2|G_2) + D_i(a_1, e|G_i) + 1] \]
\[ = m_1 d(a_2|G_2) + n_2 D_i(a_1|G_i) + n_2 m_1. \]

The fifth sum \( S_5 \) consists of contributions to \( W_{v_i}(G) \) of vertices from \( V(G_i) \) and the edge \( a_i a_j \) of \( G \). By Lemma 2.5, we obtain:

\[ S_5 = \sum_{u \in V(G_i)} \sum_{e \in E(G_i)} D_i(u, e|G) = \begin{cases} 
\sum_{u \in V(G_i)} d(u, a_i|G_i) & i = 1 \\
\sum_{u \in V(G_i)} (d(u, a_i|G_i) + 1) & i = 2 
\end{cases} \]
\[ = \begin{cases} 
d(a_i|G_i) & i = 1 \\
d(a_i|G_i) + n_i & i = 2 
\end{cases}. \]

The last sum \( S_6 \) consists of contributions to \( W_{v_i}(G) \) of vertices from \( V(G_i) \) and the edge \( a_i a_j \) of \( G \). Similar to the previous case, we obtain:

\[ S_6 = \sum_{u \in V(G_i)} \sum_{e \in E(G_i)} D_i(u, e|G) = \begin{cases} 
d(a_2|G_2) & i = 1 \\
d(a_2|G_2) + n_2 & i = 2 
\end{cases}. \]

Now the formula of \( W_{v_i}(G) \), \( i \in \{1, 2\} \), is obtained by adding the quantities \( S_1, S_2, S_3, S_4, S_5 \) and \( S_6 \).

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\section*{References}