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Trigonometrically fitted two-step Obrechkoff methods for the numerical solution of periodic initial value problems

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ABSTRACT In this paper, we present a new two-step trigonometrically fitted symmetric Obrechkoff method. The method is based on the symmetric two-step Obrechkoff method, with eighth algebraic order, high phase-lag order and is constructed to solve IVPs with periodic solutions such as orbital problems. We compare the new method to some recently constructed optimized methods from the literature. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

KEYWORDS Obrechkoff methods • Trigonometrically-fitting • Initial value problems • Symmetric multistep methods • oscillating solution

1. INTRODUCTION

In this paper, the symmetric Obrechkoff methods for solving special classes of initial value problems associated with second order ordinary differential equations of the type

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_{0,} \tag{1}$$

in which the first order derivatives do not occur explicitly, are discussed. The numerical integration methods for (1) can be divided into two distinct classes:

1. Problems for which the solution period is known (even approximately) in advance.
2. Problems for which the period is not known.

For several decades, there has been strong interest in searching for better numerical methods to integrate first-order and second-order initial value problems, because these problems are usually encountered in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics. Generally, the solution of (1) is periodic, so it is expected that the result produced by some numerical methods preserves the analogical periodicity of the analytic solution [1-22]. Computational methods involving a parameter proposed by Gautschi [8], Alolyan and et al [2], Jain et al. [12] and Steifel and Bettis [24] yield numerical solution of problems of class 1. Chawla and et al [4], Ananthakrishnaiah [3], Shokri and et al. [21, 22, 23], Simos [24], Dahlquist [5], Franco and et al [6, 7], Lambert and Watson [14], Hairer [9], Saldanha and et al [20], Wang and et al. [27, 28, 29] and Daele and Vanden Berghe [25, 26] have developed methods to solve problems of class 2.

Consider Obrechkoff method of the form

$$\sum_{j=0}^{k} \alpha_i y_{n-j+1} = \sum_{i=1}^{l} h^{2i} \sum_{j=0}^{k} \beta_i y^{(2i)}{}_{n-j+1}, \tag{2}$$

for the numerical integration of the problem (1). The method (2) is symmetric when $\alpha_j = \alpha_{k-j}, \beta_j = \beta_{k-j}, j = 0,1,2,k,$ and it is of order q if the truncation error associated with the linear difference operator is given as

$$TE = C_{q+2} h^{q+2} y^{(q+2)}, \quad x_{n-k+1} < \eta < x_{n+1},$$

where C_{q+2} is a constant dependent on h. To investigate the stability properties of the methods for solving the initial value problem (1), Lambert and Watson [14] introduced the scalar test equation $y'' = -\omega^2 y, \omega \in \mathbb{R}$. When the method (2) is applied to the test equation, we get the characteristic equation as

$$\rho(\xi) - \sum_{i=1}^{l} (-1)^i v^{2i} \sigma_i \ (\xi) = 0, \tag{3}$$

Where $v = \lambda h$ and

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^{k-j}, \quad \sigma_i(\xi) = \sum_{j=0}^{k} \beta_{ij} \xi^{k-j}, i = 1,2,\dots,l. \tag{4}$$

Definition 1.1. The method (2) is said to have interval of periodicity $\left(0, v_0{}^2\right)$ if for all $v^2 \in \left(0, v_0{}^2\right)$ the roots of Eq. (3) are complex and at least two of them lie on the unit circle and the others lie inside the unit circle.

Definition 1.2. The method (2) is said to be P-stable if its interval of periodicity is $(0,\infty)$.

Definition 1.3. For any symmetric multistep methods, the phase-lag (frequency distortion) of order q is given by

$$t(v) = v - \theta(v) = Cv^{q+1} + O\left(v^{q+2}\right), \tag{5}$$

where C is the phase lag constant and q is the phase-lag order.

The characteristic equation of the method (2) is given by

$$\Omega\left(s : v^2\right) = A(v)s^2 - 2B(v)s + A(v) = 0, \tag{6}$$

where

$$A(v) = 1 + \sum_{i=1}^{m} (-1)^i \beta_{i0} v^{2i}, \quad B(v) = 1 + \sum_{i=1}^{m} (-1)^i \beta_{i1} v^{2i}, \tag{7}$$

ψ contains polynomial functions together with trigonometric polynomials

$$\psi_{trig} = \{1, t, \ldots, t^k, \cos(r\omega t), \sin(r\omega t), \quad r = 1, 2, \ldots, p\}. \tag{8}$$

The resulting methods are then based on a hybrid set of polynomials and trigonometric functions. If P is limited to $P = M/2 - 1$, we called method with zero phase-lag. We present here the trigonometric versions of the set. In case ω is purely imaginary one obtains the hyperbolic description of this set. This set is characterized by two integer parameters K and P. The set in which there is no polynomial part is identified by $K = -1$ while the set in which there is no trigonometric polynomial component is identified by $P = -1$. For each problem one has $K + 2P = M - 3$, where $M - 1$ is the maximum exponent present in the full polynomial basis for the same problem (see [22, 23]).

2. CONSTRUCTION OF THE NEW METHOD

From the form (2) and without loss of generality we assume

$$\alpha_j = \alpha_{m-j}, \beta_{i,j} = \beta_{i,m-j}, \qquad j = 0(1)\left\lfloor \frac{m}{2} \right\rfloor$$

and we can write

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^{2i}\left[\beta_{i_0} y_{n+1}^{(2i)} + \beta_{i_1} y_n^{(2i)} + \beta_{i_0} y_{n-1}^{(2i)}\right], \tag{9}$$

when $m = 2$ we get

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} = h^2\left[\beta_{10}({y_{n+1}}^{(2)} + {y_{n-1}}^{(2)}) + \beta_{11}{y_n}^{(2)}\right] \\ + h^4\left[\beta_{20}({y_{n+1}}^{(4)} + {y_{n-1}}^{(4)}) + \beta_{21}{y_n}^{(4)}\right]. \end{aligned} \tag{10}$$

$M-3$ for method (10) is 7 so that if $P=-1$, $K=9$ we obtain classic method and the coefficients of this method are

$$\beta_{1,0} = \frac{11}{252},\quad \beta_{1,1} = \frac{115}{126},\quad \beta_{2,0} = -\frac{13}{15120},\quad \beta_{2,1} = \frac{313}{7560}, \tag{11}$$

and its local truncation error is given by

$$LTE_{clas} = \frac{59}{76204800} y^{(10)} h^{10} + O(h^{12}).$$

If $P=4$, $K=-1$ we obtain the method with zero phase-lag (PL), and the coefficients of this case are given by

$$\beta_{1,0} = -\frac{1}{3600}\frac{\beta_{1,0,num}}{A},\quad \beta_{1,1} = \frac{1}{1800}\frac{\beta_{1,1,num}}{A},\quad \beta_{2,0} = -\frac{1}{18}\frac{\beta_{2,0,num}}{A},\quad \beta_{2,1} = \frac{-1}{36}\frac{\beta_{2,1,num}}{A},$$

where $A = v^4(28\cos^3 v + 48\cos^2 v + 25\cos v + 4)$ and

$$\begin{aligned} \beta_{1,0,num} &= 10935\cos(3v) + 5440\cos(5v)\cos(3v) - 12000\cos(3v)\cos(v) - 4375\cos(3v)\cos(4v) \\ &+ 38250\cos(v)\cos(4v) + 2187\cos(5v) - 42\cos(v) - 32768\cos(4v) \\ &- 1107\cos(5v)\cos(4v) - 26208\cos(5v)\cos(v), \end{aligned}$$

$$\beta_{1,1,num} = 35(\cos^2 v - 4\cos v - 31)\cos v,$$

$$\beta_{2,0,num} = 70\cos^5 v - 123\cos^3 v - 18\cos^2 v + 55\cos v + 16,$$

$$\beta_{2,1,num} = 7\cos^3 v - 8\cos^2 v - 5\cos v + 6.$$

The Taylor series expansions, used when $v \to 0$ are given below

$$\begin{aligned} \beta_{1,0} = \frac{11}{252} + \frac{59}{10584}v^2 - \frac{157}{444528}v^4 - \frac{69301}{124467840}v^6 - \frac{20437283}{78414739200}v^8 \\ - \frac{636568907}{6586838092800}v^{10} - \frac{2135266381189}{65934249308928000}v^{12} - \cdots, \end{aligned}$$

$$\begin{aligned} \beta_{1,1} = \frac{115}{126} - \frac{59}{5292}v^2 + \frac{157}{222264}v^4 + \frac{69301}{62233920}v^6 - \frac{590503}{7841473920}v^8 \\ + \frac{97256377}{36227609510400}v^{10} - \frac{32506309663}{65934249308928000}v^{12} - \cdots, \end{aligned}$$

$$\beta_{2,0} = -\frac{13}{15120} - \frac{59}{127008}v^2 - \frac{97049}{533433600}v^4 - \frac{280529}{4480842240}v^6 - \frac{189458741}{9409768704000}v^8 - \frac{1802008091}{289820876083200}v^{10} - \frac{4552039054177}{2434495359098880000}v^{12} - \cdots,$$

$$\beta_{2,1} = \frac{313}{7560} - \frac{295}{63504}v^2 + \frac{191249}{266716800}v^4 + \frac{21115}{448084224}v^6 + \frac{77372441}{4704884352000}v^8 + \frac{109705741}{28982087608320}v^{10} + \frac{14524363782301}{15824219834142720000}v^{12} + \cdots,$$

The local truncation error for symmetric, Obrechkoff two-step method with zero phase-lag is given by

$$LTE_{ZeroPL} = (1-\beta_{1,1}-2\beta_{1,0})y_n^{(2)}h^2 - \left(-\frac{1}{12}+\beta_{1,0}+\beta_{2,1}+2\beta_{2,0}\right)y_n^{(4)}h^4 - \frac{1}{360}(30\beta_{1,0}+360\beta_{2,0}-1)y_n^{(6)}h^6 - \frac{1}{360}\left(-\frac{1}{56}+30\beta_{2,0}+\beta_{1,0}\right)y_n^{(8)}h^8 - \frac{1}{20160}\left(-\frac{1}{90}+56\beta_{2,0}+\beta_{1,0}\right)y_n^{(10)}h^{10} + O(h^{12})$$

where $v = \omega h$, ω is the frequency and h is the step length.

2.1. THE FIRST FORMULA

If $P = 0$, $K = 7$, so we called PL''', we have

$$\beta_{1,0} = -\frac{1}{30v^2}\frac{\beta_{1,0,num}}{A}, \quad \beta_{1,1} = \frac{1}{15v^2}\frac{\beta_{1,1,num}}{A}, \quad \beta_{2,0} = \frac{1}{120v^2}\frac{\beta_{2,0,num}}{A}, \quad \beta_{2,1} = \frac{1}{60v^2}\frac{\beta_{2,1,num}}{A},$$

$$A = -12 + 12\cos v + v^2\cos v + 5v^2,$$

and

$$\beta_{1,0,num} = -18v^2 + v^4\cos v + 14v^4 - 360\cos v + 360,$$

$$\beta_{1,1,num} = 18v^2\cos v + 61v^4 + -360\cos v - 360 + 14v^4\cos v,$$

$$\beta_{2,0,num} = 4v^2\cos v - 3v^4 + 56v^2 + 120\cos v - 120,$$

$$\beta_{2,1,num} = 244v^2 + 56v^2\cos v + 3v^4\cos v + 600\cos v - 600.$$

For small values of v the above formulae are subject to heavy cancelations. In this case the following Taylor series expansion must be used:

$$\beta_{1,0}=\frac{11}{252}+\frac{59}{317520}v^2+\frac{233}{88016544}v^4+\frac{451037}{14417109907200}v^6+\frac{111679}{363311169661440}v^8+\frac{1010455379}{4280168889781424640000}v^{10}+\frac{159482401}{157641912648257393664000}v^{12}+\ldots,$$

$$\beta_{1,1}=\frac{115}{126}-\frac{59}{158760}v^2-\frac{233}{44008272}v^4-\frac{451037}{7208554953600}v^6-\frac{111679}{181655584830720}v^8-\frac{1010455379}{2140084444890712320000}v^{10}-\frac{159482401}{78820956324128696832000}v^{12}+\ldots,$$

$$\beta_{2,0}=-\frac{13}{15120}-\frac{59}{3810240}v^2-\frac{233}{1056198528}v^4-\frac{451037}{173005318886400}v^6-\frac{111679}{4359734035937280}v^8-\frac{1010455379}{51362026677377095680000}v^{10}-\frac{159482401}{1891702951779088723968000}v^{12}+\cdots,$$

$$\beta_{2,1}=\frac{313}{7560}-\frac{59}{3810240}v^2-\frac{1156}{528099264}v^4-\frac{451037}{17300531888640}v^6-\frac{111679}{435973403593728}v^8-\frac{1010455379}{5136202667737709568000}v^{10}-\frac{159482401}{189170295177908872396800}v^{12}+\cdots,$$

The phase-lag and the local truncation error for the PL''', method are given by

$$LTE_{PL'''}=(1-\beta_{1,1}-2\beta_{1,0})y_n^{(2)}h^2-\left(-\frac{1}{12}+\beta_{1,0}+\beta_{2,1}+2\beta_{2,0}\right)y_n^{(4)}h^4$$
$$-\frac{1}{360}(30\beta_{1,0}+360\beta_{2,0}-1)y_n^{(6)}h^6-\frac{1}{360}\left(-\frac{1}{56}+30\beta_{2,0}+\beta_{1,0}\right)y_n^{(8)}h^8$$
$$-\frac{1}{20160}\left(-\frac{1}{90}+56\beta_{2,0}+\beta_{1,0}\right)y_n^{(10)}h^{10}+O(h^{12}),$$

$$pl_{PL'''}=\left|-\frac{1010455379}{205448106709508382720000}v^{16}+O(v^{18})\right|,$$

$$LTE_{PL'''}=\frac{59}{76204800}\left(D^{(10)}y(x)+\omega^2D^{(8)}y(x)\right)h^{10},$$

where $v=\omega h$, ω is the frequency and h is the step length. The behavior of the coefficients of the PL''' method are shown in Figure 1. In Figure 2, we present the $s-v$ plane (stability region) for the method PL''.

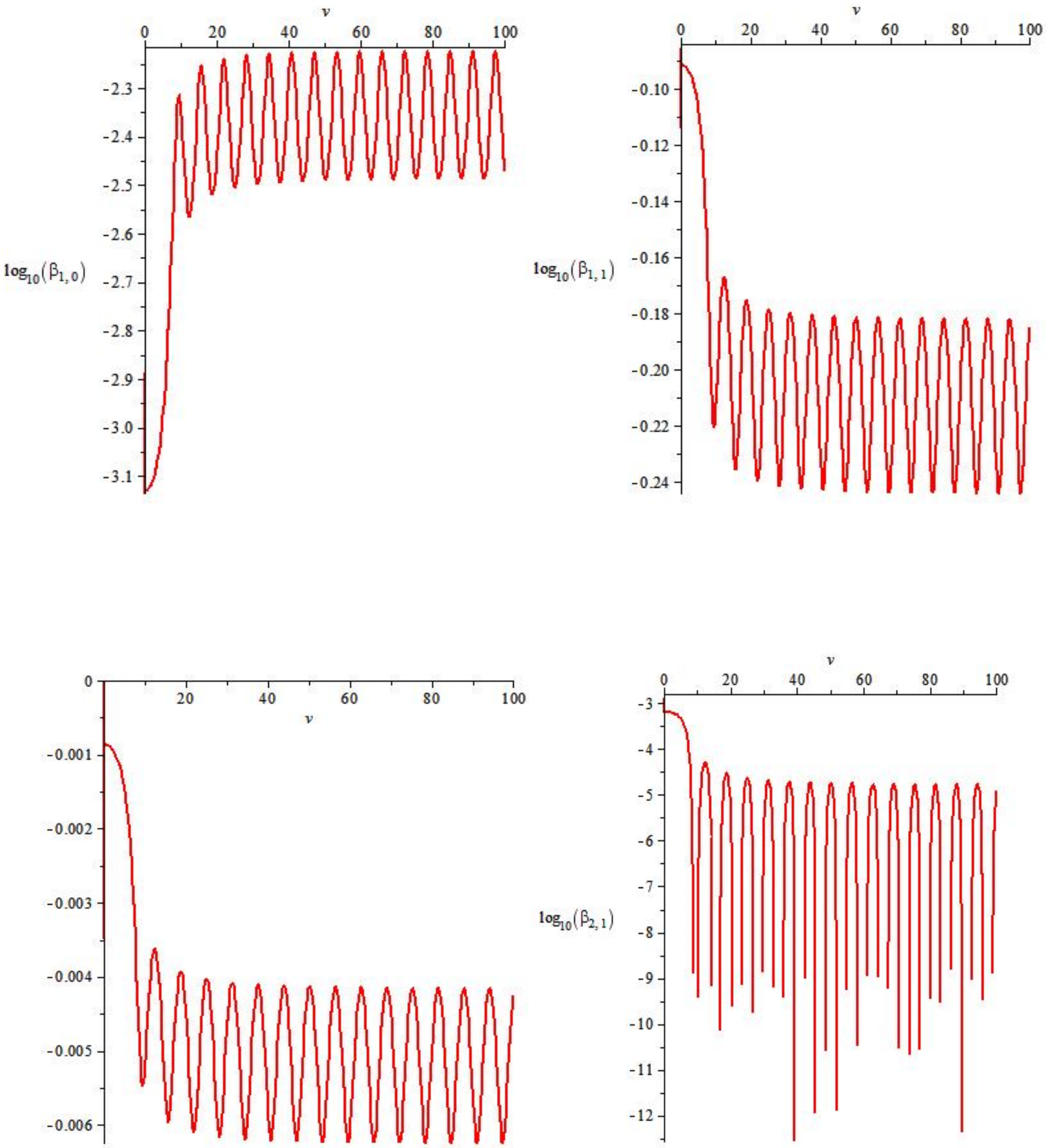

Figure 1. Behavior of the coefficients of the new proposed method PL‴ for several values of $v = \omega h$.

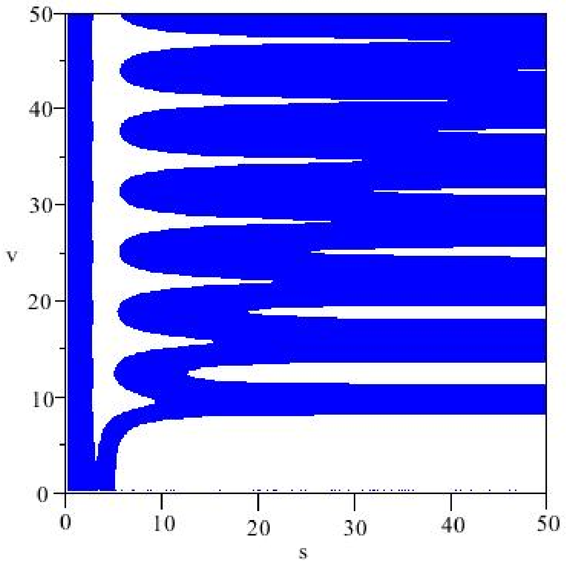

Figure 2. $s-v$ plane of the new obtained method PL$'''$.

2.2. THE SECOND FORMULA

If $P=1$, $K=5$, so we called PL'', we have

$$\beta_{1,0}=\frac{14}{15}+24\beta_{2,1},\quad \beta_{1,1}=\frac{1}{30}-12\beta_{2,1},\quad \beta_{2,0}=\frac{1}{20}+10\beta_{2,1},$$

$$\beta_{2,1}=\frac{1}{120}\frac{-3v^4+4v^2\cos v+56v^2-120+120\cos v}{v^2(5v^2+v^2\cos v+12\cos v-12)},$$

For small values of v the above formulae are subject to heavy cancelations. In this case the following Taylor series expansion must be used:

$$\beta_{1,0}=\frac{11}{252}+\frac{59}{63504}v^2+\frac{2297}{88016544}v^4+\frac{3600761}{2883421981440}v^6+\frac{25113727}{363311169661440}v^8$$
$$+\frac{13102592879}{3424135111825139712}v^{10}+\frac{43167790232461}{204934486442734611763200}v^{12}+\cdots,$$

$$\beta_{1,1}=\frac{115}{126}-\frac{59}{31752}v^2-\frac{2297}{44008272}v^4-\frac{3600761}{1441710990720}v^6-\frac{25113727}{181655584830720 20}v^8$$
$$-\frac{13102592879}{1712067555912569856}v^{10}-\frac{43167790232461}{102467243221367305881600}v^{12}+\cdots,$$

$$\beta_{2,0} = -\frac{13}{15120} - \frac{59}{762048}v^2 - \frac{139199}{26404963200}v^4 - \frac{11233841}{34601063777280}v^6 - \frac{2072169709}{108993350898432000}v^8$$
$$-\frac{222352522039}{205448106709508382720}v^{10} - \frac{745610462460887}{1229606918656407670579200 0}v^{12} + \cdots,$$

$$\beta_{2,1} = \frac{313}{7560} - \frac{295}{381024}v^2 - \frac{205351}{13202481600}v^4 - \frac{2074145}{3460106377728}v^6 - \frac{1694889341}{54496675449216000}v^8$$
$$-\frac{170725264331}{102724053354754191360}v^{10} - \frac{549423244512943}{614803459328203835289600 0}v^{12} + \cdots,$$

The phase-lag and the local truncation error for the PL''', method are given by

$$pl_{PL''} = -\frac{3328231}{8719468071874560}v^{14} + O\left(v^{16}\right),$$

and

$$LTE_{PL'''} = \frac{59}{76204800}\left(4\omega^4 D^{(6)}y(x) + D^{(10)}y(x) + 5\omega^2 D^{(8)}y(x)\right)h^{10},$$

where $v = \omega h$, ω,is the frequency and h,is the step length. The behavior of the coefficients of the PL''', method are shown in Fig. 3. In Fig. 4 we present the $s-v$ plane (stability region) for the method PL'''.

The characteristic equation $\Omega\left(s : v^2\right) = A(v)s^2 - 2B(v)s + A(v) = 0$ has complex roots of unit magnitude when

$$\left|\cos(\theta(v))\right| = \left|\frac{B(v)}{A(v)}\right| < 1, \text{ or } A(v)^2 \pm B(v)^2 > 0.$$

Substituting $A(v)$ and $B(v)$ for these the two-step methods, the interval of periodicity of the classical Obrechkoff method, PL'' and PL''' methods when $\sigma \to 0$ or $v \to 0$ are obtained [0, 25.2004], [0,36.7236] and [0,134.56] , respectively.

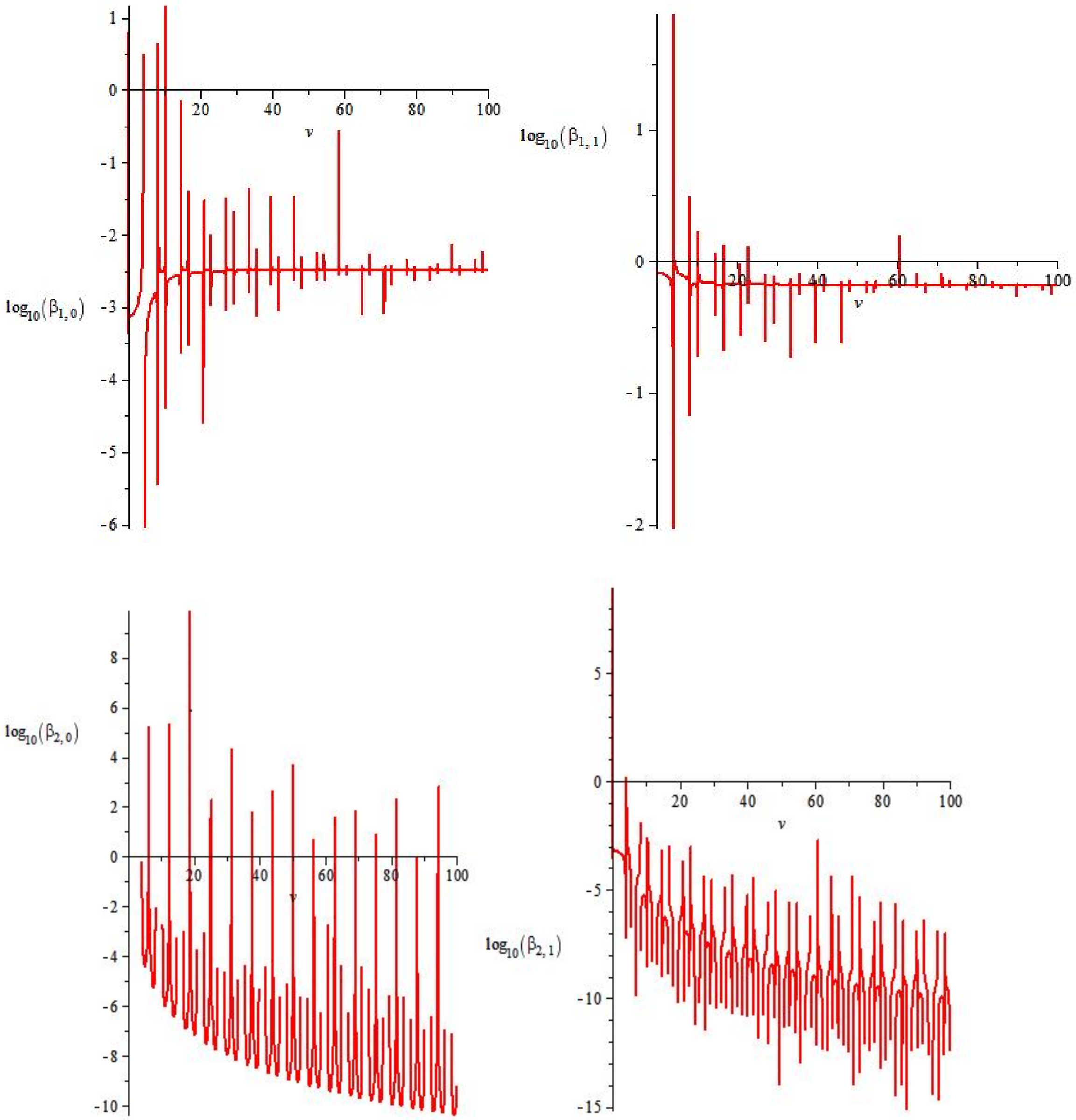

Figure 3. Behavior of the coefficients of the new proposed method PL'' for several values of $\nu = \omega h$.

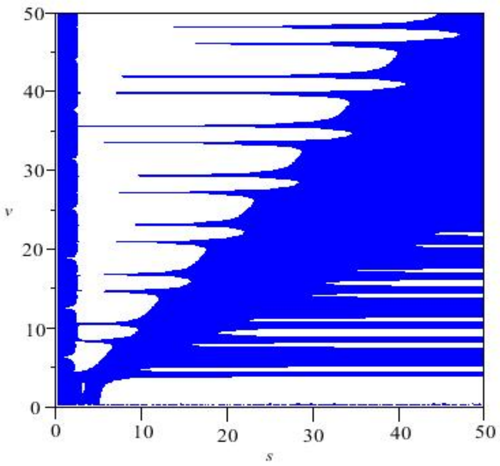

Figure 4. $s-v$ plane of the the new obtained method PL″.

3. NUMERICAL EXAMPLES

In this section, we present some numerical results obtained by our new two-step trigonometrically-fitted Obrechkoff methods and compare them with those from other multistep methods as

Simos: The 12th order Obrechkoff method of Simos [24].
Daele: The 12th order Obrechkoff method of Van Daele [26].
Achar: The 8th order Obrechkoff method of Achar [1].
Wang: The 12th order Obrechkoff method of Wang [30].

Example 3.1. We consider the nonlinear undamped Duffingequation

$$y'' = -y - y^{3+}B\cos(\omega x), \qquad y(0) = 0.200426728067, \quad y'(0) = 0, \tag{12}$$

Where $B = 0.002, \quad \omega = 1.01$ and $x \in \left[0, \frac{40.5\pi}{1.01}\right]$. We use the following exact solution for (12), $g(x) = \sum_{i=0}^{3} K_{2i+1}\cos((2i+1)\omega x)$, where

$$\{K_1, K_3, K_5, K_7\} = \{\ 0.200179477536, 0.246946143\times 10^{-3},$$
$$0.304016\times 10^{-6}, 0.374\times 10^{-9}\}\ .$$

Table 1: Comparison of the end-point absolute error in the approximations obtained by using Methods: the classical method, methods of Simos, Daele, Achar, Wang, zero phase-lag and the new methods for Example 3.1.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar	Wang
$\frac{M}{500}$	1.81e-10	9.31e-11	1.05e-5	1.42e-6	3.15e-4	4.06e-5	4.09e-5	4.08e-5
$\frac{M}{1000}$	8.02e-12	8.03e-12	2.35e-6	1.44e-6	1.81e-5	1.87e-6	1.27e-6	1.27e-6
$\frac{M}{2000}$	5.52e-12	5.52e-12	2.97e-6	1.79e-6	1.08e-6	3.83e-8	3.94e-8	3.93e-8
$\frac{M}{3000}$	7.28e-12	7.28e-12	7.88e-6	2.26e-6	2.09e-7	5.13e-9	5.18e-9	5.17e-9
$\frac{M}{4000}$	6.99e-12	6.99e-12	2.02e-7	1.93e-7	6.55e-8	3.19e-9	1.23e-9	1.23e-9
$\frac{M}{5000}$	6.65e-12	6.65e-12	5.95e-7	2.81e-7	2.67e-8	9.89e-10	4.09e-10	4.07e-10

In order to integrate this equation by a Obrechkoff method, one needs the values of y', which occur in calculating $y^{(4)}$. These higher order derivatives can all be expressed in terms of $y(x)$ and $y'(x)$ through (12), i.e.

$$y^{(3)}(x) \ = \ -\left(1+3y^2(x)\right)y'(x) - B\omega\sin(\omega x),$$

$$y^{(4)}(x) \ = \ -\left(1+3y^2(x)\right)y''(x) - 6y(x)y'(x)^2 - B\omega^2\cos(\omega x),$$

The absolute errors at $x = 40.5\pi/(1.01)$, for the new method, in comparison with methods of classical method, zero phase-lag method, Simos, Daele, Achar, Wang and the new methods are given in table 1 and the CPU times are listed in Table 2.

Example 3.2. Consider the initial value problem $y'' = -100y + 99\sin(x)$, $y(0) = 1, y'(0) = 11$, with the exact solution $y(t) = \sin(t) + \sin(10\,t) + \cos(10\,t)$. This equation has been solved numerical for $0 \le x \le 10\pi$ using exact starting values. In the numerical experiment, we take the step lengths $h = \pi/50, \pi/100, \pi/200, \pi/300, \pi/400$ and $\pi/500$. In Table 3, we present the absolute errors at the end-point and the CPU times are listed in Table 4.

Example 3.3. Consider the initial value problem

$$y'' = \frac{8y^2}{1+2x}, y(0)=1, y'(0)=-2, x \in [0,4.5],$$

Table 2: CPU time for the example 3.1, are calculated for comparison among eight methods: the classical method, methods of Simos, Daele, Achar, Wang, zero phase-lag and the new methods.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar	Wang
$\frac{M}{500}$	1.1	1.1	1.1	1	1.4	1.5	1.2	1.4
$\frac{M}{1000}$	2.2	2.1	2.2	2.2	2.9	2.9	2.3	2.9
$\frac{M}{2000}$	4.3	4.6	4.4	4.4	6.2	6.3	4.8	6.2
$\frac{M}{3000}$	7.2	7.8	7.5	7.5	9.8	9.7	7.5	9.5
$\frac{M}{4000}$	10	10.6	9.8	10	13.5	13.3	10	13
$\frac{M}{5000}$	12	13.3	14	13	17	17	12.9	16.5

Table 3: Comparison of the end-point absolute error in the approximations obtained by using Methods: the classical method, methods of Simos, Daele, Achar, zero phase-lag and the new methods for Example 3.2.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar
$\frac{\pi}{50}$	9.63e-16	1.82e-10	3.03e-06	2.76e-04	3.05e-11	1.20e-11	5.79e-13
$\frac{\pi}{100}$	1.47e-19	1.10e-14	1.15e-08	2.76e-07	2.28e-13	7.34e-13	5.79e-13
$\frac{\pi}{200}$	9.07e-23	7.02e-19	4.50e-11	2.72e-10	4.40e-13	8.62e-13	1.32e-12
$\frac{\pi}{300}$	1.44e-24	2.73e-21	1.76e-12	4.74e-12	2.11e-12	2.63e-12	1.96e-12
$\frac{\pi}{400}$	4.77e-26	5.98e-23	1.76e-13	2.67e-13	1.38e-12	2.93e-12	4.78e-12
$\frac{\pi}{500}$	4.14e-27	3.44e-24	2.95e-14	2.87e-14	6.46e-12	2.89e-12	7.50e-12

Table 4: CPU time for the example 3.2, are calculated for comparison among seven methods: the classical method, methods of Simos, Daele, Achar, zero phase-lag and the new methods.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar
$\frac{\pi}{50}$	0.14	0.23	0.12	0.11	0.17	0.25	0.19
$\frac{\pi}{100}$	0.44	0.37	0.33	0.45	0.51	0.53	0.45
$\frac{\pi}{200}$	0.89	0.87	0.9	0.89	0.86	0.83	0.75
$\frac{\pi}{300}$	1.35	1.3	1.4	1.3	1.14	1.15	0.95
$\frac{\pi}{400}$	1.8	1.8	1.9	1.8	1.39	1.40	1.23
$\frac{\pi}{500}$	2.3	2.3	2.3	2.3	1.70	1.78	1.47

Table 5: Comparison of the end-point absolute error in the approximations obtained by using Methods: the classical method, methods of Simos, Daele, Achar, Wang, zero phase-lag and the new methods for Example 3.3.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	6.61e-14	6.72e-14	1.85e-06	1.52e-06	1.24e-7	1.26e-7	1.26e-7	1.24e-7
$\frac{4.5}{1000}$	4.81e-16	4.88e-16	1.65e-06	1.45e-06	3.82e-9	3.90e-9	3.85e-9	3.82e-9
$\frac{4.5}{2000}$	2.60e-18	2.63e-18	3.94e-07	1.41e-06	1.19e-10	1.23e-10	1.20e-10	1.19e-10
$\frac{4.5}{3000}$	1.13e-19	1.15e-19	3.76e-07	2.67e-07	1.92e-11	2.02e-11	1.40e-11	1.92e-11
$\frac{4.5}{4000}$	1.20e-20	1.21e-20	3.05e-08	2.56e-07	7.85e-12	7.85e-12	2.68e-12	7.85e-12
$\frac{4.5}{5000}$	2.07e-21	2.10e-21	2.71e-08	1.31e-08	1.63e-12	1.63e-12	7.47e-14	1.63e-12

Table 6: CPU time for the example 3.3, are calculated for comparison among eight methods: the classical method, methods of Simos, Daele, Achar, Wang, zero phase-lag and the new methods.

h	PL'''	PL''	Classic	Zero PL	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	0.19	0.19	0.2	0.17	0.369	0.34	0.19	0.31
$\frac{4.5}{1000}$	0.44	0.34	0.37	0.39	0.62	0.61	0.76	1.23
$\frac{4.5}{2000}$	0.83	0.72	0.72	0.75	0.62	0.61	0.76	1.23
$\frac{4.5}{3000}$	1.1	1.1	1	1.1	1.23	1.92	1.20	1.87
$\frac{4.5}{4000}$	1.5	1.4	1.5	1.4	1.89	2.59	1.62	2.56
$\frac{4.5}{5000}$	1.8	1.8	1.8	1.8	2.59	3.29	2.06	3.24

with the exact solution

$$y(x) = \frac{1}{1+2x}.$$

The absolute errors at $x = 4.5$ for the new methods, in comparison with methods of Wang, Simos, Daele, Achar, classical method and zero phase-lag method are given in Table 5. The relative CPU times of computation of the new methods in comparison with the other six our referred methods are given in Table 6.

4. CONCLUSIONS

In this paper, we have presented the new trigonometrically-fitted two-step symmetric Obrechkoff method of order8. The details of the procedure adapted for the applications have been given in Section 2. With trigonometric fitting, we have improved the local truncation error, phase-lag error, interval of periodicity and CPU time for the classes of two-step Obrechkoff methods. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

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