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On Topological Properties of the n–Star Graph

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ABSTRACT

The $n-$star graph S_n is defined on the set of all n sequences $(u_1u_2 \dots u_n)$, $u_i \in \{1,2,\dots,n\}, u_i \neq u_j, i \neq j$, where edges are of the form $(u_1u_2 \dots u_n) \sim (u_iu_2 \dots u_n)$, for some $i \neq 1$. In this paper we will show that S_n is a vertex and edge transitive graph and discuss some topological properties of S_n.

1. INTRODUCTION

Let $\Gamma = (V,E)$ be a connected simple graph. For $u,v \in V$, the length of the shortest path from u to v is denoted by $d(u,v)$. The diameter of Γ is denoted by $D(\Gamma)$ and is defined by $D(\Gamma) = \text{Max}\{d(u,v), u,v \in V\}$.

The n-star graph S_n and its generalization (n,k)-star graph $S_{n,k}$, $1 \leq k < n$, is a popular network for interconnecting processors in a parallel computer. These graphs were proposed in [2] and [1] as an attractive alternative to the hypercube. Let $\Omega = \{1,2,\dots,n\}$. The n-star graph S_n has its vertices all $n!$ permutations of Ω, written in the form $u = (u_1u_2 \dots u_n), u_i \neq u_j, i \neq j, u_i, u_j \in \Omega$. For $\sigma \in S_n$ we define $u^\sigma \coloneqq (u_{(1)\sigma}u_{(2)\sigma} \dots u_{(n)\sigma})$. The edges of S_n are of the form $\{u, u^\sigma\}$, where $\sigma = (1,i), 2 \leq i \leq n$, is a transposition of

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the symmetric group $\mathbb{S}_n$. Therefore S_n is a regular graph of degree $n-1$ with $n!$ vertices and $(n-1)n!/2$ edges.

The $(n,k)-$star graph is a generalization of the n-star graph, where $1 \le k < n$. The vertices of $S_{n,k}$ are all the k permutations $u = (u_1 u_2 \dots u_k)$, where $u_i \ne u_j, i \ne j, u_i \in \{1,2,\dots,n\}$. Therefore $S_{n,k}$ has $n!/(n-k)!$ vertices. $S_{n,k}$ has two types of edges:

i. A star edge, which is an edge between u and u^σ, where $\sigma = (1\ i), 2 \le i \le k$, is a transposition of $\mathbb{S}_n$.
ii. A residual edge, which is an edge between $u = (u_1 u_2 \dots u_k)$ and $v = (u_i u_2 \dots u_k)$, where $u_i \in \Omega - \{u_1, u_2, \dots, u_k\}$. In this way u is joined to $k-1$ vertices in case (i) and to $n-k$ vertices in case (ii), resulting $n-k+k-1 = n-1$ as the degree of a vertex in $S_{n,k}$. Therefore $S_{n,k}$ has $\frac{(n-1)}{2}\frac{n!}{(n-k)!}$ Edges and $S_{n,n-1} = S_n$. By [1] and [2] the diameter of S_n is $\lfloor 3(n-1)/2 \rfloor$.

Note that we have the isomorphisms $S_{n,n-1} \cong S_n$ and $S_{n,1} \cong K_n$, where K_n is the complete graph on n vertices.

In this paper, we will show that S_n is a vertex and edge transitive and in the case of $n = 5$, we find theWiener index of S_n.

2. AUTOMORPHISM GROUP

An automorphism of the graph $\Gamma = (V,E)$ is a permutation $\sigma : V \to V$ such that $\{u,v\}$ is an edge of Γ if and only if $\{u^\sigma, v^\sigma\}$ is an edge of Γ. The set of all the automorphisms of Γ is a subgroup of the symmetric group on V and is denoted by $A = Aut(\Gamma)$. If A acts transitively on V, then Γ is called a vertex- transitive graph and if it acts transitively on the set of edges of Γ then Γ is called an edge-transitive graph. In what follows, we will show that S_n is both a vertex and edge transitive graph.

Proposition 2.1. *The automorphism group of S_n is isomorphic to the symmetric group $\mathbb{S}_n$.*

Proof. Let $\sigma \in \mathbb{S}_n$ and assume that $(u_1 \dots u_i \dots u_n) \sim (u_i \dots u_1 \dots u_n), i \ne 1$, is an edge of S_n. Then $(u_{(1)\sigma} \dots u_{(i)\sigma} \dots u_{(n)\sigma})$ and $(u_{(i)\sigma} \dots u_{(1)\sigma} \dots u_{(n)\sigma})$ is an edge of S_n. Therefore $\mathbb{S}_n$ is a subgroup of $Aut(S_n)$. Conversely, suppose that $\sigma \in Aut(S_n)$, then $\sigma : V \to V$ is a preserving edges permutation of S_n. If $\sigma: (u_1 \dots u_n) \mapsto (u_{1'}, u_{2'}, \dots u_{n'})$, then σ induces the permutation

$$\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1' & 2' & 3' & \dots & n' \end{pmatrix},$$

on the set $\{1,2,\dots,n\}$. Hence $\bar{\sigma} \in \mathbb{S}_n$, and $Aut(S_n) \le \mathbb{S}_n$. This proves that $Aut(S_n) \cong \mathbb{S}_n$.■

Proposition 2.2. *The automorphism group of $S_{n,k}$ is isomorphic to the symmetric group $\mathbb{S}_n$.*
Proof. It is easy to see that $Aut(S_{n,k})$ has a subgroup isomorphic to $\mathbb{S}_n$. Now let $\sigma \in Aut(S_{n,k})$. Then $\sigma : V \to V$ is a permutation and each σ induces a permutation on $\mathbb{S}_n$. Similarly to the proof of Proposition 2.1, we infer that $Aut(S_{n,k}) \leq \mathbb{S}_n$. Thus $Aut(S_n, k) \cong \mathbb{S}_n$. ■

Proposition 2.3. *The groups $Aut(S_n)$ and $Aut(S_{n,k})$ are both vertex and edge- transitive.*

Proof. It is straightforward. ■

3. THE NUMBER OF VERTICES AT CERTAIN DISTANCE FROM EACH OTHER

In this section, we will find the number of vertices at distances $i = 3,4,5$ and 6 from the given vertex $(u_1, \dots, u_n)$ in S_n.

Let $S_{n-1}(i)$ denote the subgraph of S_n induced by all the vertices as the n-tuples ending with the symbol i, $1 \leq i \leq n$. Then $S_{n-1}(i)$ is an $(n-1)$-star graph on the symbols $\{1,2,\dots,n\} - \{i\}$. In S_n, the total number of vertices at distance i from the given vertex $(u_1 u_2 \dots u_n)$ is denoted by $g(n,i)$ and it can be computed recursively as follows:

$$g(n,0) = 1, \ g(n,1) = n-1, \ g(n,2) = (n-1)(n-2)$$

and

$$g(n,i) = (n-1)g(n-1,i-1) + \sum_{j=1}^{n-2} j g(j,i-3) \tag{1}$$

for $n \geq 1, 3 \leq i \leq D(n) = \lfloor 3(n-1)/2 \rfloor$, see [6].

Next, we will find closed formulas for $g(n,3), g(n,4), g(n,5)$ and $g(n,6)$, by using the recursion Eq. 1. Before trying to find these formulas, let us first recall the following well-known and easy to prove equalities which are needed:

$$\sum_{k=1}^{n} k = \frac{n}{2}(n+1),$$
$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1),$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2}{4}(n+1)^2,$$
$$\sum_{k=1}^{n} k^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1).$$

In view of Eq. 1, we have $g(n,3) = (n-1)g(n-1,2) + \sum_{j=1}^{n-2} j g(j,0)$, which is equal to $(n-1)(n-2)(n-3) + \sum_{j=1}^{n-2} j$. By simplifying the latter summation, we consequently get the formula

$$g(n,3) = \frac{1}{2}(n-1)(n-2)(2n-5). \tag{2}$$

For $g(n,4)$, By using Eq. 1, we have $g(n,4) = (n-1)g(n-1,3) + \sum_{j=1}^{n-2} j g(j,1)$. Now by invoking Eq. 2, we may write

$$g(n,4) = \frac{1}{2}(n-1)(n-2)(n-3)(2n-7) + \sum_{j=1}^{n-2} j(j-1).$$

Inasmuch as

$$\begin{aligned}\sum_{j=1}^{n-2} j(j-1) &= \sum_{j=1}^{n-2} j^2 - \sum_{j=1}^{n-2} j \\ &= \frac{1}{6}(n-2)(n-1)(2n-3) - \frac{1}{2}(n-2)(n-1) \\ &= \frac{1}{3}(n-1)(n-2)(n-3),\end{aligned}$$

we infer that $g(n,4) = \frac{1}{2}(n-1)(n-2)(n-3)(2n-7) + \frac{1}{3}(n-1)(n-2)(n-3)$, which in turn implies that

$$g(n,4) = \frac{1}{6}(n-1)(n-2)(n-3)(6n-19). \qquad (3)$$

Next, we proceed to find $g(n,5)$. We have $g(n,5) = (n-1)g(n-1,4) + \sum_{j=1}^{n-2} j g(j,2)$. Now by applying Eq. 3 and replacing $g(j,2)$ by its value, which is $(j-1)(j-2)$, we may write

$$g(n,5) = \frac{1}{6}(n-1)(n-2)(n-3)(n-4)(6n-25) + \sum_{j=1}^{n-2} j(j-1)(j-2).$$

We notice that $j(j-1)(j-2) = j^3 - 3j^2 + 2j$ and

$$\sum_{j=1}^{n-2} j(j-1)(j-2) = \sum_{j=1}^{n-2} j^3 - 3\sum_{j=1}^{n-2} j^2 + 2\sum_{j=1}^{n-2} j.$$

The latter summation equals to

$$\frac{(n-2)^2(n-1)^2}{4} - \frac{3}{6}(n-2)(n-1)(2n-3) + \frac{2}{2}(n-2)(n-1).$$

In consequence, we have

$$g(n,5) = \frac{1}{12}(n-1)(n-2)(12n^3 - 131n^2 + 473n - 564). \qquad (4)$$

Finally, as for the calculation of $g(n,6)$ we may use Eq. 1 again to write:

$$g(n,6) = (n-1)g(n-1,5) + \sum_{j=1}^{n-2} j g(j,3).$$

Consequently,

$$\begin{aligned}g(n,6) &= \frac{1}{12}(n-1)(n-2)(n-3)(12n^3 - 167n^2 + 771n - 1180) \\ &\quad + \frac{1}{2}\sum_{j=1}^{n-2} j(j-1)(j-2)(2j-5).\end{aligned}$$

We should remind the reader that we are making use of the Eqs. 2 and 4, in the preceding equality for $g(n,6)$, to get the latter equality. We also notice that

$$j(j-1)(j-2)(2j-5) = 2j^4 - 11j^3 + 19j^2 - 10j,$$

and in turn

$$\begin{aligned}\sum_{j=1}^{n-2}(2j^4 - 11j^3 + 19j^2 - 10j) &= 2\sum_{j=1}^{n-2} j^4 - 11\sum_{j=1}^{n-2} j^3 + 19\sum_{j=1}^{n-2} j^2 - 10\sum_{j=1}^{n-2} j \\ &= \frac{1}{60}(n-2)(24n^4 - 297n^3 + 1296n^2 - 2283n + 1260).\end{aligned}$$

In consequence, we have

$$\begin{aligned}g(n,6) &= \frac{1}{12}(n-1)(n-2)(n-3)(12n^3 - 167n^2 + 771n - 1180) \\ &\quad + \frac{1}{120}(n-2)(24n^4 - 297n^3 + 1296n^2 - 2283n + 1260),\end{aligned}$$

which finally simplifies to

$$g(n,6) = \frac{1}{120}(n-2)(120n^5 - 2126n^4 + 14453n^3 - 46354n^2 + 68047n - 34140). \quad (5)$$

4. THE WIENER INDEX OF S_5

Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. The Wiener index of G, which is denoted by $W(G)$, is defined by $W(G) = \sum_{u,v\in V} d(u,v)$. For a fixed vertex v, the sum of distances between v and all other vertices of G is denoted by $d(v)$, i.e., $d(v) = \sum_{x\in V} d(v,x)$, then $W(G) = \frac{1}{2}\sum_{v\in V} d(v)$.

The Wiener index, which is proposed by Wiener [8], is one of the oldest descriptors of molecular graph and plays an indispensible role in this context. Among the important works on finding the Wiener index of a general graph, the authors rightly refer the reader to the papers by Gutman et al., see [4, 5]. We should remind the reader that the Wiener index of a graph is generalized in [7].

If the graph $G = (V, E)$ is vertex transitive, then by [3] we have $W(G) = \frac{1}{2}|V|d(v)$, where v is a fixed vertex of G. Here we will compute the Wiener index of the star graph S_5. The diameter of this graph is 6, and because of the vertex transitivity of S_5 the above formula applies. Let v be a fixed vertex of S_5. Noting that $g(n,i)$ is defined as the number of vertices of S_n at distance i from v. Hence, because the diameter of S_5 is 6 we have:

$$d(v) = g(5,1) + 2g(5,2) + \cdots + 6g(5,6).$$

Therefore, by using the formulas in Section 3, for $g(5,i)$ we obtain

$$g(5,1) = 4,\ g(5,2) = 12,\ g(5,3) = 30,\ g(5,4) = 44,\ g(5,5) = 26,\ g(5,6) = 3.$$

Proposition 4.1. *The Wiener index of* S_5 *is* 26520.

Proof. With respect to the above calculation we have $d(v) = 442$ for any fixed vertex v of S_5. Now by our previous observation, we immediately have

$$W(G) = \frac{1}{2}|V|d(v) = \frac{1}{2} \times 5! \times 422 = 26520,$$

hence we are done. ■

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