On Topological Properties of the $n$–Star Graph

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ABSTRACT
The $n$−star graph $S_n$ is defined on the set of all $n$ sequences $(u_1, u_2, ..., u_n)$, $u_i \in \{1, 2, ..., n\}, u_i \neq u_j, i \neq j$, where edges are of the form $(u_1, u_2, ..., u_n) \sim (u_1, u_2, ..., u_n)$, for some $i \neq 1$. In this paper we will show that $S_n$ is a vertex and edge transitive graph and discuss some topological properties of $S_n$.

1. INTRODUCTION

Let $\Gamma = (V,E)$ be a connected simple graph. For $u,v \in V$, the length of the shortest path from $u$ to $v$ is denoted by $d(u,v)$. The diameter of $\Gamma$ is denoted by $D(\Gamma)$ and is defined by $D(\Gamma) = \max\{d(u,v), u,v \in V \}$.

The $n$-star graph $S_n$ and its generalization $(n,k)$-star graph $S_{n,k}, 1 \leq k < n$, is a popular network for interconnecting processors in a parallel computer. These graphs were proposed in [2] and [1] as an attractive alternative to the hypercube. Let $\Omega = \{1, 2, ..., n\}$. The $n$-star graph $S_n$ has its vertices all $n!$ permutations of $\Omega$, written in the form $u = (u_1, u_2, ..., u_n), u_i \neq u_j, i \neq j, u_i, u_j \in \Omega$. For $\sigma \in S_n$ we define $u^\sigma := (u_{(1)}\sigma, u_{(2)}\sigma, ..., u_{(n)}\sigma)$. The edges of $S_n$ are of the form $\{u,u^\sigma\}$, where $\sigma = (1,i), 2 \leq i \leq n$, is a transposition of

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the symmetric group $\mathbb{S}_n$. Therefore $S_n$ is a regular graph of degree $n - 1$ with $n!$ vertices and $(n - 1)n!/2$ edges.

The $(n,k)$-star graph is a generalization of the $n$-star graph, where $1 \leq k < n$. The vertices of $S_{n,k}$ are all the $k$ permutations $u = (u_1 u_2 \ldots u_k)$, where $u_i \neq u_j, i \neq j, u_i \in \{1, 2, \ldots, n\}$. Therefore $S_{n,k}$ has $n!(n-k)!$ vertices. $S_{n,k}$ has two types of edges:

i. A star edge, which is an edge between $u$ and $u^\sigma$, where $\sigma = (1\ i), 2 \leq i \leq k$, is a transposition of $S_n$.

ii. A residual edge, which is an edge between $u = (u_1 u_2 \ldots u_k)$ and $v = (u_1 u_2 \ldots u_k)$, where $u_i \in \Omega - \{u_1, u_2, \ldots, u_k\}$. In this way $u$ is joined to $k - 1$ vertices in case (i) and to $n - k$ vertices in case (ii), resulting $n - k + k - 1 = n - 1$ as the degree of a vertex in $S_{n,k}$. Therefore $S_{n,k}$ has $\frac{(n-1)}{2} \cdot \frac{n!}{(n-k)!}$ Edges and $S_{n,n-1} = S_n$. By [1] and [2] the diameter of $S_n$ is $[3(n-1)/2]$.

Note that we have the isomorphisms $S_{n,n-1} \cong S_n$ and $S_{n,1} \cong K_n$, where $K_n$ is the complete graph on $n$ vertices.

In this paper, we will show that $S_n$ is a vertex and edge transitive and in the case of $n = 5$, we find the Wiener index of $S_n$.

2. AUTOMORPHISM GROUP

An automorphism of the graph $\Gamma = (V, E)$ is a permutation $\sigma : V \to V$ such that $\{u, v\}$ is an edge of $\Gamma$ if and only if $\{u^\sigma, v^\sigma\}$ is an edge of $\Gamma$. The set of all the automorphisms of $\Gamma$ is a subgroup of the symmetric group on $V$ and is denoted by $\text{Aut}(\Gamma)$. If $A$ acts transitively on $V$, then $\Gamma$ is called a vertex- transitive graph and if $A$ acts transitively on the set of edges of $\Gamma$ then $\Gamma$ is called an edge-transitive graph. In what follows, we will show that $S_n$ is both a vertex and edge transitive graph.

Proposition 2.1. The automorphism group of $S_n$ is isomorphic to the symmetric group $\mathbb{S}_n$.

Proof. Let $\sigma \in S_n$ and assume that $(u_1 \ldots u_i \ldots u_n) \sim (u_i \ldots u_1 \ldots u_n), i \neq 1$, is an edge of $S_n$. Then $(u_{(1)}_{\sigma} \ldots u_{(i)}_{\sigma} \ldots u_{(n)}_{\sigma})$ and $(u_{(i)}_{\sigma} \ldots u_{(1)}_{\sigma} \ldots u_{(n)}_{\sigma})$ is an edge of $S_n$. Therefore $S_n$ is a subgroup of $\text{Aut}(S_n)$. Conversely, suppose that $\sigma \in \text{Aut}(S_n)$, then $\sigma : V \to V$ is a preserving edges permutation of $S_n$. If $\sigma : (u_1 \ldots u_n) \mapsto (u_{1'}, u_{2'}, \ldots, u_{n'})$, then $\sigma$ induces the permutation

$$\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1' & 2' & 3' & \ldots & n' \end{pmatrix},$$

on the set $\{1, 2, \ldots, n\}$. Hence $\bar{\sigma} \in S_n$, and $\text{Aut}(S_n) \leq S_n$. This proves that $\text{Aut}(S_n) \cong S_n$.■
Proposition 2.2. The automorphism group of $S_{n,k}$ is isomorphic to the symmetric group $S_n$.

Proof. It is easy to see that $Aut(S_{n,k})$ has a subgroup isomorphic to $S_n$. Now let $\sigma \in Aut(S_{n,k})$. Then $\sigma : V \to V$ is a permutation and each $\sigma$ induces a permutation on $S_n$. Similarly to the proof of Proposition 2.1, we infer that $Aut(S_{n,k}) \leq S_n$. Thus $Aut(S_{n,k}) \cong S_n$.

Proposition 2.3. The groups $Aut(S_n)$ and $Aut(S_{n,k})$ are both vertex and edge-transitive.

Proof. It is straightforward.

3. The Number of Vertices at Certain Distance from Each Other

In this section, we will find the number of vertices at distances $i = 3, 4, 5$ and $6$ from the given vertex $(u_1, \ldots, u_n)$ in $S_n$.

Let $S_{n-1}(i)$ denote the subgraph of $S_n$ induced by all the vertices as the $n$-tuples ending with the symbol $i$, $1 \leq i \leq n$. Then $S_{n-1}(i)$ is an $(n - 1)$-star graph on the symbols $\{1, 2, \ldots, n\} - \{i\}$. In $S_n$, the total number of vertices at distance $i$ from the given vertex $(u_1u_2\ldots u_n)$ is denoted by $g(n,i)$ and it can be computed recursively as follows:

$$g(n,0) = 1, \quad g(n,1) = n - 1, \quad g(n,2) = (n - 1)(n - 2)$$

and

$$g(n,i) = (n - 1)g(n - 1,i - 1) + \sum_{j=1}^{n-2} jg(j,i - 3)$$

for $n \geq 1, 3 \leq i \leq D(n) = \lceil 3(n - 1)/2 \rceil$, see [6].

Next, we will find closed formulas for $g(n,3), g(n,4), g(n,5)$ and $g(n,6)$, by using the recursion Eq. 1. Before trying to find these formulas, let us first recall the following well-known and easy to prove equalities which are needed:

$$\sum_{k=1}^{n} k = \frac{n}{2}(n + 1),$$

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n + 1)(2n + 1),$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2}{4}(n + 1)^2,$$

$$\sum_{k=1}^{n} k^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1).$$

In view of Eq. 1, we have $g(n,3) = (n - 1)g(n - 1,2) + \sum_{j=1}^{n-2} jg(j,0)$, which is equal to $(n - 1)(n - 2)(n - 3) + \sum_{j=1}^{n-2} j$. By simplifying the latter summation, we consequently get the formula

$$g(n,3) = \frac{1}{2}(n - 1)(n - 2)(2n - 5).$$

(2)

For $g(n,4)$, by using Eq. 1, we have $g(n,4) = (n - 1)g(n - 1,3) + \sum_{j=1}^{n-2} jg(j,1)$. Now by invoking Eq. 2, we may write

$$g(n,4) = \frac{1}{2}(n - 1)(n - 2)(n - 3)(2n - 7) + \sum_{j=1}^{n-2} j(j - 1).$$
Inasmuch as
\[ \sum_{j=1}^{n-2} j(j-1) = \sum_{j=1}^{n-2} j^2 - \sum_{j=1}^{n-2} j \]
\[ = \frac{1}{6} (n-2)(n-1)(2n-3) - \frac{1}{2} (n-2)(n-1) \]
\[ = \frac{1}{3} (n-1)(n-2)(n-3), \]
we infer that \( g(n,4) = \frac{1}{2} (n-1)(n-2)(n-3)(2n-7) + \frac{1}{3} (n-1)(n-2)(n-3), \)
which in turn implies that
\[ g(n,4) = \frac{1}{6} (n-1)(n-2)(n-3)(6n-19). \tag{3} \]
Next, we proceed to find \( g(n,5). \) We have \( g(n,5) = (n-1)g(n-1,4) + \sum_{j=1}^{n-2} jg(j,2). \)
Now by applying Eq. 3 and replacing \( g(j,2) \) by its value, which is \( (j-1)(j-2), \) we may write
\[ g(n,5) = \frac{1}{6} (n-1)(n-2)(n-3)(n-4)(6n-25) + \sum_{j=1}^{n-2} j(j-1)(j-2). \]
We notice that \( j(j-1)(j-2) = j^3 - 3j^2 + 2j \)
and
\[ \sum_{j=1}^{n-2} j(j-1)(j-2) = \sum_{j=1}^{n-2} j^3 - 3 \sum_{j=1}^{n-2} j^2 + 2 \sum_{j=1}^{n-2} j. \]
The latter summation equals to
\[ \frac{(n-2)^2(n-1)^2}{4} - \frac{3}{6} (n-2)(n-1)(2n-3) + \frac{2}{2} (n-2)(n-1). \]
In consequence, we have
\[ g(n,5) = \frac{1}{12} (n-1)(n-2)(12n^3 - 131n^2 + 473n - 564). \tag{4} \]
Finally, as for the calculation of \( g(n,6) \) we may use Eq. 1 again to write:
\[ g(n,6) = (n-1)g(n-1,5) + \sum_{j=1}^{n-2} jg(j,3). \]
Consequently,
\[ g(n,6) = \frac{1}{12} (n-1)(n-2)(n-3)(12n^3 - 167n^2 + 771n - 1180) \]
\[ + \frac{1}{2} \sum_{j=1}^{n-2} j(j-1)(j-2)(2j-5). \]
We should remind the reader that we are making use of the Eqs. 2 and 4, in the preceding equality for \( g(n,6), \) to get the latter equality. We also notice that
\[ j(j-1)(j-2)(2j-5) = 2j^4 - 11j^3 + 19j^2 - 10j, \]
and in turn
\[ \sum_{j=1}^{n-2} (2j^4 - 11j^3 + 19j^2 - 10j) = 2 \sum_{j=1}^{n-2} j^4 - 11 \sum_{j=1}^{n-2} j^3 + 19 \sum_{j=1}^{n-2} j^2 - 10 \sum_{j=1}^{n-2} j \]
\[ = \frac{1}{60} (n-2)(24n^4 - 297n^3 + 1296n^2 - 2283n + 1260). \]
In consequence, we have
\[ g(n,6) = \frac{1}{12} (n-1)(n-2)(n-3)(12n^3 - 167n^2 + 771n - 1180) \]
\[ + \frac{1}{120} (n-2)(24n^4 - 297n^3 + 1296n^2 - 2283n + 1260), \]
which finally simplifies to
\[ g(n, 6) = \frac{1}{120}(n - 2)(120n^5 - 2126n^4 + 14453n^3 - 46354n^2 + 68047n - 34140). \quad (5) \]

4. **The Wiener Index of \( S_5 \)**

Let \( G = (V, E) \) be a simple connected graph with vertex set \( V \) and edge set \( E \). The Wiener index of \( G \), which is denoted by \( W(G) \), is defined by \( W(G) = \sum_{u,v \in V} d(u,v) \). For a fixed vertex \( v \), the sum of distances between \( v \) and all other vertices of \( G \) is denoted by \( d(v) \), i.e., \( d(v) = \sum_{x \in V} d(v,x) \), then \( W(G) = \frac{1}{2} \sum_{v \in V} d(v) \).

The Wiener index, which is proposed by Wiener [8], is one of the oldest descriptors of molecular graph and plays an indispensable role in this context. Among the important works on finding the Wiener index of a general graph, the authors rightly refer the reader to the papers by Gutman et al., see [4, 5]. We should remind the reader that the Wiener index of a graph is generalized in [7].

If the graph \( G = (V, E) \) is vertex transitive, then by [3] we have \( W(G) = \frac{1}{2} |V|d(v) \), where \( v \) is a fixed vertex of \( G \). Here we will compute the Wiener index of the star graph \( S_5 \). The diameter of this graph is 6, and because of the vertex transitivity of \( S_5 \) the above formula applies. Let \( v \) be a fixed vertex of \( S_5 \). Noting that \( g(n,i) \) is defined as the number of vertices of \( S_n \) at distance \( i \) from \( v \). Hence, because the diameter of \( S_5 \) is 6 we have:

\[ d(v) = g(5,1) + 2g(5,2) + \cdots + 6g(5,6). \]

Therefore, by using the formulas in Section 3, for \( g(5,i) \) we obtain

\[ g(5,1) = 4, \quad g(5,2) = 12, \quad g(5,3) = 30, \quad g(5,4) = 44, \quad g(5,5) = 26, \quad g(5,6) = 3. \]

**Proposition 4.1.** The Wiener index of \( S_5 \) is 26520.

**Proof.** With respect to the above calculation we have \( d(v) = 442 \) for any fixed vertex \( v \) of \( S_5 \). Now by our previous observation, we immediately have

\[ W(G) = \frac{1}{2} |V|d(v) = \frac{1}{2} \times 5! \times 422 = 26520, \]

hence we are done. \( \blacksquare \)

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