On the Revised Edge-Szeged Index of Graphs

HECHAO LIU¹, LIHUA YOU² AND ZIKAI TANG¹,*

¹Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), College of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, P. R. China
²School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China

ARTICLE INFO

Article History:
Received: 3 September 2019
Accepted: 12 November 2019
Published online: 30 December 2019
Academic Editor: Ivan Gutman

Keywords:
Revised edge-Szeged index
Conjugated unicyclic graph
Join graph

ABSTRACT

The revised edge-Szeged index of a connected graph $G$ is defined as $Sz^*_r(G) = \sum_{e=uv \in E(G)} \left( m_u(e|G) + \frac{m_u(e|G)}{2} \right) \left( m_v(e|G) + \frac{m_v(e|G)}{2} \right)$, where $m_u(e|G)$, $m_v(e|G)$ and $m_{uv}(e|G)$ are, respectively, the number of edges of $G$ lying closer to vertex $u$ than to vertex $v$, the number of edges of $G$ lying closer to vertex $v$ than to vertex $u$, and the number of edges equidistant to $u$ and $v$. In this paper, we give an effective method for computing the revised edge-Szeged index of unicyclic graphs and using this result we identify the minimum revised edge-Szeged index of conjugated unicyclic graphs which is defined as the unicyclic graphs with a perfect matching. We also give a method of calculating revised edge-Szeged index of the joint graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple, and refer to [2] for notations and terminologies used but not defined here.

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ ($N(v)$ for short) denote the set of all the adjacent vertices of $v$ in $G$ and $d_G(v) = |N_G(v)|$, the degree of $v$ in $G$. Let $w \in N_G(u)$, $d^2_G(u) = \sum_{w \in N_G(u)} d(w)$. Denote $t_G(u)$ the number of triangles in graph $G$ that contain the vertex $u$. Call $u$ a pendant vertex of $G$, if $d_G(u) = 1$ and $uv$ a pendant edge of $G$, if one of its endpoints is a pendant vertex. Denote by $PV$ the set of pendant vertices of $G$. The distance, $d(u,v|G)$ (or $d(u,v)$ for short), between vertices $u$ and $v$ of $G$ is the length of the shortest $u,v$ path in $G$. Let $D(u|G) = \sum_{v \in V(G)} d(u,v|G)$ and $G - uv$, $G + uv$ denote the graph obtained from $G$ by deleting the edge of $uv$ and adding an edge between $u$ and $v$, respectively. An edge $e$ is called a cut

*Corresponding Author (Email address: zikaitang@163.com)
DOI: 10.22052/ijmc.2019.200349.1460
edge of a connected graph $G$ if $G - e$ is disconnect. Let $P_n$, $C_n$, $S_n$ and $K_n$ be the path, cycle, star and complete graph of order $n$, respectively.

A matching $M$ in a graph $G$ is a set of edges of $G$ such that no two edges from $M$ share a vertex. If every vertex of $G$ is incident with an edge of $M$, the matching $M$ is called a perfect matching.

In chemical graph theory, graph invariants are numbers related to graphs with invariant structure. These invariants are also called topological indices. Topological indices provide correlations with physical, chemical and thermodynamic parameters of chemical compounds, see [17, 18, 26]. Among all the topological indices, the most well-known is the Wiener index [29], which is defined as the sum of distances over all unordered vertex pairs in $G$, namely, $W(G) = \sum_{(u, v) \in V(G)} d_G(u, v)$.

A long time known property of the Wiener index is the formula [29]

$$W(G) = \sum_{e=uv \in E(G)} n_u(e|G)n_v(e|G),$$

where $n_u(e|G)$ and $n_v(e|G)$ are, respectively, the number of vertices of $G$ lying closer to vertex $u$ than to vertex $v$ and the number of vertices of $G$ lying closer to vertex $v$ than to vertex $u$. It is applicable for trees. Using the above formula, another topological index related, named by Szeged index, was introduced by Gutman [13], which is an extension of the Wiener index and defined by $Sz(G) = \sum_{e=uv \in E(G)} n_u(e|G)n_v(e|G)$. In addition, some properties and applications of Wiener index and Szeged index have been investigated [1, 5-7, 11, 12, 15, 16, 19, 20, 22, 24, 26, 29, 31, 32].

Given an edge $e = uv \in E(G)$, the distance between the vertex $x$ and the edge $e$, denoted by $d(x, e)$, is defined as $d(x, e) = \min\{d(x, u), d(x, v)\}$. Denote $M_u(e|G) = \{e \in E(G) : d(u, e) < d(v, e)\}$, $M_v(e|G) = \{e \in E(G) : d(v, e) < d(u, e)\}$ and $M_0(e|G) = \{e \in E(G) : d(u, e) = d(v, e)\}$. Let $m = |E(G)|$, $m_u(e|G) = |M_u(e|G)|$, $m_v(e|G) = |M_v(e|G)|$ and $m_0(e|G) = |M_0(e|G)|$, we have $m_u(e|G) + m_v(e|G) + m_0(e|G) = m$. Then, the edge-Szeged index [14] and revised edge-Szeged index [8] of $G$ are defined as

$$Sz_0(G) = \sum_{e=uv \in E(G)} m_u(e|G)m_v(e|G),$$

$$Sz_0'(G) = \sum_{e=uv \in E(G)} (m_u(e|G) + \frac{m_0(e|G)}{2})(m_v(e|G) + \frac{m_0(e|G)}{2}).$$

For the sake of simplicity, we consider the contribution $\phi(e)$ of an edge $e = uv$ defined as

$$\phi(e) = (m_u(e|G) + \frac{m_0(e|G)}{2})(m_v(e|G) + \frac{m_0(e|G)}{2}).$$

Up until now, much work has been done on revised edge-Szeged index. Faghami and Ashrafi [12] computed an exact formula for the revised edge-Szeged index of Cartesian product of graphs. Liu and Wang [23] gave a lower bound of the edge revised Szeged index among all $m$-edges cactus graphs with $k$ cycles. Wang et al. [28] characterized the $n$-vertex unicyclic graphs with a given diameter having the minimum edge-Szeged index. They used a unified approach to identify the $n$-vertex unicyclic graphs with the minimum, the second minimum, the third minimum and the fourth minimum edge-Szeged indices. Other results
see [3, 4, 9, 21], and the references cited therein.

In this paper, we give an effective method for computing the revised edge-Szeged index of unicyclic graphs and identify the minimum revised edge-Szeged index of conjugated unicyclic graphs. And we also give a method of calculating revised edge Szeged index of the joint graph.

2. REVISED EDGE-SZEGED INDEX OF CONJUGATED UNICYCLIC GRAPHS

For nonnegative integer $\beta \geq 2$, let $T_{2\beta,\beta}$ (see Figure 1) be the tree obtained by attaching a pendent edge to each of some $\beta - 1$ non-central vertices of the star $S_{\beta+1}$. Let $T_{2\beta+1,\beta}$ (see Figure 1) be the tree obtained by attaching a pendent edge to each of some $\beta$ noncentral vertices of the star $S_{\beta+1}$. Let $T(2\beta, \beta)$ and $U(2\beta, \beta)$ denote the set of conjugated trees (trees with a perfect matching) and conjugated unicyclic graphs (unicyclic graphs with a perfect matching) of order $2\beta$, respectively, where $\beta$ is the number of matchings in $G$.

First, we introduce some lemmas that will be useful in the proof of the main result.

Figure 1. The graphs $T_{2\beta,\beta}$ and $T_{2\beta+1,\beta}$.

**Lemma 2.1.** [25] Let $G$ be a graph of order $2\beta$ with a perfect matching. If $PV \neq \emptyset$, then for any vertex $u \in V(G)$, $|N(u) \cap PV| \leq 1$.

**Lemma 2.2.** [8] Let $G$ be a unicyclic graph with $n$ vertices. Then $Sz^*_e(G) \leq \frac{n^3}{4}$ with equality if and only if $G$ is the cycle $C_n$.

From Lemma 2.2, it is obvious that $Sz^*_e(U(2\beta, \beta)) \leq 2\beta^3$ with equality if and only if $U(2\beta, \beta)$ is the cycle $C_{2\beta}$. Therefore, in the following, we only need to consider the lower bound of revised edge-Szeged index of conjugated unicyclic graphs. First, we introduce some useful graph transformations.
Lemma 2.3. (The edge-lifting transformation) Let $G$ be a connected graph which obtained from a connected graph $G_1(u \in V(G_1), |G_1| \geq 2)$ and a tree $T_1(v \in V(T_1), |T_1| \geq 3)$ by adding an edge $e = uv$ and with perfect matchings.

(i) If $e = uv \in M$ ($M$ is a perfect matching), let $G'$ (see Figure 2(i)) be the graph obtained from $G$ by deleting $e$ from $G$, identifying $u$ and $v$ into a new vertex $x$ and adding a vertex $y$ connected to $x$. Let the edge connecting $x$ and $y$ in $G'$ be again denoted by $e$. Then $Sz_G^*(G) > Sz_G^*(G')$.

(ii) If $e = uv \notin M$ ($M$ is a perfect matching), there exists a cut edge $e_1 = vw \in M$ in $T_1$. Then, obviously, $T_1$ (see Figure 2(i)) can be seen as the graph obtained from a tree $T_2$ and a tree $T_3$ by adding an edge between a vertex $w$ of $T_2$ and a vertex $v$ of $T_2$. Let $G'$ (see Figure 2(ii)) be the graph obtained from $G$ by deleting $e$ and $e_1$ from $G$, identifying $u$, $v$, and $w$ into a new vertex $x$ and adding an edge $yz$ connected to $x$. Let the edge connecting $x$ and $y$, $y$ and $z$ in $G'$ be again denoted by $e$ and $e_1$. Then $Sz_G^*(G) > Sz_G^*(G')$.

Proof. Note that $G'$ is a graph with perfect matchings, since $G$ is a graph with perfect matchings.

(i) Observe that after the modification of the graph, for every edge $f$, distinct from $e$, the contribution $\phi(f)$ stays unchanged. For edge $e$, we have that $\phi_G(e) = \frac{1}{2}(|E(G')| - \frac{1}{2})$

$$\phi_G(e) = \left(\frac{1}{2}(|E(G_1)| + \frac{1}{2})\right) \left(|E(T_1)| + \frac{1}{2}\right) = |E(G_1)| |E(T_1)| + \frac{1}{2}(|E(G_1)| + |E(T_1)|) + \frac{1}{4}$$

$$\geq (|E(G')| - 2) + \frac{1}{2}(|E(G')| - 1) + \frac{1}{4} = \frac{3}{2}|E(G')| - \frac{9}{4} > \phi_G(e).$$

Thus $Sz_G^*(G) > Sz_G^*(G')$.

(ii) Observe that after the modification of the graph, for every edge $f$, distinct from $e$ and $e_1$, the contribution $\phi(f)$ stays unchanged. For edge $e$ and $e_1$ we have that
On the Revised Edge–Szeged Index of Graph

283

\[ \phi_{G'}(e) = \frac{3}{2} \left( |E(G')| - 2 + \frac{1}{2} \right) = \frac{3}{2} |E(G')| - \frac{9}{4}, \]
\[ \phi_{G'}(e_1) = \frac{1}{2} \left( |E(G')| - 1 + \frac{1}{2} \right) = \frac{1}{2} |E(G')| - \frac{1}{4}, \]
\[ \phi_G(e) = \left( |E(G_1)| + \frac{1}{2} \right) \left( |E(T_2)| + |E(T_3)| + \frac{3}{2} \right) \]
\[ = |E(G_1)| \left( |E(T_2)| + |E(T_3)| + 1 \right) + \frac{1}{2} \left( |E(G_1)| + |E(T_2)| + |E(T_3)| + 1 \right) + \frac{1}{4} \]
\[ \geq |E(G')| - 2 + \frac{1}{2} \left( |E(G')| - 1 \right) + \frac{1}{4} = \frac{3}{2} |E(G')| - \frac{9}{4}, \]
\[ \phi_G(e_1) = \left( |E(T_3)| + \frac{1}{2} \right) \left( |E(T_2)| + |E(G_1)| + \frac{3}{2} \right) \]
\[ = |E(T_3)| \left( |E(T_2)| + |E(G_1)| + 1 \right) + \frac{1}{2} \left( |E(T_3)| + |E(T_2)| + |E(G_1)| + 1 \right) + \frac{1}{4} \]
\[ \geq |E(G')| - 2 + \frac{1}{2} \left( |E(G')| - 1 \right) + \frac{1}{4} = \frac{3}{2} |E(G')| - \frac{9}{4}. \]

Then \( \phi_G(e) + \phi_G(e_1) > \phi_{G'}(e) + \phi_{G'}(e_1). \) Thus \( Sz_e^*(G) > Sz_e^*(G') \) and the proof is completed.

By Lemma 2.3, we have the following result.

**Lemma 2.4.** Let \( G \in \mathcal{T}(2\beta, \beta) \) where \( \beta \geq 2. \) Then \( Sz_e^*(G) \geq 4\beta^2 - \frac{15}{2} \beta + \frac{15}{4} \) with equality if and only if \( G \equiv \mathcal{T}(2\beta, \beta). \)

Let \( g \geq 3 \) be an integer, and let \( C_g = v_1 v_2 \cdots v_g v_1 \) be a cycle on \( g \) vertices. Let \( T_1, T_2, \ldots, T_g \) be vertex-disjoint trees, and let the root vertex of \( T_i \) be \( v_i \) for \( 1 \leq i \leq g. \)

Denote by \( C_g(T_1, T_2, \ldots, T_g) \) the unicyclic graph obtained from of \( C_g \) by identifying the root vertex \( u_i \) of \( T_i \) with \( v_i \) for \( 1 \leq i \leq g. \) Any unicyclic graph \( G \) with a \( g \)-cycle can be denoted by the form \( C_g(T_1, T_2, \ldots, T_g) \), where \( |T_i| = t_i \) (\( i = 1, 2, \ldots, g \)) and \( \sum_{i=1}^{g} t_i = n. \)

By Lemma 2.3, we can repeat the edge-lifting transformation to the unicyclic graphs \( C_g(T_1, T_2, \ldots, T_g) \) and we have

**Lemma 2.5.** If \( C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(2\beta, \beta), \) then

\[ Sz_e^*(C_g(T_1, T_2, \ldots, T_g)) \geq Sz_e^*(C_g(T_1', T_2', \ldots, T_g')) \]

with equality if and only if \( T_k \cong T_k', \) for all \( k \) \( (1 \leq k \leq g). \) where \( |T_k'| = |T_k| = t_k. \)

\( T_k' \cong T_2 \beta_k \) if \( t_k = 2\beta_k + 1 \) and \( T_k' \cong T_2 \beta_k + 1 \) if \( t_k = 2\beta_k + 1. \)

In the following, we give an effective method for computing the revised edge-Szeged index among unicyclic graphs \( G = C_g(T_1, T_2, \ldots, T_g). \)

**Theorem 2.6.** If \( G = C_g(T_1, T_2, \ldots, T_g), \) then

\[ Sz_e^*(G) = \sum_{i=1}^{g} W(T_i) + \sum_{i=1}^{g} (|G| - |T_i| + 1) D(v_i|T_i) \sum_{i=1}^{g} \sum_{j=1}^{g} |T_i| |T_j| d(v_i, v_j|C_g) \]
\[ - \delta(g) (\sum_{i<j} |T_i| |T_j| + \frac{1}{4} \sum_{i=1}^{g} |T_i|^2) - \frac{1}{2} \delta(g) - 1 |G|^2 + \frac{1}{2} (2 |G| + 1) g - \frac{1}{4} |G|. \]
where $\delta(g) = 0$ for even $g$, $\delta(g) = 1$ for odd $g$.

**Proof.** We divide the edge of $G = C_g(T_1, T_2, \ldots, T_g)$ into the following groups:

(a) the edges belonging to the tree $T_i$, $i = 1, 2, \ldots, g$; (b) the edges belonging to the cycle $C_g$. For the edge $e = uv \in E(T_i)$, we assume that $d(u, v_i | T_i) > d(v, v_i | T_i)$ for $i = 1, 2, \ldots, g$. For any vertex $w \in V(T_i)$, it is counted $d(w, v_i | T_i)$ times in the sum $\sum_{e \in E(T_i)} n_u(e | T_i)$ for the edges in the path from $w$ to $v_i$. Thus $\sum_{e \in E(T_i)} n_u(e | T_i) = D(v_i | T_i)$ for $i = 1, 2, \ldots, g$, see [15]. Note that $m_u(e | T_i) = n_u(e | T_i) - 1$ and $m_v(e | T_i) = n_v(e | T_i) - 1$. The contributions to $S_{z_i}^+(G)$ pertaining to the edges of type (a) are

$A = \Sigma_{i=1}^g \sum_{e \in E(T_i)} \left( m_u(e | G) + \frac{m_0(e | G)}{2} \right) \left( m_v(e | G) + \frac{m_0(e | G)}{2} \right)$

$B = \sum_{e \in E(C_g)} \left( m_u(e | G) + \frac{m_0(e | G)}{2} \right) \left( m_v(e | G) + \frac{m_0(e | G)}{2} \right)$

For the edge $e \in E(T_i)$ pertaining to the edges of type (b), we obviously,

$m_u(e | G) = n_u(e | G) - 1$ and $m_v(e | G) = n_v(e | G) - 1$.

The contributions to $S_{z_i}^+(G)$ pertaining to the edges of type (b) are

$B = \sum_{e \in E(C_g)} \left( m_u(e | G) + \frac{m_0(e | G)}{2} \right) \left( m_v(e | G) + \frac{m_0(e | G)}{2} \right)$

$A = \sum_{j=1}^g \sum_{T_j \not\subset T_i} d(v_i, v_j | C_g)$.

If $g$ is even, then obviously, $m_u(e | G) = n_u(e | G) - 1$ and $m_v(e | G) = n_v(e | G) - 1$. The contributions to $S_{z_i}^+(G)$ pertaining to the edges of type (b) are

$B = \sum_{e \in E(C_g)} \left( m_u(e | G) + \frac{m_0(e | G)}{2} \right) \left( m_v(e | G) + \frac{m_0(e | G)}{2} \right)$

As $S_{z_i}^+(G) = A + B$, the result follows easily.

**Lemma 2.7.** (The branch transformation) Let $G = C_g(T_1, T_2, \ldots, T_k, \ldots, T_l, \ldots, T_g) \in U(2\beta, \beta)$ with its unique cycle $C_g = v_1v_2 \cdots v_g$ and $N_i = \Sigma_{j=1}^g t_j d(v_i, v_j | C_g)$, where $v_i, v_j \in C_g$ and $|T_j| = t_j$. Suppose that there exist a path $v_k w u$ with root vertex $v_k$ in $T_k$.

Let $G' = G - v_k w + v_i w$, Figure 3. If $(N_k + \frac{\delta(g)}{4} t_k) \geq (N_l + \frac{\delta(g)}{4} t_l)(1 \leq k < l \leq g)$. Then $S_{z_i}^+(G) > S_{z_i}^+(G')$. 

\[\square\]
**Figure 3.** The branch transformation of Lemma 2.7.

**Proof.** Note that \( t'_k = |T'_k| = |T_k| - 2 = t_k - 2 \) and \( t'_i = |T'_i| = |T_i| + 2 = t_i + 2 \), by Theorem 2.6, we have that

\[
S_{e}^*(G) - S_{e}^*(G') = 4 \left[ \left( N_k + \frac{\delta(g)}{4} t_k \right) - \left( N_i + \frac{\delta(g)}{4} t_i \right) \right] + 8 d(v_k, v_i|C_g) - 2 \delta(g) > 4 \left[ \left( N_k + \frac{\delta(g)}{4} t_k \right) - \left( N_i + \frac{\delta(g)}{4} t_i \right) \right] \geq 0.
\]

Hence the proof is completed.

As \( \left( N'_k + \frac{\delta(g)}{4} t'_k \right) - \left( N'_i + \frac{\delta(g)}{4} t'_i \right) = \left( N_k + \frac{\delta(g)}{4} t_k \right) - \left( N_i + \frac{\delta(g)}{4} t_i \right) + 4 d(v_k, v_i|C_g) - 2 \delta(g) > 0 \) if \( N_k + \frac{\delta(g)}{4} t_k \geq (N_i + \frac{\delta(g)}{4} t_i) \), we still have \( \left( N'_k + \frac{\delta(g)}{4} t'_k \right) > \left( N'_i + \frac{\delta(g)}{4} t'_i \right) \)

for the new unicyclic graph \( G' \), and \( S_{e}^*(G) > S_{e}^*(G') \). By Lemmas 2.5 and 2.7, we have that:

**Lemma 2.8.** Let \( G = C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(2\beta, \beta) \). Then there exists a unicyclic graph \( G' = C_g(T'_1, T'_2, \ldots, T'_g) \in \mathcal{U}(2\beta, \beta) \) such that \( T'_i = K_1 \) or \( K_2 \) \((1 \leq i \leq g - 1)\) and \( S_{e}^*(C_g(T_1, T_2, \ldots, T_g)) \geq S_{e}^*(C_g(T'_1, T'_2, \ldots, T'_g)) \).

Next we give some transformations among \( \mathcal{U}(2\beta, \beta) \) which decrease the length of the unique cycle of the graph. By Lemma 2.8, there exist a unicyclic graph \( G' = C_g(T'_1, T'_2, \ldots, T'_g) \in \mathcal{U}(2\beta, \beta) \) such that \( T'_i = K_1 \) or \( K_2 \) \((1 \leq i \leq g - 1)\), \( S_{e}^*(G) \geq S_{e}^*(G') \) and the circuit \( C_g = v_1 v_2 \cdots v_g v_1 \) be not changed. We have that \( (d(v_1), d(v_2)) = (3, 3) \) or \( (2, 2) \) or \( (3, 2) \). Since \( G' = C_g(T'_1, T'_2, \ldots, T'_g) \in \mathcal{U}(2\beta, \beta) \), if \( (d(v_1), d(v_2)) = (2, 3) \), then \( (d(v_{g-1}), d(v_{g-2})) = (3, 3) \) or \( (2, 2) \) or \( (3, 2) \), we can reorder \( C_g \) such that \( (d(v_1), d(v_2)) = (3, 3) \) or \( (2, 2) \) or \( (3, 2) \). In the following, we consider the three cases, i.e. \( (d(v_1), d(v_2)) = (3, 3), (2, 2) \) and \( (3, 2) \), respectively.
Lemma 2.9. Let $G = C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(2\beta, \beta)$ such that $T_i = K_1$ or $K_2$, $1 \leq i \leq g - 1$, and the cycle length $g \geq 5$. If $d(v_1) = d(v_2) = 3$, let $G' = G + v_g v_3 + v_1 v_3 - v_g v_1 - v_1 v_2$, then $Sz_e^*(G) > Sz_e^*(G')$.

Proof. As $G \in \mathcal{U}(2\beta, \beta)$, then $G' \in \mathcal{U}(2\beta, \beta)$. By Theorem 2.6, we have that
\[
Sz_e^*(G) - Sz_e^*(G') \geq \left[ -12 - 6|T_3| - 4D(v_3|T_3) \right] + \left[ 16 - 8\beta + 6|T_3| + 4D(v_3|T_3) \right] + 2\beta + \frac{1}{2} + 4\sum_{j=4}^{g}|T_j|d(v_1, v_j|C_g) + 4\sum_{j=4}^{g}|T_j|d(v_2, v_j|C_g)
+ 2|T_3|\sum_{j=4}^{g}|T_j|d(v_3, v_j|C_g) + 8 + 12|T_3|
- 2(4 + |T_3|)\sum_{j=4}^{g}|T_j|d(v_3, v_j|C_{g-2})
- \delta(g)(4 + 4|T_3|) + \delta(g)(2 + 2|T_3|)
\geq \left( \frac{25}{2} - 2\delta(g) \right) + (12 - 2\delta(g))|T_3| - 6\beta + 4\sum_{j=4}^{g}|T_j|d(v_1, v_j|C_g) + 4\sum_{j=4}^{g}|T_j|d(v_2, v_j|C_g)
+ 2|T_3|\sum_{j=4}^{g}|T_j|d(v_3, v_j|C_g)
- 2(4 + |T_3|)\sum_{j=4}^{g}|T_j|d(v_3, v_j|C_{g-2})
\]

(i) If the cycle length $g$ is odd, then
\[
Sz_e^*(G) - Sz_e^*(G') \geq \frac{21}{2} + 10|T_3| - 6\beta + 12\sum_{j=4}^{g+1}\frac{|T_j|}{2} + 8|T_{\frac{g+3}{2}}| + (4 + 2|T_3|)|T_{\frac{g+5}{2}}|
+ (4 + 4|T_3|)\sum_{j=\frac{g+7}{2}}^{g+1}|T_j|
= -\frac{3}{2} + 7|T_3| + 9\sum_{j=4}^{\frac{g+1}{2}}|T_j| + 5|T_{\frac{g+3}{2}}| + (1 + 2|T_3|)|T_{\frac{g+5}{2}}|
+ (1 + 4|T_3|)\sum_{j=\frac{g+7}{2}}^{g+1}|T_j| > 0.
\]

(ii) If the cycle length $g$ is even, then
\[
Sz_e^*(G) - Sz_e^*(G') \geq \frac{25}{2} + 12|T_3| - 6\beta + 12\sum_{j=4}^{g+1}|T_j| + 4|T_{\frac{g+2}{2}}|
+ (4 + 4|T_3|)\sum_{j=\frac{g+3}{2}}^{g+1}|T_j|
= \frac{1}{2} + 9|T_3| + 9\sum_{j=4}^{\frac{g+1}{2}}|T_j| + |T_{\frac{g+2}{2}}| + (1 + 4|T_3|)\sum_{j=\frac{g+3}{2}}^{g+1}|T_j| > 0.
\]
So, the proof is completed.

Lemma 2.10. Let $G = C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(2\beta, \beta)$, such that $T_i = K_1$ or $K_2, 1 \leq i \leq g - 1$ and the cycle length $g \geq 5$. If $d(v_1) = d(v_2) = 2$, let $G' = G + v_g v_3 - v_g v_1$, then $Sz_e^*(G) > Sz_e^*(G')$. 
Proof. As $G \in \mathcal{U}(2\beta, \beta)$, then $G' \in \mathcal{U}(2\beta, \beta)$. By Theorem 2.6, we have that
\[
S_{Z_e}^{(g)}(G) - S_{Z_e}^{(g)}(G') \geq [-1 - 3|T_3| - 2D(v_3|T_3)| + [3 - 6\beta + 3|T_3| + 2D(v_3|T_3)]
\]
\[
+ 2\beta + \frac{1}{2} + 2\sum_{j=4}^{g} |T_j| d(v_1,v_j|C_g) + 2 \sum_{j=4}^{g} |T_j| d(v_2,v_j|C_g)
\]
\[
+ 2|T_3| \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_g) + 2 + 6|T_3|
\]
\[
- 2(2 + |T_3|) \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_{g-2})
\]
\[
- \delta(g)(1 + 2|T_3|) + \delta(g) \left( \frac{1}{2} + |T_3| \right)
\]
\[
\geq \left( \frac{g}{2} - \frac{1}{2} \delta(g) \right) + (6 - \delta(g))|T_3| - 4\beta + 2 \sum_{j=4}^{g} |T_j| d(v_1,v_j|C_g)
\]
\[
+ 2 \sum_{j=4}^{g} |T_j| d(v_2,v_j|C_g) + 2 |T_3| \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_g)
\]
\[
- 2(2 + |T_3|) \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_{g-2})
\].

(i) If the cycle length $g$ is odd.
\[
S_{Z_e}^{(g)}(G) - S_{Z_e}^{(g)}(G') \geq 4 + 5|T_3| - 4\beta + 6 \sum_{j=4}^{g+1} \frac{2}{T_j} + 4|T_{\frac{g+3}{2}}| + (2 + 2|T_3|)|T_{\frac{g+5}{2}}|
\]
\[
+ (2 + 4|T_3|) \sum_{j=4}^{g+1} \frac{2}{T_j}
\]
\[
= 3|T_3| + 4 \sum_{j=4}^{g+1} \frac{2}{T_j} + 2|T_{\frac{g+3}{2}}| + 2|T_3| |T_{\frac{g+5}{2}}| + 4|T_3| \sum_{j=4}^{g+1} \frac{2}{T_j} > 0.
\]

(ii) If the cycle length $g$ is even.
\[
S_{Z_e}^{(g)}(G) - S_{Z_e}^{(g)}(G') \geq \frac{g}{2} + 6|T_3| - 4\beta + 6 \sum_{j=4}^{g+1} \frac{2}{T_j} + 2|T_{\frac{g+2}{2}}| + (2 + 4|T_3|) \sum_{j=4}^{g+1} \frac{2}{T_j}
\]
\[
= \frac{1}{2} + 4|T_3| + 4 \sum_{j=4}^{g+1} \frac{2}{T_j} + 4|T_3| |T_{\frac{g+2}{2}}| + 4|T_3| \sum_{j=4}^{g+1} \frac{2}{T_j} > 0.
\]

Hence, the proof is completed.

Lemma 2.11. Let $G = C_g(T_1,T_2,...,T_g) \in \mathcal{U}(2\beta, \beta)$ such that $T_i = K_1$ or $K_2, 1 \leq i \leq g - 1, and the cycle length $g \geq 5$. If $d(v_1) = 3, d(v_2) = 2$, let $G' = G + v_g v_3 + v_1 v_3 - v_g v_1 - v_1 v_2$, then $S_{Z_e}^{(g)}(G) > S_{Z_e}^{(g)}(G')$.

Proof. As $G \in \mathcal{U}(2\beta, \beta)$, then $G' \in \mathcal{U}(2\beta, \beta)$. By Theorem 2.6,
\[
S_{Z_e}^{(g)}(G) - S_{Z_e}^{(g)}(G') \geq -9 + [11 - 6\beta] + 4 \sum_{j=4}^{g} |T_j| d(v_1,v_j|C_g) + 2 \sum_{j=4}^{g} |T_j| d(v_2,v_j|C_g)
\]
\[
+ 2 \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_g) + 14 - 8 \sum_{j=4}^{g} |T_j| d(v_3,v_j|C_{g-2})
\]
\[
- 5\delta(g) + \frac{5}{2} \delta(g) + 2\beta + \frac{1}{2}
\]
\[
\geq \left( \frac{33}{2} - \frac{5}{2} \delta(g) \right) - 4\beta + 4 \sum_{j=4}^{g} |T_j| d(v_1,v_j|C_g)
\]
\[ + 2 \sum_{j=4}^{g} |T_j| d(v_2, v_j|c_g) + 2 \sum_{j=4}^{g} |T_j| d(v_3, v_j|c_g) - 8 \sum_{j=4}^{g} |T_j| d(v_3, v_j|c_{g-2}). \]

(i) If the cycle length \( g \) is odd, then
\[
Sz^*_e(G) - Sz^*_e(G') \geq \frac{28}{2} - 4\beta + 10 \sum_{j=4}^{\frac{g+1}{2}} |T_j| + 6|T_{\frac{g+3}{2}}| + 4|T_{\frac{g+5}{2}}| + 6 \sum_{j=\frac{g+7}{2}}^{g} |T_j|.
\]
\[
= 6 + 8 \sum_{j=4}^{\frac{g+1}{2}} |T_j| + 4|T_{\frac{g+3}{2}}| + 2|T_{\frac{g+5}{2}}| + 4 \sum_{j=\frac{g+7}{2}}^{g} |T_j| > 0.
\]

(ii) If the cycle length \( g \) is even, then
\[
Sz^*_e(G) - Sz^*_e(G') \geq \frac{33}{2} - 4\beta + 12 \sum_{j=4}^{\frac{g+1}{2}} |T_j| + 4|T_{\frac{g+2}{2}}| + 8 \sum_{j=\frac{g+3}{2}}^{g} |T_j|.
\]
\[
= \frac{17}{2} + 10 \sum_{j=4}^{\frac{g+1}{2}} |T_j| + 2|T_{\frac{g+2}{2}}| + 6 \sum_{j=\frac{g+3}{2}}^{g} |T_j| > 0.
\]

The proof is now completed. \( \blacksquare \)

![Figure 4](image-url)

**Figure 4.** Seven conjugated unicyclic graphs with \( \beta = 3 \).

**Theorem 2.12.** Let \( G = C_g(T_1,T_2,...,T_g) \in \mathcal{U}(6,3) \). Then \( Sz^*_e(H_1) < Sz^*_e(H_2) < Sz^*_e(H_3) < Sz^*_e(H_4) < Sz^*_e(H_5) < Sz^*_e(H_6) < Sz^*_e(H_7) \), where \( H_i, 1 \leq i \leq 7 \), be shown in Figure 4.

**Proof.** There are only seven conjugated unicyclic graphs in \( \mathcal{U}(6,3) \), which was shown in Figure 4. By calculating directly, we have that
\[
Sz^*_e(H_1) = \frac{139}{4}, \quad Sz^*_e(H_2) = \frac{141}{4}, \quad Sz^*_e(H_3) = \frac{151}{4}, \quad Sz^*_e(H_4) = \frac{79}{2},
\]
\[
Sz^*_e(H_5) = \frac{83}{2}, \quad Sz^*_e(H_6) = \frac{187}{4}, \quad Sz^*_e(H_7) = 54.
\]
So we have that \( Sz^*_e(H_1) < Sz^*_e(H_2) < Sz^*_e(H_3) < Sz^*_e(H_4) < Sz^*_e(H_5) < Sz^*_e(H_6) < Sz^*_e(H_7) \). The result follows. \( \blacksquare \)
Theorem 2.13. Let $G = C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(2\beta, \beta)$ $(\beta \geq 4)$.

(i) If $4 \leq \beta \leq 7$, then $Sz_e^*(G) \geq 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4}$, with equality if and only if $G \cong G_4$;

(ii) If $\beta \geq 8$, then $Sz_e^*(G) \geq 4\beta^2 + \frac{11}{2}\beta - 11$, with equality if and only if $G \cong G_4$;

Proof. By using Lemmas 2.7, 2.9, 2.10 and 2.11 repeatedly, the final graphs are $\{G_i\}, 1 \leq i \leq 7$, see Figure 5. By calculating directly, we have that

$$Sz_e^*(G_1) = 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4},$$
$$Sz_e^*(G_2) = 5\beta^2 + \frac{1}{2}\beta - \frac{45}{4},$$
$$Sz_e^*(G_3) = 5\beta^2 - \frac{1}{2}\beta - \frac{23}{4},$$
$$Sz_e^*(G_4) = 4\beta^2 + \frac{11}{2}\beta - 11,$$
$$Sz_e^*(G_5) = 4\beta^2 + \frac{15}{2}\beta - 19,$$
$$Sz_e^*(G_6) = 4\beta^2 + \frac{35}{2}\beta - 55,$$
$$Sz_e^*(G_7) = 4\beta^2 + \frac{31}{2}\beta - 43.$$

Then, we have that

$$Sz_e^*(G) \geq Sz_e^*(G_1) = 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4}, \text{ for } 4 \leq \beta \leq 7,$$
$$Sz_e^*(G) \geq Sz_e^*(G_4) = 4\beta^2 + \frac{11}{2}\beta - 11, \text{ for } \beta \geq 8.$$

The result follows.

3. ON REVISED EDGE-SZEGED OF THE JOIN OF GRAPHS

In the section, we consider revised edge-Szeged index of the join graph. The join graph of $G$ and $H$, denoted by $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$, and with edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. For the revised edge-Szeged index of the graph $G$, let $|E(G \vee H)| = m$, we have
\[
S_z^e(G) = \sum_{e = uv \in E(G)} (m_u(e) + \frac{m_0(e)}{2})(m_v(e) + \frac{m_0(e)}{2}). \\
= \frac{1}{4} \sum_{e = uv \in E(G)} (m + m_u(e) - m_v(e))(m + m_v(e) - m_u(e)) \times \frac{m^3}{4} - \frac{1}{4} \sum_{e = uv \in E(G)} (m_u(e) - m_v(e))^2.
\]

**Theorem 3.1.** Let \( G \) and \( H \) be simple graphs, where \(|E(G \cup H)| = m, |G| = n_1, |E(G)| = m_1, |H| = n_2 \) and \(|E(H)| = m_2\). Then \( S_z^e(G \cup H) = \frac{m^3}{4} - \frac{1}{4}(S_1 + S_2 + S_3) \), where

\[
S_1 = \sum_{e = uv \in E(G)} [(d^G_v(u) + d_G(u)) - (d^G_v(v) + d_G(v))]^2, \\
S_2 = \sum_{e = uv \in E(H)} [(d^H_v(u) + d_H(u)) - (d^H_v(v) + d_H(v))]^2, \\
S_3 = \sum_{e = uv \in E'} [(d_G(v) - d_H(v)) + (n_2 - n_1) + (m_2 - m_1) + (d^G_v(u) - d^H_v(v)) + (t_H(v) - t_G(u))]^2.
\]

**Proof.** We divide the edge of \( G \cup H \) into three groups: \( E(G), E(H) \) and \( E' = \{uv : u \in V(G), v \in V(H)\}\).

**Case 1.** \( e = uv \in E(G)\). When \( e' = u'v' \in E(H) \) or \( u' \in V(G), v' \in V(H) \), \( u' \neq u, v \), then \( d_{G \cup H}(u,v,e') = 1 \). When \( e'' = u''v'' \in E(G) \) and \( d_{G \cup H}(u,e'') \geq 2, d_{G \cup H}(v,e'') \geq 2 \), then \( d_{G \cup H}(u,e'') = d_{G \cup H}(u,e'') = 2 \). Let \( N'_G(u) = N_G(u) \backslash \{v\} \) and \( N'_G(u) = N'_G(u) \cup N_G(u) \cup N_3(u) \), where

\[
N_1(u) = \{w \in N'_G(u) | w \ is \ in \ a \ triangle \ that \ contains \ edge \ uv\}, \\
N_2(u) = \{w \in N'_G(u) | w \ is \ in \ a \ quadrilateral \ that \ contains \ edge \ uv\}, \\
N_3(u) = N'_G(u) \backslash \{N_1(u) \cup N_2(u)\}.
\]

Then, one known that

\[
m_u(e|G \cup H) + \sum_{w \in N_1(u)} (d_G(w) - 2) = n_2 + (d_G(u) - 1) + \sum_{w \in N_1(u)} (d_G(w) - 2) \\
+ \sum_{w \in N_2(u)} (d_G(w) - 2) + \sum_{w \in N_3(u)} (d_G(w) - 1) \\
= n_2 + (d_G(u) - 1) + \sum_{w \in N_1(u)} d_G(w) - |N_1(u)| \\
+ \sum_{w \in N_2(u)} d_G(w) - |N_2(u)| + \sum_{w \in N_3(u)} d_G(w) \\
- (|N_1(u)| + |N_2(u)| + |N_3(u)|) \\
= n_2 + d^G_v(u) - d_G(v) - |N_1(u)| - |N_2(u)|.
\]

Similarly, we have that \( m_v(e|G \cup H) + \sum_{w \in N_1(v)} (d_G(w) - 2) = n_2 + d^G_v(v) - d_G(u) - |N_1(v)| - |N_2(v)| \). It is obvious that \( \sum_{w \in N_1(u)} (d_G(w) - 2) = \sum_{w \in N_1(v)} (d_G(w) - 2) \).

Then,

\[
S_1 = \sum_{uv \in E(G)} (m_u(e) - m_v(e))^2 = \sum_{uv \in E(G)} [(d^G_v(u) + d_G(u)) - (d^G_v(v) + d_G(v))]^2.
\]

**Case 2.** \( e = uv \in E(H)\). Similarly, we have that

\[
S_2 = \sum_{uv \in E(H)} (m_u(e) - m_v(e))^2 = \sum_{uv \in E(H)} [(d^H_v(u) + d_H(u)) - (d^H_v(v) + d_H(v))]^2.
\]

**Case 3.** \( e = uv \in E'\). One known that \( m_u(e|G \cup H) = d_G(u) + (n_2 - 1) + (m_2 - d^H_v(v) + t_H(v)) \) and \( m_v(e|G \cup H) = d_H(v) + (n_1 - 1) + (m_1 - d^G_v(u) + t_G(u)) \). Thus,
\[ S_3 = \sum_{e=uv \in E'} [(d_G(u) - d_H(v)) + (n_2 - n_1) + (m_2 - m_1) + (d^2_G(u) - d^2_H(v)) + (t_H(v) - t_G(u))]^2. \]

In summary, we have that \( Sz^*_e(G \lor H) = \frac{m^3}{4} - \frac{1}{4}(S_1 + S_2 + S_3) \) and the result follows. ■

By Theorem 3.1, one can calculate revised edge-Szeged index of some special graphs, such as the complete bipartite graph \( K_{m,n} = \overline{K_m} \lor \overline{K_n} \), the wheel graph \( W_n = K_1 \lor C_{n-1}, n \geq 5 \), the fan graph \( F_n = K_1 \lor P_{n-1}, n \geq 6 \).

\[ Sz^*_e(K_{m,n}) = Sz^*_e(\overline{K_m} \lor \overline{K_n}) = \frac{1}{4}nm(n^2m^2 - (n - m)^2), \]

\[ Sz^*_e(W_n) = Sz^*_e(K_1 \lor C_{n-1}) = \frac{1}{4}(n - 1)(4n^2 + 20n - 73), (n \geq 5), \]

\[ Sz^*_e(F_n) = Sz^*_e(K_1 \lor P_{n-1}) = \frac{1}{4}(4n^3 + 8n^2 - 118n + 203), (n \geq 6). \]

ACKNOWLEDGEMENT. The research is supported by program for excellent talents in Hunan Normal University (ET13101), the National Natural Science Foundation of China (Grant Nos. 11971180, 11571123), the Guangdong Provincial Natural Science Foundation (Grant No. 2019A1515012052).

REFERENCES


