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Kato's Chaos and P–Chaos of a Coupled Lattice System given by García Guirao and Lampart which is Related with Belusov–Zhabotinskii Reaction

RISONG LI•

School of Mathematics and Computer Science, Guangdong Ocean University Zhanjiang, Guangdong, 524025, P. R. China

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ABSTRACT

In (J. Math. Chem., 48: 66-71, 2010) and (J. Math. Chem., 48: 159-164, 2010) García Guirao and Lampart presented the following lattice dynamical system stated by Kaneko in (Phys Rev Lett, 65: 1391-1394, 1990) which is related to the Belusov-Zhabotinskii reaction: $z_v^{u+1} = (1-\eta)\Theta(z_v^u) + \frac{1}{2}\eta[\Theta(z_{v-1}^u) - \Theta(z_{v+1}^u)]$, where u is discrete time index, v is lattice side index with system size M, $\eta\in[0,1]$ is coupling constant and Θ is a continuous selfmap on H. They proved that for the tent map Θ defined as $\Theta(z) = 1 - |1\text{-}2z|$ for any $z\in H$, the above system with η=0 has positive topological entropy and that such a system is Li-Yorke chaotic and Devaney chaotic. In this article, we further consider the above system. In particular, we give a sufficient condition under which the above system is Kato chaotic for η=0 and a necessary condition for the above system to be Kato chaotic for η=0. Moreover, it is deduced that for η=0, if Θ is P-chaotic then so is this system, where a continuous map Θ from a compact metric space Z to itself is said to be P-chaotic if it has the pseudo-orbit-tracing property and the closure of the set of all periodic points for Θ is the space Z. Also, an example and three open problems are presented.

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1. INTRODUCTION

In the whole article, a dynamical system is a pair (Z, Θ) where Z is a compact metric space and $\Theta: Z \to Z$ is a continuous map. Since Li and Yorke [1] gave the first definition of chaos in 1975, topological dynamical systems have been highly explored in the literature [2,3]

•Corresponding Author (Email address: gdoulrs@163.com)
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because these systems may be very good examples of problems coming from the theory of topological dynamics and model many phenomena from various disciplines of engineering and science.

From physical and chemical engineering applications we know that Lattice Dynamical Systems or 1d Spatiotemporal Discrete Systems which are generalizations of classical discrete dynamical systems have recently appeared as an important subject for investigation. We can find the importance of these type of systems [4]. In [5] García Guirao and Lampart presented the following lattice dynamical system which is stated by Kaneko in [6] and is related to the Belusov-Zhabotinskii reaction:

$$z_v^{u+1} = (1-\eta)\Theta(z_v^u) + \frac{1}{2}\eta[\Theta(z_{v-1}^u) - \Theta(z_{v+1}^u)], \tag{1}$$

where u is discrete time index, v is lattice side index with system size M, $\eta \in H = [0,1]$ is coupling constant and Θ is a continuous self map on H. They proved that the above system with $\eta = 0$ and $\Theta = \Lambda$ has positive topological entropy, where Λ is the tent map defined by $\Lambda(z) = 1 - |1-2z|$ for any $x \in [0,1]$. To understand whether a given coupled lattice system has a complicated dynamics or not by the observation of one topological dynamical property is an open problem (see [5]). In [7], by the concept of chaos, the authors characterized the dynamical complexity of a coupled lattice system which is stated by Kaneko in [6] and is related to the Belusov-Zhabotinskii reaction. They got that this system with $\Theta = \Lambda$ is Devaney chaotic and Li-Yorke chaotic for $\eta = 0$, where Λ is the tent map. Also, some problems on the dynamics of this system with $\Theta = \Lambda$ were given by them for $\eta \neq 0$ [5].

Inspired by [5, 7], we will further study the dynamical properties of the above system (1). In particular, it is deduced that for $\eta = 0$, if the above system (1) is Kato chaotic then so is Θ. We also prove that if Θ is Kato chaotic, and if Θ satisfies that for any $\kappa > 0$, if $\left|\Theta^{n_i}(s_{1,i}) - \Theta^{n_i}(s_{2i})\right| < \kappa$ for any $i \in \{1,2,\dots,M\}$ and some integers $n_i > 0$ $(i = 1,2,\dots,M)$ then there exists an integer $l(n_1, n_2, \dots, n_M, \kappa) > 0$ with

$$\left|\Theta^{l(n_1,n_2,\dots,n_M,\kappa)}(s_{1,i}) - \Theta^{l(n_1,n_2,\dots,n_M,\kappa)}(s_{1,i})\right| < \kappa,$$

for any $i \in \{1,2,\dots,M\}$, then, for $\eta = 0$, the system (1) is Kato chaotic. Moreover, we obtain a sufficient condition for the system (1) to be P-chaotic when $\eta = 0$. Also, an example and three open problems are given.

2. PRELIMINARIES

In the whole article, Z is a compact metric space with metric ρ, (Z,Θ) is a dynamical system and $H = [0,1]$. Assume that ρ is the product metric on the product space H^M which is defined as

$$\rho((z_1,z_2,\dots,z_M),(z'_1,z'_2,\dots,z'_M))=(\sum_{k=1}^{M}(z_k-z'_k)^2)^{\frac{1}{2}},$$

for any $(z_1,z_2,\dots,z_M),(z'_1,z'_2,\dots,z'_M)\in H^M$.

Suppose that (Z,ρ) is a metric space. A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is transitive if for any nonempty open subsets $U_1,U_2\subset Z$, $\Theta^k(U_1)\cap U_2\neq\emptyset$ for some integer $\kappa>0$.

A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is topologically mixing if for any nonempty open subsets $U_1,U_2\subset Z$, $\Theta^k(U_1)\cap U_2\neq\emptyset$ for some integer $\kappa>0$ and any integer $\rho>\kappa$.

A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is sensitive if there is a $\kappa>0$ such that for any given $v>0$ and any given $z\in Z$, there is a point $z'\in Z$ with $\rho(z,z')<v$ and

$$\rho\left(\Theta^K(z),\Theta^K(z')\right)>\kappa$$

for some integer $\kappa>0$, where κ is called a sensitivity constant of Θ. A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is accessible if for any $\kappa>0$ and any two nonempty open subsets $U_1,U_2\subset Z$, there are two points $z\in U_1$ and $z'\in U_2$ with

$$\rho\left(\Theta^K(z),\Theta^K(z')\right)<\kappa$$

for some integer $\kappa>0$. A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is chaotic in the sense of Ruelle and Takens [12] if it is transitive and sensitive. A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ is Kato chaotic if it is sensitive and accessible. Note that a topologically mixing dynamic system (Z,Θ) or a topologically mixing map $\Theta: Z\to Z$ is Kato chaotic [13]. In [14] Gu showed that for a continuous self map with a fixed point on a complete metric space without isolated point, the chaoticity in the sense of Ruelle-Takens implies the chaoticity in the sense of Kato, but the converse does not hold in general. This shows that Kato's chaoticity is strictly weaker than the chaoticity in the sense of Ruelle-Takens. From Theorem 12 in [15] we know that if a continuous map $\Theta:[0,1]\to[0,1]$ is topologically chaotic, i.e., has positive topological entropy, then it is chaotic in the sense of Li and Yorke, but the converse is not true. By Theorem 2.1 in [16] if a continuous map $\Theta:[0,1]\to[0,1]$ is sensitive, then it is topologically chaotic. So, Li-Yorke chaos does not imply sensitivity. Consequently, by the definition, Li-Yorke chaos does not imply Kato's chaos. A question arises: does Kato's chaos imply Li-Yorke chaos? To my knowledge the problem is still open.

A dynamic system (Z,Θ) or a map $\Theta: Z\to Z$ with the pseudo-orbit-tracing property [17] is said to be P-chaotic if $\overline{Per(\Theta)}=Z$, where $Per(\Theta)$ is the set of all periodic points of Θ, and $\overline{A}$ is the closure of A.

Let $\Theta: Z\to Z$ be a continuous selfmap on a compact metric space (Z,ρ). A sequence $\{u_j: j\geq 0\}\subset Z$ is said to be a η-pseudo-orbit for Θ [17,18] if $\rho(\Theta(u_j),u_{j+1})<\eta$ for any integer $j \geq 0$. For a fixed $\lambda > 0$, a given sequence $\{u_j : j \geq 0\} \subset Z$ is said to be λ-traced by $u \in Z$ [17,18] if $\rho(\Theta^j(u), u_j) < \lambda$ for any integer $j \geq 0$. A self map $\Theta: Z \to Z$ is called to have the pseudo-orbit-tracing property [17,18] if for any given $\lambda > 0$ there is $\eta > 0$ such that any η-pseudo-orbit for Θ can be λ-traced by some point of Z.

The state space of the system (1) is the set

$$Z = \{z: z = \{z_k\}, z \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|z_k\| < \infty\},$$

where $a \geq 1$ is the dimension of the range space of the map of state z_k, $b \geq 1$ is the dimension of the lattice and the l^2 norm

$$\|z\|_2 = (\sum_{k \in z^b} |z_k|^2)^{\frac{1}{2}},$$

is usually taken ($|z_k|$ is the length of the vector z_k), see [5, 7].

We will continue to explore the above system (1) which is stated by Kaneko in [6] and is related to the Belusov-Zhabotinskii reaction (for this point one can refer to [8], and for experimental study of chemical turbulence by this method one can find in [9−11]).

In general, one always supposes that one of the following periodic boundary conditions of the system (1) holds:

1. $z_v^u = Z_{v+M}^u$,
2. $z_v^u = Z_v^{u+M}$,
3. $z_v^u = Z_{v+M}^{u+M}$,

standardly, we assume that the first case of the boundary conditions is satisfied.

3. MAIN RESULTS

Motivated by the results in [5, 7] we have the following result.

Theorem 3.1. For zero coupling constant, a necessary condition for the system (1) to be Kato chaotic is that Θ is Kato chaotic.

Proof. Clearly, the system (1) is equivalent to the system $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ for $\eta = 0$ where $\Theta_k = \Theta$ for every $k \in \{1, 2, \dots, M\}$. It is easily seen that the system $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ is sensitive if and only if so is Θ, where $\Theta_k = \Theta$ for every $k \in \{1,2,\dots,M\}$. By the definition one can easily verify that if the system $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ is accessible then so is Θ_k for every $k \in \{1,2,\dots,M\}$. Consequently, by hypothesis and the definition one can easily see that Theorem 3.1 is true. ∎

The following theorem is from [13]. For completeness, we give its proof.

Theorem 3.2. If Θ is topologically mixing on a metric space (Z, ρ), then it is Kato chaotic.

Proof. It is well known that any topologically mixing map is sensitive. As Θ is topologically mixing, it is sensitive. Let $a \in Z$ be a given point. Then, for any given $\varepsilon > 0$ and any given nonempty open sets $U, V \subset Z$, by the topological mixing of Θ, there are an integer $n > 0$, $u \in U$ and $v \in V$ such that $\Theta^n(u), \Theta^n(v) \in B(a, \frac{1}{2}\varepsilon)$, where

$$B(a, \tfrac{1}{2}\varepsilon) =. \left\{b \in Z: \rho(a, b) < \tfrac{1}{2}\varepsilon\right\}.$$

This implies that

$$\rho\big(\Theta^n(u), \Theta^n(v)\big) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

So, by the definition, Θ is accessible. Consequently, by the definition, Θ is Kato chaotic. ■

Theorem 3.3. Let Θ be a topologically transitive continuous map on [0,1]. Then Θ is Kato chaotic.

Proof. It is well known that if Θ is a topologically transitive continuous map on [0,1], then one of the following holds:

1. Θ is topologically mixing.
2. There is $a \in (0,1)$ such that $\Theta^2|_{[0,a]}$and $\Theta^2|_{[a,1]}$ are topologically mixing.

If Θ is topologically mixing, by Theorem 3.2 we know that Θ is Kato chaotic. If there is $a \in (0,1)$ such that $\Theta^2|_{[0,a]}$and $\Theta^2|_{[a,1]}$ are topologically mixing, by Theorem 3.2 we know that $\Theta^2|_{[0,a]}$and $\Theta^2|_{[a,1]}$ are Kato chaotic. As a topologically transitive continuous maps on [0,1] is sensitive, by the definition it is enough to show that Θ is accessible. For any $\kappa > 0$ and any nonempty open sets $[U, V] \subset [0,1]$, by the topological mixing of $\Theta^2|_{[0,a]}$and $\Theta^2|_{[a,1]}$ there are a positive integer $m > 0$, $u \in U$ and $v \in V$ such that $\Theta^m(u) \in B(a, \frac{1}{2}\kappa)$ and $\Theta^m(v) \in B(a, \frac{1}{2}\kappa)$. This implies that $|\Theta^m(u) - \Theta^m(v)| < \frac{1}{2}\kappa + \frac{1}{2}\kappa = \kappa$. So, by the definition and the above argument, Θ is accessible. Consequently, by the definition and the above argument, Θ is Kato chaotic. ■

Lemma 3.1 [16]. Let Θ be a sensitive continuous map on $[0,1]$. Then Θ is topologically chaotic.

Lemma 3.2 [15]. Let Θ be a continuous maps on $[0,1]$. Then Θ is topologically chaotic if and only if Θ is Devaney chaotic.

Theorem 3.4. Let Θ be a continuous map on $[0,1]$. Then Θ is Kato chaotic if and only if Θ is Devaney chaotic.

Proof. By the definitions, Lemma 3.1 and Lemma 3.2, Theorem 3.4 holds. For a continuous map Θ on $[0,1]$, we do not know whether the chaoticity of Θ in the sense of Kato implies the same property of the product map $\Theta \times \Theta$. However, we have the following result. ■

Theorem 3.5. Suppose that Θ is Kato chaotic, and that Θ satisfies that for any $\kappa > 0$, if

$$\left|\Theta^{n_i}\left(S_{1,i}\right) - \Theta^{n_i}\left(S_{2,i}\right)\right| < \kappa,$$

for any $i \in \{1,2,\ldots,M\}$ and some integers $n_i > 0$ ($i = 1,2,\ldots,M$) then there exists an integer $l(n_1, n_2, \ldots, n_M, \kappa) > 0$ with

$$\left|\Theta^{l(n_1,n_2,\ldots,n_M,\kappa)}\left(S_{1,i}\right) - \Theta^{l(n_1,n_2,\ldots,n_M,\kappa)}\left(S_{1,i}\right)\right| < \kappa,$$

for any $i \in \{1,2,\ldots,M\}$, then, for $\eta = 0$, the system (1) is Kato chaotic.

Proof. For $\eta = 0$, it is obvious that the system (1) is equivalent to the system $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ where $\Theta_k = \Theta$ for every $k \in \{1,2,\ldots,M\}$. By hypothesis and [14] we have that $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ is sensitive where $\Theta_k = \Theta$ for every $k \in \{1,2,\ldots,M\}$. By hypothesis and the definition one easily show that $\Theta_1 \times \Theta_2 \times \cdots \times \Theta_M$ is accessible where $\Theta_k = \Theta$ for every $k \in \{1,2,\ldots,M\}$. By the definition, $\Theta_1 \times \Theta_2 \times \cdots \times \Theta_M$ is Kato chaotic where $\Theta_k = \Theta$ for every $k \in \{1,2,\ldots,M\}$. ■

Example 3.1. Suppose that $\Theta = \Lambda$ is the tent map. Then the system (1) is Kato chaotic for $\eta = 0$. As the tent map $\Theta = \Lambda$ is topologically mixing, $\Theta_1 \times \Theta_2 \times \cdots \times \Theta_M$ is topologically mixing, where $\Theta_k = \Lambda$ for every $k \in \{1,2,\ldots,M\}$. As the system (1) is equivalent to the system $(H^M, \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M)$ for $\eta = 0$ where $\Theta_k = \Lambda$ for very $k \in \{1,2,\ldots,M\}$, by [13] or Theorem 3.2 the system (1) is Kato chaotic.

Remark 3.1. Theorem 3.5 shows that study on Kato's chaocity of the system (1) is very difficult. Especially, for $\eta \neq 0$, study on Kato's chaocity of the system (1) is rather difficult.

We have the following open problem.

Problem 3.1. For the above system (1) with any coupling constant $\eta \in (0,1]$, is the result of Theorem 3.5 valid?

The proof of following lemma is easy. For completeness, we give its proof here.

Lemma 3.3. Let $\Theta: Z \to Z$ be a continuous map on a metric space. Then $\Theta \times \Theta$ has the pseudo-orbit-tracing property if and only if Θ has the pseudo-orbit-tracing property.

Proof. Assume that $\Theta \times \Theta$ has the pseudo-orbit-tracing property. For any given $\lambda > 0$, we let $\{u_j: j \geq 0\} \subset Z$ be a η-pseudo-orbit for Θ. Clearly, $\{(u_j, u_j): j \geq 0\} \subset Z \times Z$ be a η-pseudo-orbit for $\Theta \times \Theta$. By the definition, there exists $(u,v) \in Z \times Z$ with

$$\rho'((\Theta \times \Theta)^j (u,v)(u_j, u_j)) < \eta,$$

for any integer $j \geq 0$, where $\rho'$is the usual product metric on the product metric space $Z \times Z$. This means that $\rho\left(\Theta^j(u), u_j\right) < \eta$ for any integer $j \geq 0$. So, Θ has the pseudo-orbit-tracing property. The proof of the converse is similar and is omitted. ■

Theorem 3.6. For $\eta = 0$, if Θ is P-chaotic, then so is the system (1).

Proof. Clearly, for $\eta = 0$, the system (1) is equivalent to the system (H^M, F), where $F = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M$ and $\Theta_i = \Theta$ for any $i \in \{1,2,\ldots,M\}$. Since

$$Per(F) = Per(\Theta) \times Per(\Theta) \times \cdots \times Per(\Theta),$$

by hypothesis, the definition and Lemma 3.3, F is P-chaotic. ■

Problem 3.2. For $\eta > 0$ and any given P-chaotic continuous self map Θ on $[0,1]$, is the system (1) P-chaotic?

One can easily extend the system (1) to the following non-autonomous discrete system:

$$z_{v,i}^{u+1} = (1-\eta)\Theta\left(z_{v,i}^{u}\right) + \frac{1}{2}\eta\left[\Theta\left(z_{v-1,i}^{u}\right) - \Theta\left(z_{v+1,i}^{u}\right)\right], \tag{2}$$

for any integer $i \geq 1$ where u is discrete time index, v is lattice side index with system size M, $\eta \in H = [0,1]$ is coupling constant and Θ_i is a continuous selfmap on H for any integer $i \geq 1$.

Inspired by [19], we pose the following problem.

Problem 3.3. The dynamical properties of the system (1) can be extended to the non-autonomous discrete system (2)?

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