Degree Distance Index of the Mycielskian and its Complement

ALI BEHTOEI AND MAHDI ANBARLOEI

Department of Mathematics, Imam Khomeini International University, P. O. Box: 34149-16818, Qazvin, Iran

Correspondence should be addressed to a.behtoei@sci.ikiu.ac.ir

Received 28 December 2014; Accepted 7 June 2015

ACADEMIC EDITOR: BIJAN TAERI

ABSTRACT Let G be a finite connected simple graph. The degree distance index $DD(G)$ of G is defined as $\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)(\deg_G(u)+\deg_G(v))$, where $\deg_G(u)$ is the degree of vertex u in G and $d_G(u,v)$ is the distance between two vertices u and v in G. In this paper, we determine the degree distance of the complement of arbitrary Mycielskian graphs. It is well known that almost all graphs have diameter two. We determine this graphical invariant for the Mycielskian of graphs with diameter two.

KEYWORDS Degree distance • Zagreb indices • Mycielskian.

1. INTRODUCTION

Throughout this paper we consider (non trivial) simple graphs, that are finite and undirected graphs without loops or multiple edges. Let $G=(V(G),E(G))$ be a connected graph of order n = |V(G)| and of size m = |E(G)|. The distance between two vertices u and v is denoted by $d_G(u,v)$ and is the length of a shortest path between u and v in G. The diameter of G is $\max\{d_G(u,v): u,v\in V(G)\}$. It is well known that almost all graphs have diameter two. The degree of vertex u is the number of edges adjacent to u and is denoted by $\deg_G(u)$.

A ***chemical graph*** is a graph whose vertices denote atoms and edges denote bonds between those atoms of the underlying chemical structure. A ***topological index*** for a (chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications.

The concept of topological index came from work done by Harold Wiener in 1947 while he was working on boiling point of paraffin. The ***Wiener index*** of G is defined as $W(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$. Two important topological indices introduced about forty years ago by Ivan Gutman and Trinajstić [5] are the ***first Zagreb index*** $M_1(G)$ and the ***second Zagreb index*** $M_2(G)$ which are defined as

$$M_1(G)=\sum_{uv\in E(G)}(\deg_G(u)+\deg_G(v))=\sum_{u\in V(G)}(\deg_G(u))^2,\ M_2(G)=\sum_{uv\in E(G)}\deg_G(u)\ \deg_G(v).$$

The ***degree distance*** was introduced by Dobrynin and Kochetova [1] and Gutman [4] as a weighted version of the Wiener index. The degree distance of G, denoted by $DD(G)$, is defined as follows and it is computed for important families of graphs (see[8] and [12] for instance):

$$DD(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)\ (\deg_G(u)+\deg_G(v)).$$

For a graph $G=(V,E)$, the ***Mycielskian*** of G is the graph $\mu(G)$ (or simply, μ) with the disjoint union $V\cup X\cup\{x\}$ as its vertex set and $E\cup\{v_ix_j : v_iv_j\in E\}\cup\{xx_i : 1\le i\le n\}$ as its edge set, where $V=\{v_1,v_2,...,v_n\}$ and $X=\{x_1,x_2,...,x_n\}$, see [9]. The Mycielskian and generalized Mycielskians have fascinated graph theorists a great deal. This has resulted in studying several graph parameters of these graphs. Fisher et al. [3] determine the domination number of the Mycielskian in 1998, Taeri et al. [2] determine the Wiener index of the Mycielskian in 2012, and Ashrafi et al. [6] determine Zagreb coindices of the Mycielskian in 2012.

In this paper we determine the degree distance index of the Mycielskian of each graph with diameter two. Also, we determine the degree distance of the complement of Mycielskian of arbitrary graphs.

2. DEGREE DISTANCE OF THE MYCIELSKIAN

In order to determine the degree distance index of Mycielskian graphs, we need the following observations. From now on we will always assume that G is a connected graph,

$$V(G)=\{v_1,v_2,...,v_n\},\ X=\{x_1,x_2,...,x_n\},\ V(G)\cap X=\phi,\ x\notin V(G)\cup X,$$

and μ is the Mycielskian of G, where

$$V(\mu)=V(G)\cup X\cup\{x\},\ E(\mu)=E(G)\cup\{v_ix_j : v_iv_j\in E(G)\}\cup\{xx_i : 1\le i\le n\}.$$

Observation 1. *Let μ be the Mycielskian of G. Then for each $v \in V(\mu)$ we have*

$$\deg_{\mu}(v) = \begin{cases} n & v = x \\ 1 + \deg_G(v_i) & v = x_i \\ 2\deg_G(v_i) & v = v_i \end{cases}$$

Observation 2. *In the Mycielskian μ of G, the distance between two vertices $u, v \in V(\mu)$ are given as follows.*

$$d_{\mu}(u,v) = \begin{cases} 1 & u = x, v = x_i \\ 2 & u = x, v = v_i \\ 2 & u = x_i,\ v = x_j \\ d_G(v_i, v_j) & u = v_i,\ v = v_j,\ d_G(v_i, v_j) \le 3 \\ 4 & u = v_i,\ v = v_j,\ d_G(v_i, v_j) \ge 4 \\ 2 & u = v_i, v = x_j, i = j \\ d_G(v_i, v_j) & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) \le 2 \\ 3 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) \ge 3. \end{cases}$$

Specially, the diameter of the Mycielskian graph is at most four.

There are $|E(G)|$ unordered pairs of vertices in $V = V(G)$ whose distance is one, and

$$\sum_{\substack{(u,v) \in V \times V \\ d_G(u,v) = 1}} (\deg_G(u) + \deg_G(v)) = 2 \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v)) = 2M_1(G).$$

Lemma 1. *Let G be a graph of size m whose vertex set is $V = \{v_1, v_2, ..., v_n\}$. Then,*

$$\sum_{\{v_i, v_j\} \subseteq V} (\deg_G(u) + \deg_G(v)) = (n-1)2m.$$

Proof. For each $i \in [n] = \{1,2,...,n\}$, $|\{ \{i, j\} \subseteq [n] : j \neq i\}| = n-1$. Therefore,

$$\sum_{\{i,j\} \subseteq [n]} (\deg_G(v_i) + \deg_G(v_j)) = \sum_{i=1}^{n} (n-1) \deg_G(v_i) = (n-1)2m.$$

□

Lemma 2. *For each graph G of size m we have*

$$\sum_{\{v_i, v_j\} \notin E(G)} (\deg_G(v_i) + \deg_G(v_j)) = 2m(n-1) - M_1(G).$$

Proof. Since each vertex $v_i \in V(G)$ has $\deg_G(v_i)$ neighbors in G, the number of non-adjacent vertices to v_i in G equals $n-1-\deg_G(v_i)$. This implies that

$$\begin{aligned}\sum_{\{v_i,v_j\}\notin E(G)}(\deg_G(v_i)+\deg_G(v_j)) &= \sum_{i=1}^{n}(n-1-\deg_G(v_i))\ \deg_G(v_i)\\ &= (n-1)\sum_{i=1}^{n}\deg_G(v_i)-\sum_{i=1}^{n}(\deg_G(v_i))^2\\ &= 2m(n-1)-M_1(G).\end{aligned}$$

□

It is a well known fact that almost all graphs have diameter two. This means that graphs of diameter two play an important role in the theory of graphs and their applications.

Theorem 1. *Let G be an n-vertex graph of size m whose diameter is 2. If μ is the Mycielskian of G, then the degree distance index of μ is given by*

$$DD(\mu)=4DD(G)-M_1(G)+(7n-1)n+(8n+12)m.$$

Proof. By the definition of degree distance index, we have

$$DD(\mu(G))=\sum_{\{u,v\}\subseteq V(\mu)}d_\mu(u,v)\ (\deg_\mu(u)+\deg_\mu(v)).$$

Regarding to the different possible cases which u and v can be chosen from the set $V(\mu)$, the following cases are considered. In what follows, the notations are as before and two observations 1 and 2 are applied for computing degrees and distances in μ.

Case 1. $u=x$ and $v\in X$:

$$\sum_{i=1}^{n}d_\mu(x,x_i)\ (\deg_\mu(x)+\deg_\mu(x_i))=\sum_{i=1}^{n}(n+1+\deg_G(v_i))=n(n+1)+2m.$$

Case 2. $u=x$ and $v\in V(G)$:

$$\sum_{i=1}^{n}d_\mu(x,v_i)\ (\deg_\mu(x)+\deg_\mu(v_i))=\sum_{i=1}^{n}2(n+2\deg_G(v_i))=2(n^2+4m).$$

Case 3. $\{u,v\}\subseteq X$:

Using Lemma 1 we see that

$$\begin{aligned}\sum_{\{x_i,x_j\}\subseteq X}d_\mu(x_i,x_j)(\deg_\mu(x_i)+\deg_\mu(x_j)) &= \sum_{\{x_i,x_j\}\subseteq X}2(2+\deg_G(v_i)+\deg_G(v_j))\\ &= 4\binom{n}{2}+2\sum_{\{i,j\}\subseteq[n]}(\deg_G(v_i)+\deg_G(v_j))\\ &= 2n^2-2n+4(n-1)m.\end{aligned}$$

Case 4. $\{u,v\}\subseteq V(G)$. Since the diameter of G is two, Observation 2 implies that $d_\mu(v_i,v_j)=d_G(v_i,v_j)$. Hence,

$$\sum_{\{v_i,v_j\}\subseteq V(G)} d_\mu(v_i,v_j)\ (\deg_\mu(v_i)+\deg_\mu(v_j)) = \sum_{\{v_i,v_j\}\subseteq V(G)} d_G(v_i,v_j)\ (2\deg_G(v_i)+2\deg_G(v_j))$$
$$=2\,DD(G).$$

Case 5. $u=v_i$ and $v=x_i$, $1\le i\le n$.

$$\sum_{i=1}^{n} d_\mu(v_i,x_i)\ (\deg_\mu(v_i)+\deg_\mu(x_i)) = \sum_{i=1}^{n} 2\ (3\deg_G(v_i)+1)$$
$$=2n+12m.$$

Case 6. $u=v_i$ and $v=x_j$, $i\neq j$.

$$\sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_\mu(v_i)+\deg_\mu(x_j)) = \sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (2\deg_G(v_i)+\deg_G(v_j)+1)$$
$$= \sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_G(v_i)+\deg_G(v_j))$$
$$+ \sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_G(v_i)+1).$$

Since $d_\mu(v_i,x_j)=d_\mu(v_j,x_i)$, $d_\mu(v_i,v_i)=0$, and using Observation 2, we have

$$\sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_G(v_i)+\deg_G(v_j)) = 2 \sum_{\substack{\{v_i,v_j\}\subseteq V(G)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_G(v_i)+\deg_G(v_j))$$
$$=2 \sum_{\{v_i,v_j\}\subseteq V(G)} d_G(v_i,v_j)\ (\deg_G(v_i)+\deg_G(v_j))$$
$$=2\,DD(G).$$

Each edge $v_iv_j=v_jv_i\in E(G)$ corresponds to two pairs $\{v_i,x_j\}$ and $\{v_j,x_i\}$ of distance 1 in the Mycielskian graph μ. Since the diameter of G is two and using Lemma 2 we get

$$\sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ i\neq j}} d_\mu(v_i,x_j)\ (\deg_G(v_i)+1) = \sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ v_iv_j\in E(G)}} 1\,(1+\deg_G(v_i)) + \sum_{\substack{\{v_i,x_j\}\subseteq V(\mu)\\ v_iv_j\notin E(G)}} 2\ (1+\deg_G(v_i))$$
$$=2m+\sum_{v_iv_j\in E(G)} (\deg_G(v_i)+\deg_G(v_j)).$$
$$+4(\binom{n}{2}-m)+2\sum_{v_iv_j\notin E(G)} (\deg_G(v_i)+\deg_G(v_j))$$
$$=2n(n-1)+2m(2n-3)-M_1(G).$$

Now the result follows through these six cases.

□

3. DEGREE DISTANCE OF THE COMPLEMENT OF MYCIELSKIAN

In order to determine the degree distance index of the complement of Mycielskian graphs, we need two following observations.

Observation 3. *Let $\overline{\mu}$ be the complement of Mycielskian μ of G. Then for each $v \in V(\overline{\mu})$ we have*

$$\deg_{\overline{\mu}}(v) = \begin{cases} n & v = x \\ 2n - (1 + \deg_G(v_i)) & v = x_i \\ 2n - 2\deg_G(v_i) & v = v_i \end{cases}$$

Observation 4. *In the complement of Mycielskian μ of G, the distance between two vertices $u, v \in V(\overline{\mu})$ are given as follows.*

$$d_{\overline{\mu}}(u,v) = \begin{cases} 2 & u = x, v = x_i \\ 1 & u = x, v = v_i \\ 1 & u = x_i,\ v = x_j \\ 1 & u = v_i,\ v = v_j,\ d_G(v_i, v_j) > 1 \\ 2 & u = v_i,\ v = v_j,\ d_G(v_i, v_j) = 1 \\ 1 & u = v_i, v = x_j, i = j \\ 1 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) > 1 \\ 2 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) = 1. \end{cases}$$

Specially, the diameter of $\overline{\mu}$ is exactly 2.

Theorem 2. *Let G be an n-vertex graph of size m and let $\overline{\mu}$ be the complement of the Mycielskian μ of G. Then, the degree distance index of $\overline{\mu}$ is given by*

$$DD(\overline{\mu}) = n(6n^2 + 10n - 5) - 4m - 5M_1(G).$$

Proof. By the definition of degree distance, we have

$$DD(\overline{\mu}) = \sum_{\{u,v\} \subseteq V(\overline{\mu})} d_{\overline{\mu}}(u,v)\ (\deg_{\overline{\mu}}(u) + \deg_{\overline{\mu}}(v)).$$

We consider the following cases. For computing degrees and distances in $\overline{\mu}$ we use two observations 3 and 4.

Case 1. $u = x$ and $v \in X$.

$$\sum_{i=1}^{n} d_{\overline{\mu}}(x, x_i)\ (\deg_{\overline{\mu}}(x) + \deg_{\overline{\mu}}(x_i)) = \sum_{i=1}^{n} 2(3n - \deg_G(v_i) - 1) = 6n^2 - 2n - 4m.$$

Case 2. $u = x$ and $v \in V(G)$.

$$\sum_{i=1}^{n} d_{\overline{\mu}}(x, v_i)\ (\deg_{\overline{\mu}}(x) + \deg_{\overline{\mu}}(v_i)) = \sum_{i=1}^{n} (3n - 2\deg_G(v_i)) = 3n^2 - 4m.$$

Case 3. $\{u, v\} \subseteq X$. Using Lemma 1 we see that

$$\sum_{\{x_i, x_j\} \subseteq X} d_{\overline{\mu}}(x_i, x_j)(\deg_{\overline{\mu}}(x_i) + \deg_{\overline{\mu}}(x_j)) = \sum_{\{i, j\} \subseteq [n]} (4n - 2 - (\deg_G(v_i) + \deg_G(v_j)))$$
$$= 4n^2 - 2n - 2m(n-1).$$

Case 4. $\{u, v\} \subseteq V(G)$. Using Lemma 2 we have

$$\sum_{\{v_i, v_j\} \subseteq V(G)} d_{\overline{\mu}}(v_i, v_j)\ (\deg_{\overline{\mu}}(v_i) + \deg_{\overline{\mu}}(v_j)) = \sum_{v_i v_j \notin E(G)} (4n - 2(\deg_G(v_i) + \deg_G(v_j)))$$
$$+ 2 \sum_{v_i v_j \in E(G)} (4n - 2(\deg_G(v_i) + \deg_G(v_j)))$$
$$= 4n(\binom{n}{2} - m) - 2(2m(n-1) - M_1(G))$$
$$+ 8mn - 4M_1(G)$$
$$= 2n^2(n-1) + 4m - 2M_1(G).$$

Case 5. $u = v_i$ and $v = x_i$, $1 \le i \le n$.

$$\sum_{i=1}^{n} d_{\overline{\mu}}(v_i, x_i)\ (\deg_{\overline{\mu}}(v_i) + \deg_{\overline{\mu}}(x_i)) = \sum_{i=1}^{n} (4n - 3\deg_G(v_i) - 1) = 4n^2 - n - 6m.$$

Case 6. $u = v_i$ and $v = x_j$, $i \ne j$. By Observation 4, $d_{\overline{\mu}}(v_i, x_j) = d_{\overline{\mu}}(v_j, x_i)$ is 1 when $v_i v_j \notin E(G)$, otherwise is 2. Thus,

$$\sum_{\substack{\{v_i, x_j\} \subseteq V(\overline{\mu}) \\ i \ne j}} d_{\overline{\mu}}(v_i, x_j)\ (\deg_{\overline{\mu}}(v_i) + \deg_{\overline{\mu}}(x_j)) = \sum_{\substack{(v_i, v_j) \\ v_i v_j \notin E(G)}} (4n - 1 - 2\deg_G(v_i) - \deg_G(v_j))$$
$$+ \sum_{\substack{(v_i, v_j) \\ v_i v_j \in E(G)}} 2(4n - 1 - 2\deg_G(v_i) - \deg_G(v_j))$$

Each vertex v_i can be paired with $n-1-\deg_G(v_i)$ vertices v_j as (v_i, v_j) with the condition $v_iv_j \notin E(G)$. Also, note that $\sum_{(v_i,v_j)}(\deg_G(v_i)+\deg_G(v_j))$ is equal to $2\sum_{\{v_i,v_j\}}(\deg_G(v_i)+\deg_G(v_j))$. Hence, using Lemma 2 we obtain

$$\sum_{\substack{(v_i,v_j)\\ v_iv_j\notin E(G)}}(4n-1-2\deg_G(v_i)-\deg_G(v_j)) = 2(\binom{n}{2}-m)(4n-1)-\sum_{\substack{(v_i,v_j)\\ v_iv_j\notin E(G)}}(\deg_G(v_i)+\deg_G(v_j))$$
$$-\sum_{\substack{(v_i,v_j)\\ v_iv_j\notin E(G)}}\deg_G(v_i)$$
$$=(n^2-n-2m)(4n-1)-2(2m(n-1)-M_1(G))-(2m(n-1)-M_1(G)).$$

Note that $|\{(v_i,v_j):\ v_iv_j \in E(G)\}| = 2m$ and

$$\sum_{(v_i,v_j):v_iv_j\in E(G)}\deg_G(v_i)=\sum_{i=1}^{n}(\deg_G(v_i))^2,$$

because each vertex v_i has $\deg_G(v_i)$ neighbors and appears $\deg_G(v_i)$ times in the desired summation. Thus, using similar arguments we see that

$$\sum_{\substack{(v_i,v_j)\\ v_iv_j\in E(G)}}2(4n-1-2\deg_G(v_i)-\deg_G(v_j)) = 4m(4n-1)-6M_1(G).$$

Now the result follows through these cases.

□

By considering Observation 3, it's not hard to check that

$$M_1(\bar{\mu}) = 5M_1(G)+8n^3-3n^2-24mn+4m+n.$$

Thus, Theorems 1 and 2 imply the following result.

Corollary 4. Let G be an n-vertex graph of size m and let H be the complement of the Mycielskian of G. Then, $DD(\mu(H)) = 16n^3+73n^2+5n+20m+56mn-25M_1(G)$.

□

REFERENCES

1. A. A. Dobrynin and A. A. Kochetova, Degree *Distance of a Graph: A Degree Analogue of the Wiener Index, J. Chem. Inf. Comput. Sci.*, **34** (1994) 1082-1086.

2. M. Eliasi, G. Raeisi, B. Taeri, *Wiener index of some graph operations*, *Discret. Appl. Math.*, **160** (2012) 1333-1344.
3. D.C. Fisher, P.A. McKena, E.D. Boyer, *Hamiltonicity, diameter, domination, packing and biclique partitions of Mycielskis graphs*, *Discret. Appl. Math.*, **84** (1998) 93-105.
4. I. Gutman, Selected *Properties of the Schultz Molecular Topological Index*, *J. Chem. Inf. Comput. Sci.*, **34** (1994) 1087-1089.
5. I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total* π*-electron energy of alternant hydrocarbons*, *Chem. Phys. Lett.*, **17** (1972) 535-538.
6. H. Hua, A. R. Ashrafi, L. Zhang, *More on Zagreb coindices of graphs*, *Filomat*, **26** (2012) 1215-1225.
7. M. H. Khalifeh, H. Yousefi Azari, A. R. Ashrafi, S. Wagner, *Some new results on distance-based graph invariants*, *Eur. J. Comb.*, **30** (2009) 1149-1163.
8. A. Ilić, S. Klavžar, D. Stevanović, *Calculating the degree distance of partial Hamming graphs*, *MATCH Commun. Math. Comput. Chem.*, **63** (2010) 411-424.
9. J. Mycielski, *Sur le colouriage des graphes*, *Colloq. Math.*, **3** (1955) 161-162.
10. M. Tavakoli, F. Rahbarnia, *Applications of some graph operations in computing some invariants of chemical graphs*, *Iranian J. Math. Chem.*, **4** (2013) 221-230.
11. K. Xu, M. Liu, K. C. Das, I. Gutman and B. Furtula, *A survey on graphs extremal with respect to distance-based topological indices*, *MATCH Commun. Math. Comput. Chem.*, **71** (2014) 461-508.
12. Z. Yarahmadi, *Computing some topological indices of tensor product of graphs*, *Iranian J. Math. Chem.*, **2** (2011) 109-118.