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On the Revised Edge-Szeged Index of Graphs

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ABSTRACT

The revised edge-Szeged index of a connected graph G is defined as $Sz_e^*(G) = \sum_{e=uv\in E(G)}\left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right)$, where $m_u(e|G)$, $m_v(e|G)$ and $m_0(e|G)$ are, respectively, the number of edges of G lying closer to vertex u than to vertex v, the number of edges of G lying closer to vertex v than to vertex u, and the number of edges equidistant to u and v. In this paper, we give an effective method for computing the revised edge-Szeged index of unicyclic graphs and using this result we identify the minimum revised edge-Szeged index of conjugated unicyclic graphs which is defined as the unicyclic graphs with a perfect matching. We also give a method of calculating revised edge-Szeged index of the joint graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple, and refer to [2] for notations and terminologies used but not defined here.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ ($N(v)$ for short) denote the set of all the adjacent vertices of v in G and $d_G(v) = |N_G(v)|$, the degree of v in G. Let $w \in N_G(u)$, $d_2^G(u) = \sum_{w\in N_G(u)} d(w)$. Denote $t_G(u)$ the number of triangles in graph G that contain the vertex u. Call u a pendent vertex of G, if $d_G(u) = 1$ and uv a pendant edge of G, if one of its endpoints is a pendent vertex. Denote by PV the set of pendent vertices of G. The distance, $d(u,v|G)$ (or $d(u,v)$ for short), between vertices u and v of G is the length of the shortest u,v path in G. Let $D(u|G) = \sum_{v\in V(G)} d(u,v|G)$ and $G - uv$, $G + uv$ denote the graph obtained from G by deleting the edge of uv and adding an edge between u and v, respectively. An edge e is called a cut

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edge of a connected graph G if $G - e$ is disconnect. Let P_n, C_n, S_n and K_n be the path, cycle, star and complete graph of order n, respectively.

A matching M in a graph G is a set of edges of G such that no two edges from M share a vertex. If every vertex of G is incident with an edge of M, the matching M is called a perfect matching.

In chemical graph theory, graph invariants are numbers related to graphs with invariant structure. These invariants are also called topological indices. Topological indices provide correlations with physical, chemical and thermodynamic parameters of chemical compounds, see [17, 18, 26]. Among all the topological indices, the most well-known is the Wiener index [29], which is defined as the sum of distances over all unordered vertex pairs in G, namely, $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$.

A long time known property of the Wiener index is the formula [29]

$$W(G) = \sum_{e=uv \in E(G)} n_u(e|G) n_v(e|G),$$

where $n_u(e|G)$ and $n_v(e|G)$ are, respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. It is applicable for trees. Using the above formula, another topological index related, named by Szeged index, was introduced by Gutman [13], which is an extension of the Wiener index and defined by $Sz(G) = \sum_{e=uv \in E(G)} n_u(e|G) n_v(e|G)$. In addition, some properties and applications of Wiener index and Szeged index have been investigated [1, 5–7, 11, 12, 15, 16, 19, 20, 22, 24, 26, 29, 31, 32].

Given an edge $e = uv \in E(G)$, the distance between the vertex x and the edge e, denoted by $d(x, e)$, is defined as $d(x, e) = \min\{d(x, u), d(x, v)\}$. Denote $M_u(e|G) = \{e \in E(G): d(u, e) < d(v, e)\}$, $M_v(e|G) = \{e \in E(G): d(v, e) < d(u, e)\}$ and $M_0(e|G) = \{e \in E(G): d(u, e) = d(v, e)\}$. Let $m = |E(G)|$, $m_u(e|G) = |M_u(e|G)|$, $m_v(e|G) = |M_v(e|G)|$ and $m_0(e|G) = |M_0(e|G)|$, we have $m_u(e|G) + m_v(e|G) + m_0(e|G) = m$. Then, the edge-Szeged index [14] and revised edge-Szeged index [8] of G are defined as

$$Sz_e(G) = \sum_{e=uv \in E(G)} m_u(e|G) m_v(e|G),$$

$$Sz_e^*(G) = \sum_{e=uv \in E(G)} \left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right).$$

For the sake of simplicity, we consider the contribution $\phi(e)$ of an edge $e = uv$ defined as $\phi(e) = \left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right)$.

Up until now, much work has been done on revised edge-Szeged index. Faghani and Ashrafi [12] computed an exact formula for the revised edge-Szeged index of Cartesian product of graphs. Liu and Wang [23] gave a lower bound of the edge revised Szeged index among all m-edges cactus graphs with k cycles. Wang et al. [28] characterized the n-vertex unicyclic graphs with a given diameter having the minimum edge-Szeged index. They used a unified approach to identify the n-vertex unicyclic graphs with the minimum, the second minimum, the third minimum and the fourth minimum edge-Szeged indices. Other results see [3, 4, 9, 21], and the references cited therein.

In this paper, we give an effective method for computing the revised edge-Szeged index of unicyclic graphs and identify the minimum revised edge-Szeged index of conjugated unicyclic graphs. And we also give a method of calculating revised edge Szeged index of the joint graph.

2. Revised Edge-Szeged Index of Conjugated Unicyclic Graphs

For nonnegative integer $\beta \geq 2$, let $T_{2\beta,\beta}$ (see Figure 1) be the tree obtained by attaching a pendent edge to each of some $\beta - 1$ non-central vertices of the star $S_{\beta+1}$. Let $T_{2\beta+1,\beta}$ (see Figure 1) be the tree obtained by attaching a pendent edge to each of some β noncentral vertices of the star $S_{\beta+1}$. Let $\mathcal{T}(2\beta,\beta)$ and $\mathcal{U}(2\beta,\beta)$ denote the set of conjugated trees (trees with a perfect matching) and conjugated unicyclic graphs (unicyclic graphs with a perfect matching) of order 2β, respectively, where β is the number of matchings in G.

First, we introduce some lemmas that will be useful in the proof of the main result.

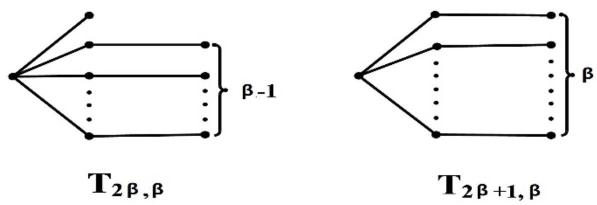

Figure 1. The graphs $T_{2\beta,\beta}$ and $T_{2\beta+1,\beta}$.

Lemma 2.1. [25] *Let G be a graph of order 2β with a perfect matching. If $PV \neq \emptyset$, then for any vertex $u \in V(G)$, $|N(u) \cap PV| \leq 1$.*

Lemma 2.2. [8] *Let G be a unicyclic graph with n vertices. Then $Sz_e^*(G) \leq \frac{n^3}{4}$ with equality if and only if G is the cycle C_n.*

From Lemma 2.2, it is obvious that $Sz_e^*(\mathcal{U}(2\beta,\beta)) \leq 2\beta^3$ with equality if and only if $\mathcal{U}(2\beta,\beta)$ is the cycle $C_{2\beta}$. Therefore, in the following, we only need to consider the lower bound of revised edge-Szeged index of conjugated unicyclic graphs. First, we introduce some useful graph transformations.

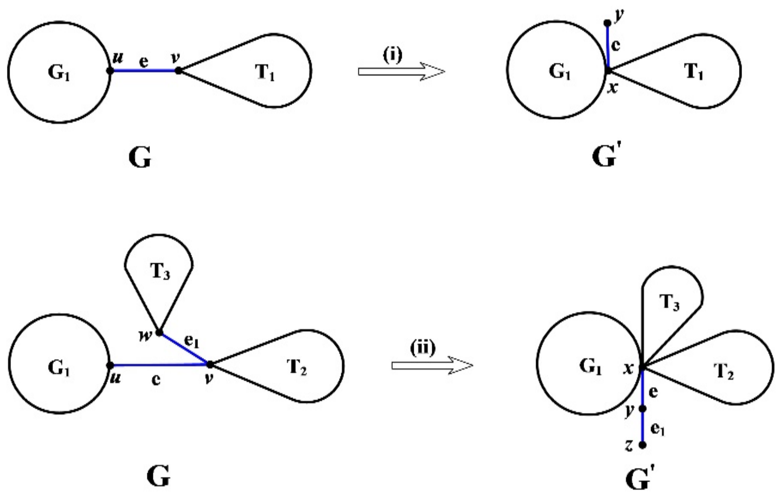

Figure 2. The edge-lifting transformation.

Lemma 2.3. (The edge-lifting transformation) *Let G be a connected graph which obtained from a connected graph $G_1(u \in V(G_1), |G_1| \geq 2)$ and a tree $T_1(v \in V(T_1), |T_1| \geq 3)$ by adding an edge $e = uv$ and with perfect matchings.*

(i) *If $e = uv \in M$ (M is a perfect matching), let G'* (see Figure $2(i)$) *be the graph obtained from G by deleting e from G, identifying u and v into a new vertex x and adding a vertex y connected to x. Let the edge connecting x and y in G' be again denoted by e. Then $Sz_e^*(G) > Sz_e^*(G')$.*

(ii) *If $e = uv \notin M$ (M is a perfect matching), there exists a cut edge $e_1 = vw \in M$ in T_1. Then, obviously, T_1 (see Figure $2(i)$) can be seen as the graph obtained from a tree T_2 and a tree T_3 by adding an edge between a vertex w of T_3 and a vertex v of T_2. Let G' (see Figure $2(ii)$) be the graph obtained from G by deleting e and e_1 from G, identifying u, v and w into a new vertex x and adding an edge yz connected to x. Let the edge connecting x and y, y and z in G' be again denoted by e and e_1. Then $Sz_e^*(G) > Sz_e^*(G')$.*

Proof. Note that G' is a graph with perfect matchings, since G is a graph with perfect matchings.

(i) Observe that after the modification of the graph, for every edge f, distinct from e, the contribution $\phi(f)$ stays unchanged. For edge e, we have that $\phi_{G'}(e) = \frac{1}{2}(|E(G')| - \frac{1}{2})$,

$$\phi_G(e) = \left(|E(G_1)| + \frac{1}{2}\right)\left(|E(T_1)| + \frac{1}{2}\right) = |E(G_1)||E(T_1)| + \frac{1}{2}(|E(G_1)| + |E(T_1)|) + \frac{1}{4}$$
$$\geq (|E(G')| - 2) + \frac{1}{2}(|E(G')| - 1) + \frac{1}{4} = \frac{3}{2}|E(G')| - \frac{9}{4} > \phi_{G'}(e).$$

Thus $Sz_e^*(G) > Sz_e^*(G')$.

(ii) Observe that after the modification of the graph, for every edge f, distinct from e and e_1, the contribution $\phi(f)$ stays unchanged. For edge e and e_1 we have that

$$
\begin{aligned}
\phi_{G'}(e) &= \tfrac{3}{2}\left(|E(G')| - 2 + \tfrac{1}{2}\right) = \tfrac{3}{2}|E(G')| - \tfrac{9}{4},\\
\phi_{G'}(e_1) &= \tfrac{1}{2}\left(|E(G')| - 1 + \tfrac{1}{2}\right) = \tfrac{1}{2}|E(G')| - \tfrac{1}{4},\\
\phi_G(e) &= \left(|E(G_1)| + \frac{1}{2}\right)\left(|E(T_2)| + |E(T_3)| + \frac{3}{2}\right)\\
&= |E(G_1)|(|E(T_2)| + |E(T_3)| + 1) + \tfrac{1}{2}(|E(G_1)| + |E(T_2)| + |E(T_3)| + 1) + \tfrac{1}{4}\\
&\geq |E(G')| - 2 + \tfrac{1}{2}(|E(G')| - 1) + \tfrac{1}{4} = \tfrac{3}{2}|E(G')| - \tfrac{9}{4},\\
\phi_G(e_1) &= \left(|E(T_3)| + \frac{1}{2}\right)\left(|E(T_2)| + |E(G_1)| + \frac{3}{2}\right)\\
&= |E(T_3)|(|E(T_2)| + |E(G_1)| + 1) + \tfrac{1}{2}(|E(T_3)| + |E(T_2)| + |E(G_1)| + 1) + \tfrac{1}{4}\\
&\geq |E(G')| - 2 + \tfrac{1}{2}(|E(G')| - 1) + \tfrac{1}{4} = \tfrac{3}{2}|E(G')| - \tfrac{9}{4}.
\end{aligned}
$$

Then $\phi_G(e) + \phi_G(e_1) > \phi_{G'}(e) + \phi_{G'}(e_1)$. Thus $Sz_e^*(G) > Sz_e^*(G')$ and the proof is completed. ■

By Lemma 2.3, we have the following result.

Lemma 2.4. *Let $G \in \mathcal{T}(2\beta, \beta)$ where $\beta \geq 2$. Then $Sz_e^*(G) \geq 4\beta^2 - \frac{15}{2}\beta + \frac{15}{4}$ with equality if and only if $G \cong T(2\beta, \beta)$.*

Let $g \geq 3$ be an integer, and let $C_g = v_1 v_2 \cdots v_g v_1$ be a cycle on g vertices. Let $T_1, T_2, \dots, T_g$ be vertex-disjoint trees, and let the root vertex of T_i be v_i for $1 \leq i \leq g$. Denote by $C_g(T_1, T_2, \dots, T_g)$ the unicyclic graph obtained from of C_g by identifying the root vertex u_i of T_i with v_i for $1 \leq i \leq g$. Any unicyclic graph G with a g-cycle can be denoted by the form $C_g(T_1, T_2, \dots, T_g)$, where $|T_i| = t_i$, $(i = 1, 2, \dots, g)$ and $\sum_{i=1}^{g} t_i = n$. By Lemma 2.3, we can repeat the edge-lifting transformation to the unicyclic graphs $C_g(T_1, T_2, \dots, T_g)$ and we have

Lemma 2.5. *If $C_g(T_1, T_2, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$, then*

$$Sz_e^*\left(C_g(T_1, T_2, \dots, T_g)\right) \geq Sz_e^*\left(C_g(T_1', T_2', \dots, T_g')\right)$$

with equality if and only if $T_k \cong T_k'$, for all k $(1 \leq k \leq g)$, where $|T_k'| = |T_k| = t_k$, $T_k' \cong T_{2\beta_k, \beta_k}$ if $t_k = 2\beta_k$ and $T_k' \cong T_{2\beta_k + 1, \beta_k}$ if $t_k = 2\beta_k + 1$.

In the following, we give an effective method for computing the revised edge-Szeged index among unicyclic graphs $G = C_g(T_1, T_2, \dots, T_g)$.

Theorem 2.6. *If $G = C_g(T_1, T_2, \dots, T_g)$, then*

$$
\begin{aligned}
Sz_e^*(G) &= \sum_{i=1}^{g} W(T_i) + \sum_{i=1}^{g}(|G| - |T_i| + 1)\, D(v_i|T_i) \sum_{i=1}^{g}\sum_{j=1}^{g} |T_i||T_j| d(v_i, v_j|C_g)\\
&\quad - \delta(g)\left(\sum_{i<j} |T_i||T_j| + \tfrac{1}{4}\sum_{i=1}^{g}|T_i|^2\right) - \tfrac{1}{2}|\delta(g) - 1||G|^2 + \tfrac{1}{4}(2|G| + 1)g - \tfrac{1}{4}|G|,
\end{aligned}
$$

where $\delta(g) = 0$ for even g, $\delta(g) = 1$ for odd g.

Proof. We divide the edge of $G = C_g(T_1, T_2, \dots, T_g)$ into the following groups:

(a) the edges belonging to the tree T_i, $i = 1, 2, \dots, g$; (b) the edges belonging to the cycle C_g. For the edge $e = uv \in E(T_i)$, we assume that $d(u, v_i|T_i) > d(v, v_i|T_i)$ for $i = 1, 2, \dots, g$. For any vertex $w \in V(T_i)$, it is counted $d(w, v_i|T_i)$ times in the sum $\sum_{e\in E(T_i)} n_u(e|T_i)$ for the edges in the path from w to v_i. Thus $\sum_{e\in E(T_i)} n_u(e|T_i) = D(v_i|T_i)$ for $i = 1, 2, \dots, g$, see [15]. Note that $m_u(e|T_i) = n_u(e|T_i) - 1$ and $m_v(e|T_i) = n_v(e|T_i) - 1$. The contributions to $Sz_e^*(G)$ pertaining to the edges of type (a) are $A = \sum_{i=1}^{g} \sum_{e\in E(T_i)} \left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right)$.

$= \sum_{i=1}^{g} \sum_{e\in E(T_i)} m_u(e|T_i)(m_v(e|T_i) + |E(G)| - |E(T_i)|) + \sum_{i=1}^{g} \sum_{e\in E(T_i)} (\frac{1}{2}|G| - \frac{1}{4})$

$= \sum_{i=1}^{g} \sum_{e\in E(T_i)} m_u(e|T_i) m_v(e|T_i)$

$+ \sum_{i=1}^{g} (|E(G)| - |E(T_i)|) \sum_{e\in E(T_i)} m_u(e|T_i) + \left(\frac{1}{2}|G| - \frac{1}{4}\right)(|G| - g)$

$= \sum_{i=1}^{g} W(T_i) - \sum_{i=1}^{g} (|T_i| - 1)^2 + \sum_{i=1}^{g} (|G| - |T_i| + 1) \sum_{e\in E(T_i)} (n_u(e|T_i) - 1)$

$+ \left(\frac{1}{2}|G| - \frac{1}{4}\right)(|G| - g)$

$= \sum_{i=1}^{g} W(T_i) + \sum_{i=1}^{g} (|G| - |T_i| + 1) D(v_i|T_i) - \frac{1}{2}|G|^2 + \frac{1}{2}|G|g - \frac{1}{4}|G| + \frac{1}{4}g.$

If g is even, then obviously, $m_u(e|G) = n_u(e|G) - 1$ and $m_v(e|G) = n_v(e|G) - 1$. The contributions to $Sz_e^*(G)$ pertaining to the edges of type (b) are

$B = \sum_{e\in E(C_g)} \left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right). = \sum_{e\in E(C_g)} n_u(e|G) n_v(e|G).$

$= \sum_{i=1}^{g} \sum_{j=1}^{g} |T_i||T_j| d(v_i, v_j|C_g).$

If g is odd, then $m_u(e|G) = n_u(e|G)$ and $m_v(e|G) = n_v(e|G)$. The contributions to $Sz_e^*(G)$ pertaining to the edges of type (b) are

$$B = \sum_{e\in E(C_g)} \left(m_u(e|G) + \frac{m_0(e|G)}{2}\right)\left(m_v(e|G) + \frac{m_0(e|G)}{2}\right)$$
$$= \sum_{e\in E(C_g)} \left(n_u(e|G) + \frac{1}{2} n_0(e|G)\right)\left(n_v(e|G) + \frac{1}{2} n_0(e|G)\right)$$
$$= \sum_{i=1}^{g} \sum_{j=1}^{g} |T_i||T_j| d(v_i, v_j|C_g) - \sum_{i<j} |T_i||T_j| - \frac{1}{4} \sum_{i=1}^{g} |T_i|^2 + \frac{1}{2}|G|^2.$$

As $Sz_e^*(G) = A + B$, the result follows easily. ■

Lemma 2.7. (The branch transformation) *Let* $G = C_g(T_1, T_2, \dots, T_k, \dots, T_l, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$ *with its unique cycle* $C_g = v_1 v_2 \cdots v_g$ *and* $N_i = \sum_{j=1}^{g} t_j d(v_i, v_j|C_g)$, *where* $v_i, v_j \in C_g$ *and* $|T_j| = t_j$. *Suppose that there exist a path* $v_k w u$ *with root vertex* v_k *in* T_k. *Let* $G' = G - v_k w + v_l w$, Figure 3. *If* $(N_k + \frac{\delta(g)}{4} t_k) \geq (N_l + \frac{\delta(g)}{4} t_l)(1 \leq k < l \leq g)$. *Then* $Sz_e^*(G) > Sz_e^*(G')$.

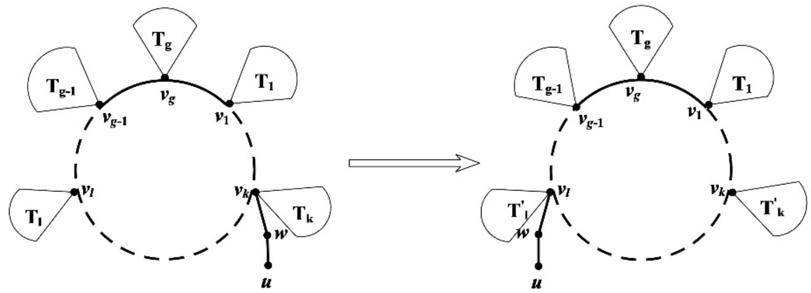

Figure 3. The branch transformation of Lemma 2.7.

Proof. Note that $t'_k = |T'_k| = |T_k| - 2 = t_k - 2$ and $t'_l = |T'_l| = |T_l| + 2 = t_l + 2$, by Theorem 2.6, we have that

$$Sz_e^*(G) - Sz_e^*(G') = 4\left[\left(N_k + \frac{\delta(g)}{4}t_k\right) - \left(N_l + \frac{\delta(g)}{4}t_l\right)\right] + 8d\left(v_k, v_l|C_g\right) - 2\delta(g) > 4\left[\left(N_k + \frac{\delta(g)}{4}t_k\right) - \left(N_l + \frac{\delta(g)}{4}t_l\right)\right] \geq 0.$$

Hence the proof is completed. ■

As $\left(N'_k + \frac{\delta(g)}{4}t'_k\right) - \left(N'_l + \frac{\delta(g)}{4}t'_l\right) = \left(N_k + \frac{\delta(g)}{4}t_k\right) - \left(N_l + \frac{\delta(g)}{4}t_l\right) + 4d\left(v_k, v_l|C_g\right) - 2\delta(g) > 0$ if $(N_k + \frac{\delta(g)}{4}t_k) \geq (N_l + \frac{\delta(g)}{4}t_l)$, we still have $\left(N'_k + \frac{\delta(g)}{4}t'_k\right) > \left(N'_l + \frac{\delta(g)}{4}t'_l\right)$ for the new unicyclic graph G', and $Sz_e^*(G) > Sz_e^*(G')$. By Lemmas 2.5 and 2.7, we have that:

Lemma 2.8. *Let $G = C_g(T_1, T_2, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$. Then there exists a unicyclic graph $G' = C_g(T'_1, T'_2, \dots, T'_g) \in \mathcal{U}(2\beta, \beta)$ such that $T'_i = K_1$ or K_2 $(1 \leq i \leq g-1)$ and $Sz_e^*\left(C_g(T_1, T_2, \dots, T_g)\right) \geq Sz_e^*\left(C_g(T'_1, T'_2, \dots, T'_g)\right)$.*

Next we give some transformations among $\mathcal{U}(2\beta, \beta)$ which decrease the length of the unique cycle of the graph. By Lemma 2.8, there exist a unicyclic graph $G' = C_g(T'_1, T'_2, \dots, T'_g) \in \mathcal{U}(2\beta, \beta)$ such that $T'_i = K_1$ or K_2 $(1 \leq i \leq g-1)$, $Sz_e^*(G) \geq Sz_e^*(G')$ and the circuit $C_g = v_1v_2 \cdots v_gv_1$ be not changed. We have that $\left(d(v_1), d(v_2)\right) = (3,3)$ or $(2,2)$ or $(3,2)$. Since $G' = C_g\left(T'_1, T'_2, \dots, T'_g\right) \in \mathcal{U}(2\beta, \beta)$, if $\left(d(v_1), d(v_2)\right) = (2,3)$, then $\left(d\left(v_{g-1}\right), d\left(v_{g-2}\right)\right) = (3,3)$ or $(2,2)$ or $(3,2)$, we can reorder C_g such that $\left(d(v_1), d(v_2)\right) = (3,3)$ or $(2,2)$ or $(3,2)$. In the following, we consider the three cases, i.e. $\left(d(v_1), d(v_2)\right) = (3,3)$, $(2,2)$ and $(3,2)$, respectively.

Lemma 2.9. *Let* $G = C_g(T_1, T_2, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$ *such that* $T_i = K_1$ or K_2, $1 \le i \le g-1$, *and the cycle length* $g \ge 5$. *If* $d(v_1) = d(v_2) = 3$, *let* $G' = G + v_g v_3 + v_1 v_3 - v_g v_1 - v_1 v_2$, *then* $Sz_e^*(G) > Sz_e^*(G')$.

Proof. As $G \in \mathcal{U}(2\beta, \beta)$, then $G' \in \mathcal{U}(2\beta, \beta)$. By Theorem 2.6, we have that

$$\begin{aligned} Sz_e^*(G) - Sz_e^*(G') \ge\ & [-12 - 6|T_3| - 4D(v_3|T_3)] \\ & + [16 - 8\beta + 6|T_3| + 4D(v_3|T_3)] + 2\beta + \tfrac{1}{2} \\ & + 4\textstyle\sum_{j=4}^{g} |T_j| d(v_1, v_j|C_g) + 4\sum_{j=4}^{g} |T_j| d(v_2, v_j|C_g) \\ & + 2|T_3| \textstyle\sum_{j=4}^{g} |T_j| d(v_3, v_j|C_g) + 8 + 12|T_3| \\ & - 2(4 + |T_3|) \textstyle\sum_{j=4}^{g} |T_j| d(v_3, v_j|C_{g-2}) \\ & - \delta(g)(4 + 4|T_3|) + \delta(g)(2 + 2|T_3|) \\ \ge\ & \left(\tfrac{25}{2} - 2\delta(g)\right) + (12 - 2\delta(g))|T_3| - 6\beta \\ & + 4\textstyle\sum_{j=4}^{g} |T_j| d(v_1, v_j|C_g) + 4\sum_{j=4}^{g} |T_j| d(v_2, v_j|C_g). \\ & + 2|T_3| \textstyle\sum_{j=4}^{g} |T_j| d(v_3, v_j|C_g) \\ & - 2(4 + |T_3|) \textstyle\sum_{j=4}^{g} |T_j| d(v_3, v_j|C_{g-2}) \end{aligned}$$

(i) If the cycle length g is odd, then

$$\begin{aligned} Sz_e^*(G) - Sz_e^*(G') \ge\ & \tfrac{21}{2} + 10|T_3| - 6\beta + 12\textstyle\sum_{j=4}^{\frac{g+1}{2}} |T_j| + 8|T_{\frac{g+3}{2}}| + (4 + 2|T_3|)|T_{\frac{g+5}{2}}| \\ & + (4 + 4|T_3|) \textstyle\sum_{j=\frac{g+7}{2}}^{g} |T_j| \\ =\ & -\tfrac{3}{2} + 7|T_3| + 9\textstyle\sum_{j=4}^{\frac{g+1}{2}} |T_j| + 5|T_{\frac{g+3}{2}}| + (1 + 2|T_3|)|T_{\frac{g+5}{2}}| \\ & + (1 + 4|T_3|) \textstyle\sum_{j=\frac{g+7}{2}}^{g} |T_j| > 0. \end{aligned}$$

(ii) If the cycle length g is even, then

$$\begin{aligned} Sz_e^*(G) - Sz_e^*(G') \ge\ & \tfrac{25}{2} + 12|T_3| - 6\beta + 12\textstyle\sum_{j=4}^{\frac{g}{2}+1} |T_j| + 4|T_{\frac{g}{2}+2}| \\ & + (4 + 4|T_3|) \textstyle\sum_{j=\frac{g}{2}+3}^{g} |T_j| \\ =\ & \tfrac{1}{2} + 9|T_3| + 9\textstyle\sum_{j=4}^{\frac{g}{2}+1} |T_j| + |T_{\frac{g}{2}+2}| + (1 + 4|T_3|) \textstyle\sum_{j=\frac{g}{2}+3}^{g} |T_j| > 0. \end{aligned}$$

So, the proof is completed. ■

Lemma 2.10. *Let* $G = C_g(T_1, T_2, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$, *such that* $T_i = K_1$ or K_2, $1 \le i \le g - 1$) *and the cycle length* $g \ge 5$. *If* $d(v_1) = d(v_2) = 2$, *let* $G' = G + v_g v_3 - v_g v_1$, *then* $Sz_e^*(G) > Sz_e^*(G')$.

Proof. As $G \in \mathcal{U}(2\beta,\beta)$, then $G' \in \mathcal{U}(2\beta,\beta)$. By Theorem 2.6, we have that

$$
\begin{aligned}
Sz_e^*(G) - Sz_e^*(G') \geq & [-1-3|T_3|-2D(v_3|T_3)] + [3-6\beta+3|T_3|+2D(v_3|T_3)] \\
& + 2\beta + \tfrac{1}{2} + 2\textstyle\sum_{j=4}^{g}|T_j|d(v_1,v_j|C_g) + 2\sum_{j=4}^{g}|T_j|d(v_2,v_j|C_g) \\
& + 2|T_3|\textstyle\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_g) + 2 + 6|T_3| \\
& - 2(2+|T_3|)\textstyle\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_{g-2}) \\
& - \delta(g)(1+2|T_3|) + \delta(g)\left(\tfrac{1}{2}+|T_3|\right) \\
& \geq \left(\tfrac{9}{2}-\tfrac{1}{2}\delta(g)\right) + (6-\delta(g))|T_3| - 4\beta + 2\textstyle\sum_{j=4}^{g}|T_j|d(v_1,v_j|C_g) \\
& + 2\textstyle\sum_{j=4}^{g}|T_j|d(v_2,v_j|C_g) + 2|T_3|\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_g) \\
& -2(2+|T_3|)\textstyle\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_{g-2}).
\end{aligned}
$$

(i) If the cycle length g is odd.

$$
\begin{aligned}
Sz_e^*(G) - Sz_e^*(G') \geq & 4 + 5|T_3| - 4\beta + 6\textstyle\sum_{j=4}^{\frac{g+1}{2}}|T_j| + 4|T_{\frac{g+3}{2}}| + (2+2|T_3|)|T_{\frac{g+5}{2}}|. \\
& +(2+4|T_3|)\textstyle\sum_{j=\frac{g+7}{2}}^{g}|T_j| \\
& = 3|T_3| + 4\textstyle\sum_{j=4}^{\frac{g+1}{2}}|T_j| + 2|T_{\frac{g+3}{2}}| + 2|T_3||T_{\frac{g+5}{2}}| + 4|T_3|\sum_{j=\frac{g+7}{2}}^{g}|T_j| > 0.
\end{aligned}
$$

(ii) If the cycle length g is even.

$$
\begin{aligned}
Sz_e^*(G) - Sz_e^*(G') \geq & \tfrac{9}{2} + 6|T_3| - 4\beta + 6\textstyle\sum_{j=4}^{\frac{g}{2}+1}|T_j| + 2|T_{\frac{g}{2}+2}| + (2+4|T_3|)\sum_{j=\frac{g}{2}+3}^{g}|T_j| \\
& = \tfrac{1}{2} + 4|T_3| + 4\textstyle\sum_{j=4}^{\frac{g}{2}+1}|T_j| + 4|T_3|\sum_{j=\frac{g}{2}+3}^{g}|T_j| > 0.
\end{aligned}
$$

Hence, the proof is completed. ■

Lemma 2.11. *Let* $G = C_g(T_1,T_2,\ldots,T_g) \in \mathcal{U}(2\beta,\beta)$ *such that* $T_i = K_1$ *or* $K_2, 1 \leq i \leq g-1$, *and the cycle length* $g \geq 5$. *If* $d(v_1) = 3$, $d(v_2) = 2$, *let* $G' = G + v_g v_3 + v_1 v_3 - v_g v_1 - v_1 v_2$, *then* $Sz_e^*(G) > Sz_e^*(G')$.

Proof. As $G \in \mathcal{U}(2\beta,\beta)$, then $G' \in \mathcal{U}(2\beta,\beta)$. By Theorem 2.6,

$$
\begin{aligned}
Sz_e^*(G) - Sz_e^*(G') \geq & -9 + [11-6\beta] + 4\textstyle\sum_{j=4}^{g}|T_j|d(v_1,v_j|C_g) + 2\sum_{j=4}^{g}|T_j|d(v_2,v_j|C_g) \\
& +2\textstyle\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_g) + 14 - 8\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_{g-2}) \\
& -5\delta(g) + \tfrac{5}{2}\delta(g) + 2\beta + \tfrac{1}{2} \\
& \geq \left(\tfrac{33}{2}-\tfrac{5}{2}\delta(g)\right) - 4\beta + 4\textstyle\sum_{j=4}^{g}|T_j|d(v_1,v_j|C_g)
\end{aligned}
$$

$$+ 2\sum_{j=4}^{g}|T_j|d(v_2,v_j|C_g) + 2\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_g) - 8\sum_{j=4}^{g}|T_j|d(v_3,v_j|C_{g-2}).$$

(i) If the cycle length g is odd, then

$$Sz_e^*(G) - Sz_e^*(G') \geq \frac{28}{2} - 4\beta + 10\sum_{j=4}^{\frac{g+1}{2}}|T_j| + 6|T_{\frac{g+3}{2}}| + 4|T_{\frac{g+5}{2}}| + 6\sum_{j=\frac{g+7}{2}}^{g}|T_j|.$$

$$= 6 + 8\sum_{j=4}^{\frac{g+1}{2}}|T_j| + 4|T_{\frac{g+3}{2}}| + 2|T_{\frac{g+5}{2}}| + 4\sum_{j=\frac{g+7}{2}}^{g}|T_j| > 0.$$

(ii) If the cycle length g is even, then

$$Sz_e^*(G) - Sz_e^*(G') \geq \frac{33}{2} - 4\beta + 12\sum_{j=4}^{\frac{g}{2}+1}|T_j| + 4|T_{\frac{g}{2}+2}| + 8\sum_{j=\frac{g}{2}+3}^{g}|T_j|.$$

$$= \frac{17}{2} + 10\sum_{j=4}^{\frac{g}{2}+1}|T_j| + 2|T_{\frac{g}{2}+2}| + 6\sum_{j=\frac{g}{2}+3}^{g}|T_j| > 0.$$

The proof is now completed. ■

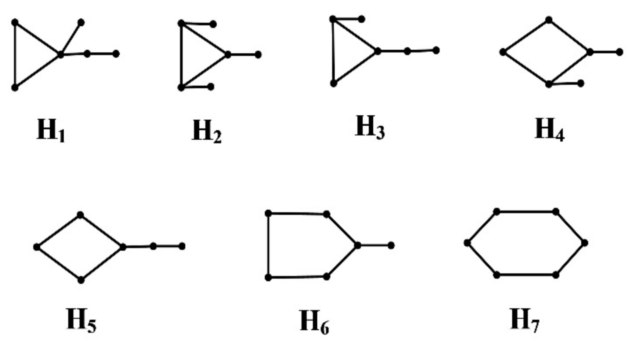

Figure 4. Seven conjugated unicyclic graphs with $\beta = 3$.

Theorem 2.12. *Let* $G = C_g(T_1, T_2, \ldots, T_g) \in \mathcal{U}(6,3)$. *Then* $Sz_e^*(H_1) < Sz_e^*(H_2) < Sz_e^*(H_3) < Sz_e^*(H_4) < Sz_e^*(H_5) < Sz_e^*(H_6) < Sz_e^*(H_7)$, where $H_i, 1 \leq i \leq 7$, *be shown in Figure* 4.

Proof. There are only seven conjugated unicyclic graphs in $\mathcal{U}(6,3)$, which was shown in Figure 4. By calculating directly, we have that

$$Sz_e^*(H_1) = \frac{139}{4},\ Sz_e^*(H_2) = \frac{141}{4},\ Sz_e^*(H_3) = \frac{151}{4},\ Sz_e^*(H_4) = \frac{79}{2},$$
$$Sz_e^*(H_5) = \frac{83}{2},\ Sz_e^*(H_6) = \frac{187}{4},\ Sz_e^*(H_7) = 54.$$

So we have that $Sz_e^*(H_1) < Sz_e^*(H_2) < Sz_e^*(H_3) < Sz_e^*(H_4) < Sz_e^*(H_5) < Sz_e^*(H_6) < Sz_e^*(H_7)$. The result follows. ■

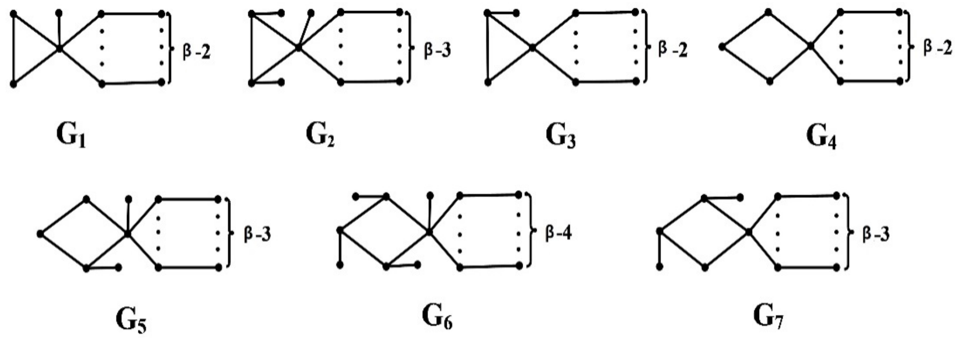

Figure 5. Seven conjugated unicyclic graphs.

Theorem 2.13. *Let* $G = C_g(T_1, T_2, \dots, T_g) \in \mathcal{U}(2\beta, \beta)$ $(\beta \geq 4)$.
(i) *If* $4 \leq \beta \leq 7$, *then* $Sz_e^*(G) \geq 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4}$, *with equality if and only if* $G \cong G_1$;
(ii) *If* $\beta \geq 8$, *then* $Sz_e^*(G) \geq 4\beta^2 + \frac{11}{2}\beta - 11$, *with equality if and only if* $G \cong G_4$;

Proof. By using Lemmas 2.7, 2.9, 2.10 and 2.11 repeatedly, the final graphs are $\{G_i\}$, $1 \leq i \leq 7$, see Figure 5. By calculating directly, we have that

$$Sz_e^*(G_1) = 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4},$$
$$Sz_e^*(G_2) = 5\beta^2 + \frac{1}{2}\beta - \frac{45}{4},$$
$$Sz_e^*(G_3) = 5\beta^2 - \frac{1}{2}\beta - \frac{23}{4},$$
$$Sz_e^*(G_4) = 4\beta^2 + \frac{11}{2}\beta - 11,$$
$$Sz_e^*(G_5) = 4\beta^2 + \frac{15}{2}\beta - 19,$$
$$Sz_e^*(G_6) = 4\beta^2 + \frac{35}{2}\beta - 55,$$
$$Sz_e^*(G_7) = 4\beta^2 + \frac{31}{2}\beta - 43.$$

Then, we have that

$$Sz_e^*(G) \geq Sz_e^*(G_1) = 5\beta^2 - \frac{7}{2}\beta + \frac{1}{4}, \text{ for } 4 \leq \beta \leq 7,$$
$$Sz_e^*(G) \geq Sz_e^*(G_4) = 4\beta^2 + \frac{11}{2}\beta - 11, \text{ for } \beta \geq 8.$$

The result follows. ■

3. On Revised Edge-Szeged of the Join of Graphs

In the section, we consider revised edge-Szeged index of the join graph. The join graph of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$, and with edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G),\ v \in V(H)\}$. For the revised edge-Szeged index of the graph G, let $|E(G \vee H)| = m$, we have

$$Sz_e^*(G) = \sum_{e=uv\in E(G)}(m_u(e) + \frac{m_0(e)}{2})\,(m_v(e) + \frac{m_0(e)}{2}).$$
$$= \frac{1}{4}\sum_{e=uv\in E(G)}(m + m_u(e) - m_v(e))\,(m + m_v(e) - m_u(e)).$$
$$= \frac{m^3}{4} - \frac{1}{4}\sum_{e=uv\in E(G)}(m_u(e) - m_v(e))^2.$$

Theorem 3.1. *Let G and H be simple graphs, where $|E(G \vee H)| = m$, $|G| = n_1$, $|E(G)| = m_1$, $|H| = n_2$* and *$|E(H)| = m_2$. Then $Sz_e^*(G \vee H) = \frac{m^3}{4} - \frac{1}{4}(S_1 + S_2 + S_3)$*, where
$S_1 = \sum_{e=uv\in E(G)}[\left(d_2^G(u) + d_G(u)\right) - \left(d_2^G(v) + d_G(v)\right)]^2$,
$S_2 = \sum_{e=uv\in E(H)}[\left(d_2^H(u) + d_H(u)\right) - \left(d_2^H(v) + d_H(v)\right)]^2$,
$S_3 = \sum_{e=uv\in E'}[\left(d_G(u) - d_H(v)\right) + (n_2 - n_1) + (m_2 - m_1) + \left(d_2^G(u) - d_2^H(v)\right) + (t_H(v) - t_G(u))]^2$.

Proof. We divide the edge of $G \vee H$ into three groups: $E(G)$, $E(H)$ and $E' = \{uv : u \in V(G),\ v \in V(H)\}$.

Case 1. $e = uv \in E(G)$. When $e' = u'v' \in E(H)$ or $u' \in V(G)$, $v' \in V(H)$, $u' \neq u, v$, then $d_{G\vee H}(u, e') = d_{G\vee H}(v, e') = 1$. When $e'' = u''v'' \in E(G)$ and $d_G(u, e'') \geq 2, d_G(v, e'') \geq 2$, then $d_{G\vee H}(u, e'') = d_{G\vee H}(v, e'') = 2$. Let $N'_G(u) = N_G(u)\backslash\{v\}$ and $N'_G(u) = N_1(u) \cup N_2(u) \cup N_3(u)$, where

$N_1(u) = \{w \in N'_G(u) |\ w$ is in a triangle that contains edge $uv\}$,
$N_2(u) = \{w \in N'_G(u) |\ w$ is in a quadrilateral that contains edge $uv\}$,
$N_3(u) = N'_G(u) \setminus \{ N_1(u) \cup N_2(u)\}$.

Then, one known that

$$\begin{aligned} m_u(e|G \vee H) + \sum_{w\in N_1(u)}(d_G(w) - 2) &= n_2 + (d_G(u) - 1) + \sum_{w\in N_1(u)}(d_G(w) - 2) \\ &+ \sum_{w\in N_2(u)}(d_G(w) - 2) + \sum_{w\in N_3(u)}(d_G(w) - 1) \\ &= n_2 + (d_G(u) - 1) + \sum_{w\in N_1(u)} d_G(w) - |N_1(u)| \\ &+ \sum_{w\in N_2(u)} d_G(w) - |N_2(u)| + \sum_{w\in N_3(u)} d_G(w) \\ &- (|N_1(u)| + |N_2(u)| + |N_3(u)|). \\ &= n_2 + d_2^G(u) - d_G(v) - |N_1(u)| - |N_2(u)|. \end{aligned}$$

Similarly, we have that $m_v(e|G \vee H) + \sum_{w\in N_1(v)}(d_G(w) - 2) = n_2 + d_2^G(v) - d_G(u) - |N_1(v)| - |N_2(v)|$. It is obvious that $\sum_{w\in N_1(u)}(d_G(w) - 2)$=$\sum_{w\in N_1(v)}(d_G(w) - 2)$. Then,

$$S_1 = \sum_{uv\in E(G)}(m_u(e) - m_v(e))^2 = \sum_{uv\in E(G)}[\left(d_2^G(u) + d_G(u)\right) - \left(d_2^G(v) + d_G(v)\right)]^2.$$

Case 2. $e = uv \in E(H)$. Similarly, we have that

$$S_2 = \sum_{uv\in E(G)}(m_u(e) - m_v(e))^2 = \sum_{uv\in E(H)}[\left(d_2^H(u) + d_H(u)\right) - \left(d_2^H(v) + d_H(v)\right)]^2.$$

Case 3. $e = uv \in E'$. One known that $m_u(e|G \vee H) = d_G(u) + (n_2 - 1) + (m_2 - d_2^H(v) + t_H(v))$ and $m_v(e|G \vee H) = d_H(v) + (n_1 - 1) + (m_1 - d_2^G(u) + t_G(u))$. Thus,

$S_3 = \sum_{e=uv\in E'}[\left(d_G(u) - d_H(v)\right) + (n_2 - n_1) + (m_2 - m_1) + \left(d_2^G(u) - d_2^H(v)\right) + (t_H(v) - t_G(u))]^2.$

In summary, we have that $Sz_e^*(G \vee H) = \frac{m^3}{4} - \frac{1}{4}(S_1 + S_2 + S_3)$ and the result follows. ∎

By Theorem 3.1, one can calculate revised edge-Szeged index of some special graphs, such as the complete bipartite graph $K_{m,n} = \overline{K_m} \vee \overline{K_n}$, the wheel graph $W_n = K_1 \vee C_{n-1}, n \geq 5$, the fan graph $F_n = K_1 \vee P_{n-1},\ n \geq 6$.

$$Sz_e^*(K_{m,n}) = Sz_e^*(\overline{K_m} \vee \overline{K_n}) = \frac{1}{4}nm(n^2m^2 - (n-m)^2),$$

$$Sz_e^*(W_n) = Sz_e^*(K_1 \vee C_{n-1}) = \frac{1}{4}(n-1)(4n^2 + 20n - 73), (n \geq 5),$$

$$Sz_e^*(F_n) = Sz_e^*(K_1 \vee P_{n-1}) = \frac{1}{4}(4n^3 + 8n^2 - 118n + 203), (n \geq 6).$$

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